Interpolatory And Orthonormal Trigonometric Wavelets

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Abstract. The aim of this paper is the detailed investigation of trigonometric polynomial spaces as a tool for approximation and signal analysis. Sample spaces are generated by equidistant translates of certain de la Vallée Poussin means. The different de la Vallée Poussin means enable us to choose between better time- or frequency-localization. For nested sample spaces and corresponding wavelet spaces, we discuss different bases and their transformations.

§1 Introduction

Trigonometric polynomials and the approximation of periodic functions by polynomials play an important role in harmonic analysis. Here we are interested in constructing time-localized bases for certain spaces of trigonometric polynomials. We use de la Vallée Poussin means of the usual Dirichlet kernel, which allow the investigation of simple projections onto these spaces. Interpolation and orthogonal projection are discussed. With the different de la Vallée Poussin means, the operator norms of these projections as well as the time-frequency localization of the basis functions can be controlled. In order to improve the time-localization, the side oscillations are reduced by including more frequencies and averaging the highest ones. Adapting basic ideas of wavelet theory, interpolatory and orthonormal bases are employed, both of which are constructed from equidistant shifts of a single polynomial. One of our main goals is the investigation of basis transformations in a form that facilitates fast algorithms. We observe that the corresponding transformation matrices have a circulant structure and can be diagonalized by Fourier matrices. The resulting diagonal matrices contain the eigenvalues, which are computed explicitly. So, the algorithms can be easily realized using the Fast Fourier Transform (FFT).

Further, we consider the nesting