Trigonometric Interpolation and Wavelet Decompositions

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CAT Report #296

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Abstract

The aim of this paper is to describe explicit decomposition and reconstruction algorithms for nested spaces of trigonometric polynomials. The scaling functions of these spaces are defined as fundamental polynomials of Lagrange interpolation. The interpolatory conditions and the construction of dual functions are crucial for the approach presented in this paper.

*This paper was completed while the first author visited the Center for Approximation Theory, Texas A&M University. Research partially supported by Deutsche Forschungsgemeinschaft.

1. Introduction

The recent rapid development of wavelet theory opens up opportunities to investigate well-known mathematical objects such as trigonometric polynomial interpolants from the point of view of multiresolution analysis and decomposition and reconstruction algorithms. Usually periodic wavelets are obtained by periodization techniques starting from a multiresolution analysis on the real axis. The approach pursued in this paper, however, consists of the direct investigation of finite dimensional nested spaces of trigonometric polynomials which represent the simplest periodic analytic functions. Hence, scaling functions and wavelets are given explicitly as trigonometric interpolants and decomposition and reconstruction algorithms can be described in simple matrix notation. The circulant structure of all relevant matrices allows the use of Fast-Fourier-Transform techniques for the actual implementation. Thus we achieve almost optimal complexity compared to other wavelet approaches derived from implicit two-scale relations, while dealing with a fully computable trigonometric multiresolution analysis with explicit algebraic formulas. This should lead to useful applications in the treatment of the numerical solution of certain operator equations and could also be important in applications as signal analysis whenever a separation into frequency bands or octaves - and not just single sine/cosine frequencies - is necessary. For a general study of the corresponding concepts of multiresolution analysis, scaling functions and wavelets, the reader is referred to [1,4,7].

The first paper dealing with a trigonometric wavelet approach for $L^2_{2\pi}$ by C. K. Chui and H. N. Mhaskar [2] describes how the concept of a multiresolution analysis can be adapted to nested spaces of trigonometric polynomials. Unfortunately, the scaling functions and wavelets of Chui and Mhaskar do not possess interpolatory properties which would facilitate the computation of the corresponding decomposition and reconstruction sequences and the projection of any given function onto a scaling function space.

Therefore it is natural to investigate the existence of trigonometric scaling functions which are fundamental functions of Lagrange interpolation. Not only does this enable us to find simple (nonorthogonal) projections onto the scaling function spaces, it also makes the explicit computation of the two-scale coefficients much easier. It turns out that such trigonometric polynomials have been investigated for a long time in approximation theory.

For a more detailed consideration of these interpolants, see [9] and e.g. [10, Chapter 10]).

Let us remark that wavelet type techniques were instrumental in several recent important contributions to the old problem of finding mutually orthogonal trigonometric polynomial bases $\{p_n\}$ of minimal degree $d(n)$ for $C_{2\pi}$, where it is known that $d(n) = (n + \imath(n))/2$ is impossible. In the paper [9], A. A. Privalov constructed a basis with $d(n) = 4n/3$. This result was improved to $d(n) = n(1+\epsilon)$ by D. Offin and K. Oskolkov ([8]) using a periodized wavelet basis and finally settled as $d(n) = n(1+\epsilon)/2$ by R. A. Lorentz and A. A. Sahakian [6] who adopted a wavelet packet approach.

As we are specifically interested in algorithmic questions, we follow Privalov's construction of trigonometric scaling functions and wavelets interpolating on a suitable equally spaced dyadic grid. In Section 2, all relevant results from [9] are summarized in the framework of a trigonometric multiresolution analysis.

The new investigations in this paper are concerned with the explicit computation of the basis transformations connecting the spaces of scaling functions and wavelets. The corresponding transformation matrices are studied in more detail, in particular their circulant structure can be used to obtain efficient algorithms. The two-scale relations for reconstruction are proven in Section 3. In Section 4, the biorthogonal bases of dual functions are computed explicitly. The usefulness of dual scaling functions and wavelets – as described in [3] for functions on the real axis – is seen in Section 5, where the more complicated decomposition relations are established. Then, Section 6 is devoted to the presentation of the decomposition and reconstruction algorithms in a suitable matrix/vector form.

Throughout the paper two different types of Lagrange interpolants are considered which yield different scaling function and wavelet spaces. For a fixed number of interpolation nodes the scaling functions differ in polynomial degree, approximation order and decay properties. A discussion of these topics concludes the paper.

2. Definitions

For $\ell \in \mathrm{N}$, the Dirichlet kernel $D_\ell \in T_\ell$ is defined as

$$D_\ell(x) = \frac{1}{2} + \sum_{k=1}^{\ell} \cos kx = \begin{cases} \frac{\sin(\ell+\frac{1}{2})x}{2\sin\frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ \ell + \frac{1}{2} & \text{for } x \in 2\pi\mathbb{Z}\ , \end{cases}$$

where T_ℓ denotes the linear space of trigonometric polynomials of degree ℓ.

In the following, two different kinds $\phi_{j,0}^D$ and $\phi_{j,0}^F$ of de la Vallée Poussin kernels are used to construct certain interpolatory operators. Therefore, let

$$\begin{aligned}\phi_{j,0}^D(x) &= \frac{1}{3\cdot 2^{2j+1}} \sum_{\ell=2^{j+1}}^{2^{j+2}-1} D_\ell(x) \\ &= \begin{cases} \frac{\sin(3\cdot 2^j x)\sin(2^j x)}{3\cdot 2^{2j+2}\sin^2(\frac{x}{2})} & \text{for } x \notin 2\pi\mathbb{Z}, \\ 1 & \text{for } x \in 2\pi\mathbb{Z} \end{cases}\end{aligned}$$

and

$$\begin{aligned}\phi_{j,0}^F(x) &= \frac{1}{3\cdot 2^{j+1}} \sum_{\ell=3\cdot 2^j-1}^{3\cdot 2^j} D_\ell(x) \\ &= \begin{cases} \frac{\sin(3\cdot 2^j x)\cot\frac{x}{2}}{3\cdot 2^{j+1}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ 1 & \text{for } x \in 2\pi\mathbb{Z}\ . \end{cases}\end{aligned}$$

While the first kernel represents the classical de la Vallée Poussin approach, the second one is only a slight modification of a simple Dirichlet kernel.

As it is well-known, the usual Fourier sum operator based on the Dirichlet kernel provides the best approximation in the $L_{2\pi}^2$-norm but has unbounded operator norms in $C_{2\pi}$ and $L_{2\pi}^1$, respectively, which leads to a slower convergence rate in the corresponding norms. Due to the importance of uniform convergence there is an interest in using means to ensure bounded operator norms. This can be achieved by the above mentioned de la Vallée Poussin mean at the expense of a higher polynomial degree. In particular, the operator norms in $C_{2\pi}$ which are equivalent to the $L_{2\pi}^1$-norm of the kernel are uniformly bounded with respect to j for $\phi_{j,0}^D$, whereas they grow linearly with j for $\phi_{j,0}^F$. This well-known theoretical result is reflected by the stronger oscillations of the fundamental functions in the Fourier case (see Fig. 1 and 2). Note that the boundedness of the operator norms for the de la Vallée Poussin case is also the starting point for the investigations in [9] regarding orthogonal polynomial bases for $C_{2\pi}$. It is feasible to consider other means of summing Dirichlet kernels, but for simplicity our algorithmic presentation is restricted to the two cases defined above. Actually, these two cases show a lot of similarities in their computational treatment. Hence, in the following the superscripts D and F will be dropped whenever a statement holds for both cases.

The crucial interpolatory property of $\phi_{j,0}$ is

$$\phi_{j,0}(\frac{k\pi}{3\cdot 2^j}) = \delta_{k,0}, \quad k = 0, 1, \ldots, 3\cdot 2^j - 1. \tag{2.1}$$

Definition 2.1. *For $j \in \mathbb{N}_0$, the spaces V_j are defined by $V_j = span\{\phi_{j,n} : n = 0, \ldots, 3 \cdot 2^{j+1} - 1\}$, where $\phi_{j,n}(x) = \phi_{j,0}(x - \frac{n\pi}{3 \cdot 2^j})$. For notational convenience, let $\phi_{j,n} = \phi_{j,n \bmod 3 \cdot 2^{j+1}}$ for any $n \in \mathbb{Z}$. Note that V_j^D as generated by $\phi_{j,n}^D$ and V_j^F generated by $\phi_{j,n}^F$ are different spaces. Nevertheless, they have the same dimension $3 \cdot 2^{j+1}$ as can be deduced from the corresponding interpolatory property $\phi_{j,n}(\frac{k\pi}{3 \cdot 2^j}) = \delta_{k,n}$, for all $k, n \in \mathbb{Z}$.*

Definition 2.2. *For any $j \in \mathbb{N}_0$, the interpolation operator L_j mapping any real-valued 2π-periodic function f into the space V_j is defined as*

$$L_j f(x) = \sum_{n=0}^{3 \cdot 2^{j+1} - 1} f(\frac{n\pi}{3 \cdot 2^j}) \phi_{j,n}(x).$$

The following properties of the operators L_j are well-known ([9],[10]):

$$(i) \qquad L_j^D f \in T_{2^{j+2}-1}, \quad L_j^F f \in T_{3 \cdot 2^j}$$

$$(ii) \qquad L_j f(\frac{k\pi}{3 \cdot 2^j}) = f(\frac{k\pi}{3 \cdot 2^j}), k \in \mathbb{Z}$$

$$(iii) \qquad L_j^D f = f \quad \text{for all} \quad f \in T_{2^{j+1}} \cup V_j^D,$$

$$L_j^F f = f \quad \text{for all} \quad f \in T_{3 \cdot 2^j - 1} \cup V_j^F,$$

hence

$$(iv) \qquad T_{2^{j+1}} \subset V_j^D \subset T_{2^{j+2}-1},$$

$$T_{3 \cdot 2^j - 1} \subset V_j^F \subset T_{3 \cdot 2^j}.$$

A more detailed analysis of these inclusions will be undertaken in Section 7. Moreover, property (iv) implies that

$$V_j \subset V_{j+1} \quad ,$$

i.e., the spaces V_j form a sequence of nested subspaces of $L^2_{2\pi}$, the space of 2π-periodic square integrable functions. Setting $V_{-1} = \{0\}$, it is also clear that

$$L^2_{2\pi} = L^2 - \text{closure of} \quad \bigcup_{j=-1}^{\infty} V_j \quad \text{and} \quad \bigcap_{j=-1}^{\infty} V_j = \{0\} \ .$$

As the next step, the orthogonal complement of V_j relative to V_{j+1}, *i.e.*, the so-called wavelet space W_j needs to be described in more detail.

Definition 2.3. *For $j \in \mathbb{N}_0$, the spaces W_j are defined by $W_j = span\{\psi_{j,n} : n = 0, \ldots, 3 \cdot 2^{j+1} - 1\}$, where*

$$\psi_{j,n}(x) = 2\phi_{j+1,2n+1}(x) - \phi_{j,n}(x - \frac{\pi}{3 \cdot 2^{j+1}}) \in V_{j+1} \quad . \tag{2.2}$$

The functions $\psi_{j,n}$ also show interpolatory properties, namely for all $k \in \mathbb{Z}$

$$\begin{aligned} \psi_{j,n}(\frac{(2k+1)\pi}{3 \cdot 2^{j+1}}) &= \delta_{k,n}, \\ \psi_{j,n}(\frac{k\pi}{3 \cdot 2^{j}}) &= -\phi_{j,n}(\frac{(2k-1)\pi}{3 \cdot 2^{j+1}}) \ . \end{aligned} \tag{2.3}$$

Therefore, $\dim W_j = 3 \cdot 2^{j+1}$. Let $\langle \cdot, \cdot \rangle$ denote the inner product of two functions f and g in $L^2_{2\pi}$, *i.e.*,

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f(x)g(x)dx.$$

The following fundamental result was established by A. A. Privalov [9]:

Theorem A. *The spaces V_j and W_j are orthogonal, i.e.,*

$$\langle \phi_{j,n}, \psi_{j,k} \rangle = 0 \quad \text{for all} \quad n, k \in \mathbb{Z}.$$

Therefore,

$$V_{j+1} = V_j \oplus W_j,$$

where $\oplus$ denotes orthogonal summation.

A relation between V_{j+1} and V_j based solely on dilation is incompatible with the periodicity of the functions involved. Notwithstanding, the following can be computed directly using the definitions of $\phi_{j,0}$ and $\psi_{j,3 \cdot 2^{j+1}-1}$

(i)
$$\begin{aligned} \phi^D_{j+1,0}(x) &= \phi^D_{j,0}(2x)\frac{1+\cos x}{2}, \\ \phi^F_{j+1,0}(x) &= \phi^F_{j,0}(2x)(\frac{1}{2} + \frac{1}{2\cos x}), \end{aligned}$$

(ii)
$$\begin{aligned} \psi^D_{j+1,3 \cdot 2^{j+2}-1}(x) &= \psi^D_{j,3 \cdot 2^{j+1}-1}(2x)\frac{1+\cos x}{2}, \\ \psi^F_{j+1,3 \cdot 2^{j+2}-1}(x) &= \psi^F_{j,3 \cdot 2^{j+1}-1}(2x)(\frac{1}{2} + \frac{1}{2\cos x}) \quad . \end{aligned}$$

Note that the corrective factors $\frac{1+\cos x}{2}$ and $\frac{1}{2}+\frac{1}{2\cos x}$, respectively , are independent of the level j.

3. Two-scale Relations

A cornerstone for the development of reconstruction and decomposition algorithms for the "trigonometric multiresolution analysis" described in Section 2 is the knowledge of the corresponding reconstruction and decomposition sequences for the spaces V_j and W_j. This section is devoted to the computation of the two-scale sequences for reconstruction. As $V_j \subset V_{j+1}$, there must be coefficients $p_{j,n,s}$ such that $\phi_{j,n} = \sum_s p_{j,n,s}\phi_{j+1,s}$. The following result establishes the precise values of these $p_{j,n,s}$.

Theorem 3.1. *For $j \in \mathbb{N}_0$ and $n = 0, 1 \ldots, 3 \cdot 2^{j+1} - 1$, it holds that*

$$\phi_{j,n}(x) = \phi_{j+1,2n}(x) + \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,n}(\frac{(2s+1)\pi}{3\cdot 2^{j+1}})\phi_{j+1,2s+1}(x).$$

Proof: The properties of the interpolatory operator introduced in Section 2 imply that the functions $\phi_{j,n}$ are reproduced by L_{j+1} , *i.e.*,

$$\phi_{j,n}(x) = \sum_{k=0}^{3\cdot 2^{j+2}-1} \phi_{j,n}(\frac{k\pi}{3\cdot 2^{j+1}})\phi_{j+1,k}(x).$$

The desired result now follows directly from the fact that for $k = 2s$, it holds that $\phi_{j,n}(\frac{2s\pi}{3\cdot 2^{j+1}}) = \delta_{s,n}$. □

From $W_j \subset V_{j+1}$, it is clear that there also must be coefficients $q_{j,n,s}$ such that $\psi_{j,n} = \sum_s q_{j,n,s}\phi_{j+1,s}$ which are determined using the following.

Theorem 3.2. *For $j \in \mathbb{N}_0$ and $n = 0, 1 \ldots, 3 \cdot 2^{j+1} - 1$, the two-scale relation for the wavelet functions $\psi_{j,n}$ is*

$$\psi_{j,n}(x) = \phi_{j+1,2n+1}(x) - \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,n}(\frac{(2s-1)\pi}{3\cdot 2^{j+1}})\phi_{j+1,2s}(x).$$

Proof: As before, the reproduction property $L_{j+1}\phi_{j,n} = \phi_{j,n}$ is used, this time to obtain

$$\phi_{j,n}(x - \frac{\pi}{3 \cdot 2^{j+1}}) = \sum_{k=0}^{3 \cdot 2^{j+2}-1} \phi_{j,n}(\frac{k\pi}{3 \cdot 2^{j+1}})\phi_{j+1,k}(x - \frac{\pi}{3 \cdot 2^{j+1}}).$$

Therefore,

$$\begin{aligned}\psi_{j,n}(x) &= 2\phi_{j+1,2n+1}(x) - \phi_{j,n}(x - \frac{\pi}{3 \cdot 2^{j+1}}) \\ &= 2\phi_{j+1,2n+1}(x) - \sum_{k=0}^{3 \cdot 2^{j+2}-1} \phi_{j,n}(\frac{k\pi}{3 \cdot 2^{j+1}})\phi_{j+1,k+1}(x).\end{aligned}$$

By $\phi_{j,n}(\frac{s\pi}{3 \cdot 2^j}) = \delta_{s,n}$ for $s = 0, \ldots, 3 \cdot 2^{j+1} - 1$, this yields

$$\psi_{j,n}(x) = \phi_{j+1,2n+1}(x) - \sum_{s=1}^{3 \cdot 2^{j+1}} \phi_{j,n}(\frac{(2s-1)\pi}{3 \cdot 2^{j+1}})\phi_{j+1,2s}(x),$$

which completes the proof because

$$\phi_{j,n}(\frac{(3 \cdot 2^{j+2} - 1)\pi}{3 \cdot 2^{j+1}})\phi_{j+1,3 \cdot 2^{j+2}}(x) = \phi_{j,n}(\frac{-\pi}{3 \cdot 2^{j+1}})\phi_{j+1,0}(x) \quad . \quad \square$$

For future use, it is appropriate at this point to compute the inner products of the scaling functions $\phi_{j,n}$ for different levels j and k.

Lemma 3.1. *For $j, k \in \mathbb{N}_0$ with $j < k$, it holds that*

$$\langle \phi_{j,0}, \phi_{k,n} \rangle = \frac{\phi_{j,0}(\frac{n\pi}{3 \cdot 2^k})}{3 \cdot 2^{k+1}} \quad .$$

Proof: The Fourier sum representation of $\phi_{j,0} \in T_\ell$ implies

$$\frac{1}{2\pi}\int_0^{2\pi} \phi_{j,0}(x)D_\ell(x - \frac{n\pi}{3 \cdot 2^k})dx = \frac{1}{2}\phi_{j,0}(\frac{n\pi}{3 \cdot 2^k})$$

and therefore

$$\langle \phi^D_{j,0}, \phi^D_{k,n} \rangle = \frac{1}{3 \cdot 2^{2k+2}} \sum_{\ell=2^{k+1}}^{2^{k+2}-1} \phi^D_{j,0}(\frac{n\pi}{3 \cdot 2^k}) = \frac{1}{3 \cdot 2^{k+1}}\phi^D_{j,0}(\frac{n\pi}{3 \cdot 2^k}).$$

Similarly,

$$\langle \phi^F_{j,0}, \phi^F_{k,n} \rangle = \frac{1}{3 \cdot 2^{k+2}} \sum_{\ell=3\cdot 2^j-1}^{3\cdot 2^j} \phi^F_{j,0}(\frac{n\pi}{3 \cdot 2^k}) = \frac{1}{3 \cdot 2^{k+1}} \phi^F_{j,0}(\frac{n\pi}{3 \cdot 2^k}),$$

which completes the proof of this lemma. □

Let us remark that the orthogonality of the spaces V_j and W_j in Theorem A or more precisely of the functions $\phi_{j,n}$ and $\psi_{j,k}$ can be shown directly by using the two-scale relations of Theorems 3.1 and 3.2 and the formula for inner products given in Lemma 3.1.

4. Dual scaling functions and wavelets

In order to find the decomposition sequences for the orthogonal sum $V_{j+1} = V_j \oplus W_j$, it is convenient to make use of the so-called dual scaling functions and wavelets. The concept of duality was introduced in [3] for multiresolution analyses of $L^2(\mathbb{R})$ and is used here in a form suitably adapted to our purposes.

Definition 4.1. *For any $j \in \mathbb{N}_0$, the functions $\tilde{\phi}_{j,r} \in V_j, r = 0, \ldots, 3 \cdot 2^{j+1} - 1$, uniquely determined by the conditions*

$$\langle \tilde{\phi}_{j,r}, \phi_{j,k} \rangle = \delta_{r,k} \quad \text{for all} \quad r, k = 0, \ldots, 3 \cdot 2^{j+1} - 1$$

are called dual scaling functions (or dual to the functions $\phi_{j,r}$).

Note that the dual scaling functions $\tilde{\phi}_{j,r}$ lie in the same space V_j as the original scaling functions $\phi_{j,k}$. Consequently, the dual functions can be written as linear combinations of these scaling functions. In fact, the following result holds.

Theorem 4.1. *For any $j \in \mathbb{N}_0$ and the dual scaling function $\tilde{\phi}_{j,r}, r = 0, \ldots, 3 \cdot 2^{j+1} - 1$, it holds that*

$$\tilde{\phi}_{j,r} = \sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,r,s} \phi_{j,s},$$

where the coefficients are given as

$$\alpha^F_{j,r,s} = 3 \cdot 2^{j+1}\delta_{r,s} + (-1)^{r-s}$$

and

$$\alpha^D_{j,r,s} = 3 \cdot 2^{j+1}\delta_{r,s} + \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1} \left(\frac{2^{2j+1}}{\ell^2 - 3\ell 2^{j+1} + 5 \cdot 2^{2j+1}} - 1 \right) \cos \frac{\ell(r-s)\pi}{3 \cdot 2^j}$$

for $r, s = 0, \dots, 3 \cdot 2^{j+1} - 1$.

The following two lemmas are essential for the proof of the theorem.

The duality conditions lead to a linear system of equations for each dual function $\tilde{\phi}_{j,r}$, $r = 0, \dots, 3 \cdot 2^{j+1} - 1$, namely

$$\sum_{s=0}^{3 \cdot 2^{j+1}-1} \alpha_{j,r,s} \langle \phi_{j,k}, \phi_{j,s} \rangle = \delta_{r,k}, \quad k = 0, \dots, 3 \cdot 2^{j+1} - 1 \quad .$$

Note that the coefficient matrix

$$G_j = (\langle \phi_{j,k}, \phi_{j,s} \rangle)_{k=0}^{3 \cdot 2^{j+1}-1} {}_{,s=0}^{,3 \cdot 2^{j+1}-1}$$

is always the same for these linear systems, only the right hand sides differ for the particular dual functions. In fact, the special nature of the right hand sides shows that the coefficients $\alpha_{j,r,s}$ are actually the entries of G_j^{-1}. Therefore, a careful structural analysis of the matrix G_j is necessary. For this, the terminology and notation in [5] is used. Setting

$$\mu_{j,r,s} = \langle \phi_{j,r}, \phi_{j,s} \rangle \quad \text{for all} \quad r, s = 0, \dots, 3 \cdot 2^{j+1} - 1, \tag{4.1}$$

it is obtained from the definition of the scaling functions that

$$\mu_{j,r,s} = \mu_{j,0,s-r \bmod 3 \cdot 2^{j+1}} \quad .$$

Lemma 4.1. *For any $j \in \mathbb{N}_0$, G_j is a circulant matrix of dimension $3 \cdot 2^{j+1}$ with*

$$G_j = \operatorname{circ}(\mu_{j,0,0}, \mu_{j,0,1}, \ldots, \mu_{j,0,3\cdot 2^{j+1}-1})$$

and its inverse is given by

$$G_j^{-1} = (\alpha_{j,r,s})_{r,s} = \frac{1}{3 \cdot 2^{j+1}} \left(\sum_{\ell=0}^{3\cdot 2^{j+1}-1} \left(\sum_{k=0}^{3\cdot 2^{j+1}-1} \mu_{j,0,k} \; exp(\frac{\ell \pi i}{3 \cdot 2^j}(k+r-s)) \right)^{-1} \right)_{r,s},$$

where i is the imaginary unit.

Proof: It is clear from (4.1) that G_j is a circulant matrix with the given entries. According to [5, Theorem 3.2.4], G_j can be written as

$$G_j = \bar{F}_j \Lambda_j F_j,$$

where

$$F_j = \frac{1}{\sqrt{3 \cdot 2^{j+1}}} \left(exp(\frac{rs\pi i}{3 \cdot 2^j}) \right)_{r,s} \quad \text{and} \quad \Lambda_j = \operatorname{diag} \left(\sum_{k=0}^{3\cdot 2^{j+1}-1} \mu_{j,0,k} exp(\frac{kr\pi i}{3 \cdot 2^j}) \right)_r .$$

This representation implies that $G_j^{-1} = \bar{F}_j \Lambda_j^{-1} F_j$ and a short computation yields the desired form of the coefficients $\alpha_{j,r,s}$ as the entries of G_j^{-1}. □

The entries of $\left(G_j^D\right)^{-1}$ warrant further investigation, and as a first step we establish

Lemma 4.2. *For any $j \in \mathbb{N}_0$ and r, s as in (4.1),*

$$\begin{aligned}
\Sigma_C &= \sum_{n=0}^{3\cdot 2^{j+1}-1} \mu_{j,0,n}^D \cos \frac{\ell(n+r-s)\pi}{3 \cdot 2^j} \\
&= \begin{cases} \frac{1}{3\cdot 2^{j+1}} \cos \frac{(r-s)\ell\pi}{3\cdot 2^j} & \text{if } \ell \in [0, 2^{j+1}], \\ & \text{or } \ell \in [2 \cdot 2^{j+1}, 3 \cdot 2^{j+1}), \\ \frac{(2^{j+2}-\ell)^2 + (\ell - 2^{j+1})^2}{3 \cdot 2^{3j+3}} \cos \frac{(r-s)\ell\pi}{3\cdot 2^j} & \text{if } \ell \in (2^{j+1}, 2^{j+2}), \end{cases} \\
\Sigma_S &= \sum_{n=0}^{3\cdot 2^{j+1}-1} \mu_{j,0,n}^D \sin \frac{\ell(n+r-s)\pi}{3 \cdot 2^j} \\
&= \begin{cases} \frac{1}{3\cdot 2^{j+1}} \sin \frac{(r-s)\ell\pi}{3\cdot 2^j} & \text{if } \ell \in [0, 2^{j+1}], \\ & \text{or } \ell \in [2 \cdot 2^{j+1}, 3 \cdot 2^{j+1}), \\ \frac{(2^{j+2}-\ell)^2 + (\ell - 2^{j+1})^2}{3 \cdot 2^{3j+3}} \sin \frac{(r-s)\ell\pi}{3\cdot 2^j} & \text{if } \ell \in (2^{j+1}, 2^{j+2}) \end{cases} .
\end{aligned}$$

Proof: Taking into account the definition of the $\mu^D_{j,0,n}$ terms as inner products, a change of the order of integration and summation yields that

$$\Sigma_C = \langle \phi^D_{j,r-s}, L^D_j(\cos \ell x)\rangle, \tag{4.2}$$

$$\Sigma_S = \langle \phi^D_{j,r-s}, L^D_j(\sin \ell x)\rangle, \tag{4.3}$$

using the interpolation operator L^D_j of Definition 2.2. Therefore, one has to examine these inner products. The inclusion $T_{2^{j+1}} \subset V_j$ implies

$$L^D_j(\cos \ell x) = \cos \ell x \quad \text{and} \quad L^D_j(\sin \ell x) = \sin \ell x \quad \text{for} \quad \ell = 0, \ldots, 2^{j+1},$$

while

$$L^D_j(\cos \ell x) = L^D_j(\cos\ (3 \cdot 2^{j+1} - \ell)x) = \cos\ (3 \cdot 2^{j+1} - \ell)x$$

and

$$L^D_j(\sin \ell x) = -L^D_j(\sin\ (3 \cdot 2^{j+1} - \ell)x) = -\sin\ (3 \cdot 2^{j+1} - \ell)x$$

for $\ell = 2 \cdot 2^{j+1}, \ldots, 3 \cdot 2^{j+1} - 1$.

The case of $\ell = 2^{j+1}+1, \ldots, 2^{j+2}-1$ necessitates more detailed computations. The results are

$$L^D_j(\cos \ell x) = (2 - \frac{\ell}{2^{j+1}}) \cos \ell x \quad + \quad (\frac{\ell}{2^{j+1}} - 1) \cos\ (3 \cdot 2^{j+1} - \ell)x \tag{4.4}$$

and

$$L^D_j(\sin \ell x) = (2 - \frac{\ell}{2^{j+1}}) \sin \ell x \quad + \quad (1 - \frac{\ell}{2^{j+1}}) \sin\ (3 \cdot 2^{j+1} - \ell)x \quad . \tag{4.5}$$

We now proceed to prove the statement (4.4) for cosine, the one for sine can be shown in an analogous fashion. Recall the trigonometric series identity

$$\sum_{k=0}^{3\cdot 2^{j+1}-1} \cos\left(x - \frac{km\pi}{3 \cdot 2^j}\right) = \begin{cases} 0 & \text{for } m \notin 3 \cdot 2^{j+1}\mathbb{Z}, \\ 3 \cdot 2^{j+1} \cos x & \text{for } m \in 3 \cdot 2^{j+1}\mathbb{Z}\ . \end{cases} \tag{4.6}$$

Therefore,

$$
\begin{aligned}
&L_j^D(\cos \ell x)(x) \\
&= \frac{1}{3\cdot 2^{2j+1}} \sum_{n=2^{j+1}}^{2^{j+2}-1} \sum_{k=0}^{3\cdot 2^{j+1}-1} \cos\left(\frac{\ell k\pi}{3\cdot 2^j}\right)\left(\frac{1}{2}+\sum_{t=1}^{n}\cos\left(tx-\frac{tk\pi}{3\cdot 2^j}\right)\right) \\
&= \frac{1}{3\cdot 2^{2j+2}} 2^{j+1} \sum_{k=0}^{3\cdot 2^{j+1}-1} \cos\left(\frac{\ell k\pi}{3\cdot 2^j}\right) \\
&\quad + \frac{1}{3\cdot 2^{2j+1}} \sum_{n=2^{j+1}}^{2^{j+2}-1} \sum_{t=1}^{n} \sum_{k=0}^{3\cdot 2^{j+1}-1} \cos\left(\frac{\ell k\pi}{3\cdot 2^j}\right)\cos\left(tx-\frac{tk\pi}{3\cdot 2^j}\right) \\
&= \frac{1}{3\cdot 2^{2j+1}} \sum_{n=2^{j+1}}^{2^{j+2}-1} \sum_{t=1}^{n} \sum_{k=0}^{3\cdot 2^{j+1}-1} \left(\frac{1}{2}\cos\left(tx-\frac{(t-\ell)k\pi}{3\cdot 2^j}\right)+\frac{1}{2}\cos\left(tx-\frac{(t+\ell)k\pi}{3\cdot 2^j}\right)\right) \\
&= \frac{1}{3\cdot 2^{2j+2}} \sum_{n=2^{j+1}}^{2^{j+2}-1} \sum_{t=1}^{n} 3\cdot 2^{j+1}\left(\delta_{t,\ell}+\delta_{t,3\cdot 2^{j+1}-\ell}\right)\cos tx \\
&= \frac{1}{3\cdot 2^{j+1}} \sum_{n=2^{j+1}}^{2^{j+2}-1} \sum_{t=1}^{n} \left(\delta_{t,\ell}+\delta_{t,3\cdot 2^{j+1}-\ell}\right)\cos tx \quad .
\end{aligned}
$$

Examining the three different cases $\ell = 3\cdot 2^j$, $\ell \in (2^{j+1}, 3\cdot 2^j)$ and $\ell \in (3\cdot 2^j, 2^{j+2})$ separately, finally yields the desired result (4.4). We now proceed to compute the inner products of $\phi_{j,r-s}^D$ and the relevant cosine and sine functions. As in the proof of Lemma 3.1, we obtain

$$\langle \phi_{j,r-s}^D, \cos \ell x\rangle = \frac{1}{3\cdot 2^{j+1}} \cos\frac{(r-s)\ell\pi}{3\cdot 2^j}$$

and

$$\langle \phi_{j,r-s}^D, \sin \ell x\rangle = \frac{1}{3\cdot 2^{j+1}} \sin\frac{(r-s)\ell\pi}{3\cdot 2^j}$$

for $\ell = 0,\ldots,2^{j+1}$. For the remaining $\ell = 2^{j+1}+1,\ldots,2^{j+2}-1$, it holds that

$$
\begin{aligned}
&\langle \phi_{j,r-s}^D, \cos \ell x\rangle \\
&= \frac{1}{2\pi}\frac{1}{3\cdot 2^{2j+1}} \sum_{n=2^{j+1}}^{2^{j+2}-1} \left(\frac{1}{2}\int_0^{2\pi} \cos\ell\left(x+\frac{(r-s)\pi}{3\cdot 2^j}\right)dx\right. \\
&\qquad \left.+ \sum_{t=1}^{n}\int_0^{2\pi} \cos tx\ \cos\ell\left(x+\frac{(r-s)\pi}{3\cdot 2^j}\right)dx\right)
\end{aligned}
$$

$$= \frac{1}{2\pi}\frac{1}{3\cdot 2^{2j+1}}\sum_{n=2^{j+1}}^{2^{j+2}-1}\sum_{t=1}^{n}\pi\delta_{t,\ell}\cos\left(\frac{\ell(r-s)\pi}{3\cdot 2^j}\right)$$
$$= \cos\left(\frac{\ell(r-s)\pi}{3\cdot 2^j}\right)\frac{2^{j+2}-\ell}{3\cdot 2^{2j+2}} \quad .$$

Similarly,

$$\langle\phi^D_{j,r-s}, \sin \ell x\rangle = \sin\left(\frac{\ell(r-s)\pi}{3\cdot 2^j}\right)\frac{2^{j+2}-\ell}{3\cdot 2^{2j+2}} \quad .$$

Collecting all partial results finally proves the assertions. □

Proof of Theorem 4.1: In the Fourier case, there exists a simple form of the inner product (see [9])

$$\langle\phi^F_{j,0}, \phi^F_{j,k}\rangle = \frac{1}{9\cdot 2^{2j+3}}\left(3\cdot 2^{j+2}\delta_{0,k} - (-1)^k\right) \quad .$$

A direct computation of the geometric series in Lemma 4.1 or a simplified version of the proof of Lemma 4.2 gives the desired result. For the de la Vallée Poussin case, we obtain from Lemma 4.1 and (4.2),(4.3), as $\alpha_{j,r,s}$ is real,

$$\alpha^D_{j,r,s} =$$
$$\frac{1}{3\cdot 2^{j+1}}\left(\sum_{\ell=0}^{2^{j+1}} + \sum_{\ell=2^{j+2}}^{3\cdot 2^{j+1}-1} + \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1}\right)\frac{\langle\phi^D_{j,r-s}, L^D_j(\cos \ell x)\rangle}{\langle\phi^D_{j,r-s}, L^D_j(\cos \ell x)\rangle^2 + \langle\phi^D_{j,r-s}, L^D_j(\sin \ell x)\rangle^2}$$
$$= S_1 + S_2 + S_3 \quad .$$

Using Lemma 4.2, the sums S_1, S_2 and S_3 can be computed as follows.

$$S_1 = \sum_{\ell=0}^{2^{j+1}} \cos\frac{(r-s)\ell\pi}{3\cdot 2^j},$$
$$S_2 = \sum_{\ell=2^{j+2}}^{3\cdot 2^{j+1}-1} \cos\frac{(r-s)\ell\pi}{3\cdot 2^j}$$

and

$$S_3 = 2^{2j+1}\sum_{\ell=2^{j+1}+1}^{2^{j+2}-1}\frac{1}{\ell^2 - 3\ell 2^{j+1} + 5\cdot 2^{2j+1}}\cos\frac{(r-s)\ell\pi}{3\cdot 2^j} \quad .$$

Adding the three terms and using the trigonometric identity (4.6) completes the proof. □

Definition 4.2. *For any $j \in \mathbb{N}_0$, the functions $\tilde{\psi}_{j,r} \in W_j, r = 0, \ldots, 3 \cdot 2^{j+1} - 1$, uniquely determined by the conditions*

$$\langle \tilde{\psi}_{j,r}, \psi_{j,k} \rangle = \delta_{r,k}$$

for all $r, k = 0, \ldots, 3 \cdot 2^{j+1} - 1$ are called dual wavelets (or dual to the functions $\psi_{j,r}$).

As for the scaling functions, the dual wavelets $\tilde{\psi}_{j,r}$ lie in the same space W_j as the original wavelets $\psi_{j,k}$. Therefore, also the dual wavelets can be written as linear combinations of the original ones, *i.e.*, for $r = 0, \ldots, 3 \cdot 2^{j+1} - 1, \tilde{\psi}_{j,r} = \sum_{s=0}^{3 \cdot 2^{j+1}-1} \beta_{j,r,s} \psi_{j,s}$. The matrix H_j with entries $\langle \psi_{j,r}, \psi_{j,s} \rangle$ is again a circulant matrix. In fact, using (2.2), it holds that

$$\begin{aligned} \langle \psi_{j,0}, \psi_{j,k} \rangle &= 4\mu_{j+1,1,2k+1} + \mu_{j,0,k} \\ &\quad - 2\langle \phi_{j+1,1}, \phi_{j,k}(x - \frac{\pi}{3 \cdot 2^{j+1}}) \rangle - 2\langle \phi_{j,0}(x - \frac{\pi}{3 \cdot 2^{j+1}}), \phi_{j+1,2k+1} \rangle \quad . \end{aligned}$$

Lemma 3.1 and the interpolatory property (2.1) yield

$$\langle \psi_{j,0}, \psi_{j,k} \rangle = 4\mu_{j+1,1,2k+1} + \mu_{j,0,k} - \frac{\delta_{k,0}}{3 \cdot 2^j}.$$

Proceeding as in the proof of Theorem 4.1, one establishes for the Fourier case that the elements $\beta^F_{j,r,s}$ of $(H^F_j)^{-1}$ can be computed as

$$\beta^F_{j,r,s} = 3 \cdot 2^{j+1} \delta_{r,s} + 1 + (-1)^{r-s} \quad .$$

The de la Vallée Poussin case can be treated similarly.

5. Decomposition sequences

The other pair of important sequences for our approach are the so-called decomposition sequences. As $V_{j+1} = V_j \oplus W_j$ for $j \in \mathbb{N}_0$, any $\phi_{j+1,n} \in V_{j+1}$ can be written as a linear combination of the basis functions of V_j and W_j, *i.e.*, $\phi_{j,k}$ and $\psi_{j,k}$. In the following, we will distinguish two cases.

Theorem 5.1. *For any $j \in \mathbb{N}_0$ and $m = 0, \ldots, 3 \cdot 2^{j+1} - 1$, it holds that*

$$\phi_{j+1,2m}(x) = \sum_{k=0}^{3\cdot 2^{j+1}-1} \left(a_{j,m,k}\phi_{j,k}(x) + b_{j,m,k}\psi_{j,k}(x) \right),$$

where for $n = 0, \ldots, 3 \cdot 2^{j+1} - 1$, the decomposition coefficients $a_{j,m,n}$ and $b_{j,m,n}$ are given by

$$\begin{aligned} a_{j,m,n} &= \frac{\alpha_{j,m,n}}{3 \cdot 2^{j+2}} \\ b_{j,m,n} &= - \sum_{s=0}^{3\cdot 2^{j+1}-1} \frac{\alpha_{j,m,s}}{3 \cdot 2^{j+2}} \phi_{j,s}\left(\frac{(2n+1)\pi}{3 \cdot 2^{j+1}} \right) \\ &= - \frac{\tilde{\phi}_{j,m}\left(\frac{(2n+1)\pi}{(3\cdot 2^{j+1})} \right)}{3 \cdot 2^{j+2}}, \end{aligned}$$

with the terms $\alpha_{j,m,n}$ being the coefficients of the dual scaling functions as defined in Theorem 4.1.

Proof: Taking inner products with the dual scaling functions $\tilde{\phi}_{j,n}$ of Theorem 4.1, one obtains

$$\begin{aligned} a_{j,m,n} &= \langle \phi_{j+1,2m}, \tilde{\phi}_{j,n} \rangle \\ &= \sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,n,s} \langle \phi_{j+1,2m}, \phi_{j,s} \rangle \\ &= \sum_{s=0}^{3\cdot 2^{j+1}-1} \frac{\alpha_{j,n,s}}{3 \cdot 2^{j+2}} \phi_{j,0}\left(\frac{(2m-2s)\pi}{3 \cdot 2^{j+1}} \right) \\ &= \sum_{s=0}^{3\cdot 2^{j+1}-1} \frac{\alpha_{j,n,s}}{3 \cdot 2^{j+2}} \delta_{m,s}, \end{aligned}$$

where Lemma 3.1 and the interpolatory property (2.1) were used for the last two steps. Finally, the symmetry of the $\alpha_{j,m,n}$ terms yields the first part of the theorem. Evaluating the decomposition equation in the points $\frac{(2n+1)\pi}{3\cdot 2^{j+1}}$ and using the interpolatory property (2.3) of the wavelets leads to

$$0 = \sum_{s=0}^{3\cdot 2^{j+1}-1} a_{j,m,k}\phi_{j,k}\left(\frac{(2n+1)\pi}{3 \cdot 2^{j+1}} \right) + b_{j,m,n},$$

and the remaining part of the theorem. □

Distinguishing the two cases, it is possible to further simplify the expressions of the decomposition coefficients.

Corollary 5.1. *For any $j \in \mathbb{N}_0$ and $m = 0, \ldots, 3 \cdot 2^{j+1} - 1$, the decomposition coefficients for the Fourier case are*

$$a^F_{j,m,n} = \frac{\delta_{m,n}}{2} + \frac{(-1)^{m-n}}{3 \cdot 2^{j+1}},$$

$$b^F_{j,m,n} = -\frac{\phi^F_{j,m}\left(\frac{(2n+1)\pi}{3 \cdot 2^{j+1}}\right)}{2} = (-1)^{m-n+1}\frac{\cot\left(\frac{(2n-2m+1)\pi}{3 \cdot 2^{j+2}}\right)}{3 \cdot 2^{j+2}} \quad .$$

In the de la Vallée Poussin case the coefficients are

$$a^D_{j,m,n} = \frac{\delta_{m,n}}{2} + \frac{1}{3 \cdot 2^{j+2}} \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1} \left(\frac{2^{2j+1}}{\ell^2 - 3\ell 2^{j+1} + 5 \cdot 2^{2j+1}} - 1\right) \cos\frac{\ell(r-s)\pi}{3 \cdot 2^j},$$

$$b^D_{j,m,n} = -\frac{\phi^D_{j,m}\left(\frac{(2n+1)\pi}{3 \cdot 2^{j+1}}\right)}{2} - \frac{1}{3 \cdot 2^{j+2}} \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1} \left(\frac{2^{2j+1}}{\ell^2 - 3\ell 2^{j+1} + 5 \cdot 2^{2j+1}} - 1\right) (3 - \frac{\ell}{2^j}) \cos\frac{\ell(2n-2m+1)\pi}{3 \cdot 2^{j+1}} \quad .$$

The treatment of the basis functions in V_{j+1} with odd index requires more effort, although the result is quite straightforward.

Theorem 5.2. *For any $j \in \mathbb{N}_0$ and $m = 0, \ldots, 3 \cdot 2^{j+1} - 1$, it holds that*

$$\phi_{j+1,2m+1}(x) = \sum_{k=0}^{3 \cdot 2^{j+1}-1} \left(\tilde{a}_{j,m,k}\phi_{j,k}(x) + \tilde{b}_{j,m,k}\psi_{j,k}(x)\right),$$

where for $n = 0, \ldots, 3 \cdot 2^{j+1} - 1$, the decomposition coefficients $\tilde{a}_{j,m,n}$ and $\tilde{b}_{j,m,n}$ are given as

$$\tilde{a}_{j,m,n} = -b_{j,n,m},$$

$$\tilde{b}_{j,m,n} = a_{j,m,n},$$

where the terms $a_{j,m,n}$ and $b_{j,m,n}$ are defined in Theorem 5.1.

Proof: Similarly to the proof of Theorem 5.1 above, taking inner products with the dual scaling functions $\tilde{\phi}_{j,n}$ yields

$$\begin{aligned}
\tilde{a}_{j,m,n} &= \langle \phi_{j+1,2m+1}, \tilde{\phi}_{j,n} \rangle \\
&= \sum_{s=0}^{3 \cdot 2^{j+1}-1} \alpha_{j,n,s} \langle \phi_{j+1,2m+1}, \phi_{j,s} \rangle \\
&= \sum_{s=0}^{3 \cdot 2^{j+1}-1} \frac{\alpha_{j,n,s}}{3 \cdot 2^{j+2}} \phi_{j,s} \left(\frac{(2m+1)\pi}{3 \cdot 2^{j+1}} \right) \quad .
\end{aligned}$$

In order to prove the second statement, we use the two-scale relation established in Theorem 3.1 for $t = 0, \ldots, 3 \cdot 2^{j+1} - 1$

$$\phi_{j,t}(x) = \phi_{j+1,2t}(x) + \sum_{s=0}^{3 \cdot 2^{j+1}-1} \phi_{j,t} \left(\frac{(2s+1)\pi}{3 \cdot 2^{j+1}} \right) \phi_{j+1,2s+1}(x).$$

Substituting the decomposition equations gives

$$\begin{aligned}
\phi_{j,t}(x) = &\sum_{k=0}^{3 \cdot 2^{j+1}-1} \left(a_{j,t,k} \phi_{j,k}(x) + b_{j,t,k} \psi_{j,k}(x) \right) \\
&+ \sum_{s=0}^{3 \cdot 2^{j+1}-1} \phi_{j,t} \left(\frac{(2s+1)\pi}{3 \cdot 2^{j+1}} \right) \sum_{k=0}^{3 \cdot 2^{j+1}-1} \left(-b_{j,k,s} \phi_{j,k}(x) + \tilde{b}_{j,s,k} \psi_{j,k}(x) \right),
\end{aligned}$$

thus comparing coefficients of $\phi_{j,q}$ yields

$$\delta_{t,q} = a_{j,t,q} - \sum_{s=0}^{3 \cdot 2^{j+1}-1} \phi_{j,t} \left(\frac{(2s+1)\pi}{3 \cdot 2^{j+1}} \right) b_{j,q,s} \quad .$$

Similarly, using the wavelet two-scale relation for $\psi_{j,t}$ of Theorem 3.2,

$$\psi_{j,t}(x) = \phi_{j+1,2t+1}(x) - \sum_{s=0}^{3 \cdot 2^{j+1}-1} \phi_{j,t} \left(\frac{(2s-1)\pi}{3 \cdot 2^{j+1}} \right) \phi_{j+1,2s}(x)$$

and substitution of the decomposition relation implies

$$\psi_{j,t}(x) = \sum_{k=0}^{3\cdot 2^{j+1}-1} \left(-b_{j,k,t}\phi_{j,k}(x) + \tilde{b}_{j,t,k}\psi_{j,k}(x)\right)$$
$$- \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,t}\left(\frac{(2s-1)\pi}{3\cdot 2^{j+1}}\right) \sum_{k=0}^{3\cdot 2^{j+1}-1} \left(a_{j,s,k}\phi_{j,k}(x) + b_{j,s,k}\psi_{j,k}(x)\right).$$

This time, by comparing coefficients of $\psi_{j,q}$,

$$\delta_{t,q} = \tilde{b}_{j,t,q} - \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,t}\left(\frac{(2s-1)\pi}{3\cdot 2^{j+1}}\right) b_{j,s,q} \quad .$$

The second part of the theorem now follows from the fact that

$$\sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,t}\left(\frac{(2s+1)\pi}{3\cdot 2^{j+1}}\right) b_{j,q,s} = \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,t}\left(\frac{(2s-1)\pi}{3\cdot 2^{j+1}}\right) b_{j,s,q},$$

which can be established by going back to the definition of $b_{j,q,s}$ and performing suitable transformations of summation indices. Hereby, a key property of the functions $\phi_{j,t}$ is

$$\begin{aligned}\phi_{j,t}\left(\frac{(2m+1)\pi}{3\cdot 2^{j+1}}\right) &= \phi_{j,0}\left(\frac{(2m-2t+1)\pi}{3\cdot 2^{j+1}}\right)\\ &= \phi_{j,0}\left(\frac{(2t-2m-1)\pi}{3\cdot 2^{j+1}}\right)\\ &= \phi_{j,m}\left(\frac{(2t-1)\pi}{3\cdot 2^{j+1}}\right)\end{aligned}$$

for any two indices t, m. □

6. Algorithms

The two-scale and decomposition relations which have been established in the preceding sections will now be rewritten in a convenient matrix form. This notation will then be used to formulate decomposition and reconstruction algorithms for the trigonometric subspaces of this paper.

Let $\underline{\phi}_j$ denote the vector $(\phi_{j,0}, \phi_{j,1}, \dots, \phi_{j,3\cdot 2^{j+1}-1})^T$ and, analogously, $\underline{\psi}_j$ the vector $(\psi_{j,0}, \psi_{j,1}, \dots, \psi_{j,3\cdot 2^{j+1}-1})^T$. Furthermore, we define a reordering for the vector of scaling functions by

$P_j\underline{\phi}_{j+1} = (\phi_{j+1,0}, \phi_{j+1,2}, \dots, \phi_{j+1,2m}, \dots, \phi_{j+1,3\cdot 2^{j+2}-2}, \phi_{j+1,1}, \phi_{j+1,3}, \dots, \phi_{j+1,2m+1}, \dots, \phi_{j+1,3\cdot 2^{j+2}-1})^T$, *i.e.*, P_j is chosen to be the suitable permutation matrix for this ordering.

By slightly modifying the symmetric and circulant duality matrix of Section 4, *i.e.*,

$$\tilde{G}_j^{-1} = \frac{1}{3\cdot 2^{j+2}} G_j^{-1} = \left(\frac{\alpha_{j,r,s}}{3\cdot 2^{j+2}}\right)_{r=0\ \ \ ,s=0}^{3\cdot 2^{j+1}-1\ ,3\cdot 2^{j+1}-1},$$

and introducing a knot evaluation matrix K_j by

$$K_j = \left(\phi_{j,r}\left(\frac{(2s+1)\pi}{3\cdot 2^{j+1}}\right)\right)_{r=0\ \ \ ,s=0}^{3\cdot 2^{j+1}-1\ ,3\cdot 2^{j+1}-1},$$

the results of the previous sections can be written in matrix notation.

Remark 6.1. *The circulant knot evaluation matrix K_j is singular.*

Proof: The eigenvalues λ_k, $k = 0, \dots, 3\cdot 2^{j+1}-1$, of K_j are given by (see [5,Theorem 3.2.2])

$$\lambda_k = \sum_{r=0}^{3\cdot 2^{j+1}-1} \phi_{j,r}\left(\frac{\pi}{3\cdot 2^{j+1}}\right) e^{\frac{rki\pi}{3\cdot 2^j}}.$$

Due to (4.4), $\cos 3\cdot 2^j\cdot \in V_j$ and therefore

$$\lambda_{3\cdot 2^j} = L_j(\cos 3\cdot 2^j\cdot)\left(\frac{\pi}{3\cdot 2^{j+1}}\right) = \cos\frac{\pi}{2} = 0 \quad .$$

Hence, K_j is singular. □

According to Section 3, the two-scale relation or *reconstruction matrix* C_j has the following form

$$C_j = \begin{pmatrix} I_j & K_j \\ -K_j^T & I_j \end{pmatrix},$$

where I_j is an identity matrix of dimension $3\cdot 2^{j+1}$. C_j is a block circulant square matrix of dimension $3\cdot 2^{j+2}$. Theorems 3.1 and 3.2 can now be expressed as

$$\begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} = C_j P_j \underline{\phi}_{j+1} \quad .$$

Consequently, the decomposition relations of Theorems 5.1 and 5.2 can be expressed as

$$P_j \underline{\phi}_{j+1} = D_j \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix},$$

using the block circulant *decomposition matrix*

$$D_j = C_j^{-1} = \begin{pmatrix} \tilde{G}_j^{-1} & \tilde{G}_j^{-1} K_j \\ -K_j^T \tilde{G}_j^{-1} & \tilde{G}_j^{-1} \end{pmatrix} \quad .$$

As $V_{j+1} = V_j \oplus W_j$, a function $f_{j+1} \in V_{j+1}$ can be written uniquely as

$$f_{j+1} = f_j + g_j, \quad \text{with} \quad f_j \in V_j \quad \text{and} \quad g_j \in W_j \quad .$$

Using the basis functions of these spaces, one obtains

$$f_{j+1}(x) = \sum_{s=0}^{3 \cdot 2^{j+2}-1} c_s^{j+1} \phi_{j+1,s}(x),$$

$$f_j(x) = \sum_{s=0}^{3 \cdot 2^{j+2}-1} c_s^j \phi_{j,s}(x) \quad \text{and} \quad g_j(x) = \sum_{s=0}^{3 \cdot 2^{j+2}-1} d_s^j \psi_{j,s}(x) \quad .$$

Denoting coefficient vectors by $\underline{c}_j^T = (c_0^j, c_1^j, \ldots, c_{3 \cdot 2^{j+1}-1}^j)$ and $\underline{d}_j^T = (d_0^j, d_1^j, \ldots, d_{3 \cdot 2^{j+1}-1}^j)$, respectively, one obtains

$$f_{j+1} = \underline{c}_{j+1}^T \underline{\phi}_{j+1}, \quad f_j = \underline{c}_j^T \underline{\phi}_j \quad \text{and} \quad g_j = \underline{d}_j^T \underline{\psi}_j \quad .$$

As $\underline{c}_{j+1}^T \underline{\phi}_{j+1} = (P_j \underline{c}_{j+1})^T P_j \underline{\phi}_{j+1}$, the matrix form of the decomposition relation yields

$$f_{j+1} = (P_j \underline{c}_{j+1})^T P_j \underline{\phi}_{j+1} = (P_j \underline{c}_{j+1})^T D_j \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} \quad .$$

On the other hand, it holds that

$$f_j + g_j = \begin{pmatrix} \underline{c}_j^T & \underline{d}_j^T \end{pmatrix} \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} \quad .$$

Comparing coefficients leads to

$$\begin{pmatrix} \underline{c}_j^T & \underline{d}_j^T \end{pmatrix} = (P_j \underline{c}_{j+1})^T D_j,$$

and taking the transpose finally gives the matrix form of one step of the *decomposition algorithm*

$$\begin{pmatrix} \underline{c}_j \\ \underline{d}_j \end{pmatrix} = D_j^T P_j \underline{c}_{j+1} \quad .$$

Multiplying by the inverse $(D_j^T)^{-1} = C_j^T$ yields the matrix representation of one step of the *reconstruction algorithm*

$$P_j \underline{c}_{j+1} = C_j^T \begin{pmatrix} \underline{c}_j \\ \underline{d}_j \end{pmatrix} \quad .$$

Altogether we obtain the following algorithms:

Algorithm 1. *(Decomposition)*

Input data: Function values for some predetermined level η

$$f\left(\frac{k\pi}{3 \cdot 2^{\eta+1}}\right) = (L_\eta f)\left(\frac{k\pi}{3 \cdot 2^{\eta+1}}\right), \quad k = 0, \ldots, 3 \cdot 2^{\eta+1} - 1$$

Step 1: Set

$$\underline{c}_\eta = \left(f\left(\frac{k\pi}{3 \cdot 2^{\eta+1}}\right)\right)_{k=0}^{3 \cdot 2^{\eta+1}-1} \quad .$$

Step 2: Repeat for $j = \eta - 1, \ldots, 0$ the computation:

$$\begin{pmatrix} \underline{c}_j \\ \underline{d}_j \end{pmatrix} = D_j^T P_j \underline{c}_{j+1} \quad .$$

Output data: The wavelet coefficients $\underline{d}_j$ for $j = 0, \ldots, \eta - 1$ and the lowest level scaling function coefficients $\underline{c}_0$. ♣

Algorithm 2. *(Reconstruction)*

Input data: The wavelet coefficients $\underline{d}_j$ for $j = 0, \ldots, \eta - 1$ and the lowest level scaling function coefficients $\underline{c}_0$.

Step 1: Repeat for $j = 0, \ldots, \eta - 1$ the computation:

$$P_j \underline{c}_{j+1} = C_j^T \begin{pmatrix} \underline{c}_j \\ \underline{d}_j \end{pmatrix} \quad .$$

Output data: The scaling function coefficients on level η, *i.e.*, $\underline{c}_\eta$. For perfect reconstruction this is the vector

$$\left((L_\eta f)\left(\frac{k\pi}{3\cdot 2^{\eta+1}}\right)\right)_{k=0}^{3\cdot 2^{\eta+1}-1} \quad . \qquad \clubsuit$$

Note that the efficiency of these two algorithms depends essentially on the proper implementation of the matrix/vector multiplications using Fast-Fourier-Transform techniques. Following [5, Chapter 3] the circulant submatrices of C_j and D_j can be factored into so-called Fourier matrices and the diagonal matrix of the eigenvalues which can be computed directly (see *e.g.* Remark 6.1). All the necessary computations for these factorizations only need to be done once for a predetermined number of levels. Thus the algorithms 1 and 2 need $\mathcal{O}(j2^j)$ operations which is the best possible for this type of matrix calculations, but this does not realize the best possible pyramid algorithms of order $\mathcal{O}(2^j)$ available for some other wavelet schemes. Thus this fully computable trigonometric multiresolution analysis with explicit algebraic formulas yields "almost optimal" complexity. Another algorithmic advantage of the interpolatory approach is the simplicity of finding a suitable projection onto V_j just by function evaluation in the dyadic nodes.

7. Discussion

For a good understanding of the wavelet spaces it is worthwhile to describe an alternative basis consisting of sine and cosine frequencies. Surprisingly, it turns out that this other basis has a rather simple structure.

Theorem 7.1. *For $j \in \mathbb{N}_0$,*

$$V_0 = span\{T_2 \cup \cos 3x\},$$
$$W_j^F = span\{T_{3\cdot 2^{j+1}-1} \backslash T_{3\cdot 2^j}; \sin 3\cdot 2^j x, \cos 3\cdot 2^{j+1}x\}$$

and

$$W_j^D = span\{(2^{j+2}-\ell)\cos \ell x \quad - \quad (\ell - 2^{j+1})\cos\,(3\cdot 2^{j+1}-\ell)x\;,$$
$$(2^{j+2}-\ell)\sin \ell x \quad + \quad (\ell - 2^{j+1})\sin\,(3\cdot 2^{j+1}-\ell)x,$$

$$\text{for} \quad \ell = 2^{j+1}+1,\ldots,3\cdot 2^j,$$
$$(2^{j+3}-\ell)\cos\ell x \quad + \quad (\ell-2^{j+2})\cos\,(3\cdot 2^{j+2}-\ell)x\ ,$$
$$(2^{j+3}-\ell)\sin\ell x \quad - \quad (\ell-2^{j+2})\sin\,(3\cdot 2^{j+2}-\ell)x\ ,$$
$$\text{for} \quad \ell = 2^{j+2},\ldots,3\cdot 2^{j+1}\} \quad .$$

Proof: The proof is by induction. The Fourier case is obvious. The properties (4.4),(4.5) of the interpolation operator L_j^D can be used to show that the functions listed for W_j^D are indeed elements of V_{j+1}^D. It can be seen directly that these functions are in fact orthogonal to all of V_j^D, *i.e.*, lie in W_j^D. Recalling that the dimension of V_{j+1}^D is $3\cdot 2^{j+1}$ completes the proof. □

In order to illustrate Theorem 7.1, the basis functions for V_0, W_0^D, and W_1^D are listed below.

$$V_0 = span\{1, \cos x, \sin x, \cos 2x, \sin 2x, \cos 3x\},$$

$$W_0^D = span\{\sin 3x, \cos 4x, \sin 4x, 3\cos 5x + \cos 7x, \cos 6x, 3\sin 5x - \sin 7x\},$$

$$W_1^D = span\{\sin 6x, 3\cos 5x - \cos 7x, 3\sin 5x + \sin 7x, \cos 8x,$$

$$\sin 8x, 7\cos 9x + \cos 15x, 3\cos 10x + \cos 14x, 5\cos 11x + 3\cos 13x,$$

$$\cos 12x, 7\sin 9x - \sin 15x, 3\sin 10x - \sin 14x, 5\sin 11x - 3\sin 13x\} \quad .$$

Figure 1 shows two scaling functions $\phi_{0,2} \in V_0$ and $\phi_{2,11}^D \in V_2^D$ and two wavelets $\psi_{0,2}^D \in W_0^D$ and $\psi_{2,11}^D \in W_2^D$ for the de la Vallée Poussin means. Recall that the scaling functions and wavelets at higher levels are not just scaled versions of the ones at level zero due to their interpolatory properties. Figure 2 shows the analogous functions $\phi_{2,11}^F \in V_2^F$, $\psi_{0,2}^F \in W_0^F$ and $\psi_{2,11}^F \in W_2^F$ for the Fourier means. As there are only two Dirichlet kernels to be summed up, the functions for this case prove to be more oscillatory than for the de la Vallée Poussin means. While the number of oscillations is the same in both cases, the height of the peaks between two zero interpolating nodes is essentially greater for the Fourier scaling functions. This decay rate of the fundamental Lagrange functions can be easily deduced from their kernel representations given at the beginning of Section 2. Namely, the local maxima of the scaling functions on the level j can be estimated in the periodic distance (*i.e.*, for $|r-k| \leq 3 \cdot 2^j$) from above and below in the following way

$$C_1 \left(|r-k|+1\right)^{-2} \quad \leq \quad \max_{x_{j,k-1} \leq x \leq x_{j,k+1}} |\phi_{j,r}^D(x)| \quad \leq \quad C_2 \left(|r-k|+1\right)^{-2}$$

and

$$C_1 \left(|r-k|+1\right)^{-1} \quad \leq \quad \max_{x_{j,k-1} \leq x \leq x_{j,k+1}} |\phi_{j,r}^F(x)| \quad \leq \quad C_2 \left(|r-k|+1\right)^{-1}.$$

Analogous results hold for the wavelets.

Closely related to these localization properties are error estimates for the nonorthogonal projection operators L_j onto the scaling function spaces. Note that the approach in [2] can be interpreted as an application of a Fourier sum operator to the Haar scaling function and Haar wavelet, respectively. Consequently, just a quasi–interpolation operator can be constructed for which only a very low order of convergence is established ([2], Theorem 3.4). On the other hand, in our approach the convergence order is comparable to $E_n(f, C_{2\pi})$, the best approximation of f in the uniform norm by trigonometric polynomials of order n. Using standard arguments (see [9]) and the properties of L_j given in Section 2, one obtains for $f \in C_{2\pi}$ that

$$\|f - L_j^D f\|_{C_{2\pi}} \leq c_1 \; E_{2^{j+1}}(f, C_{2\pi})$$

and

$$\|f - L_j^F f\|_{C_{2\pi}} \leq c_2 \; j \; E_{32^j-1}(f, C_{2\pi}).$$

Summarizing the differences between the Fourier and the de la Vallée Poussin case, we note that for a fixed number of interpolation conditions the Fourier approach offers lower polynomial degree, reproduction of higher order trigonometric polynomials and less complicated expressions for all matrices and the dual functions. On the other hand, the de la Vallée Poussin means yield better decay rate, localization as well as a superior order of convergence.

Using other means of summation, or more generally employing a periodization technique to Meyer type wavelets with compactly supported Fourier transform at the expense of giving up convenient properties such as interpolation conditions leads – among others – to the investigations of D. Offin, K. Oskolkov and R. A. Lorentz, A. A. Sahakian mentioned in the introduction which are far less algorithmically accessible.

Figures 3, 4 and 5 illustrate the use of trigonometric wavelet decompositions to detect discontinuities in higher order derivatives of a function. In this case, a cubic B-spline with equidistant knots at $\{1, 2, 3, 4, 5\}$, *i.e.*

$$f(x) := \begin{cases} 0 & x \in [0,1] \\ \frac{1}{6}(x-1)^3 & x \in (1,2] \\ \frac{1}{6}(-3(x-1)^3 + 12(x-1)^2 - 12(x-1) + 4 & x \in (2,3] \\ \frac{1}{6}(-3(5-x)^3 + 12(5-x)^2 - 12(5-x) + 4 & x \in (3,4] \\ \frac{1}{6}(5-x)^3 & x \in (4,5] \\ 0 & x \in (5, 2\pi] \end{cases}$$

(properly periodized to generate a 2π-periodic function) was interpolated by elements of V_9^D (Figure 3) and V_9^F using the operators L_9^D and L_9^F of Definition 2.2. As expected, the breakpoints of the spline, where its third derivative has jump discontinuities, can be clearly detected in wavelets components of W_8^D and W_8^F, created by one decomposition step and shown in Figure 4. Again, the effect is less localized for the Fourier case. The detection effect is more and more blurred in subsequent decomposition steps, as illustrated by the wavelet parts for the levels 7,6, and 5, shown in Figure 5 for both cases.

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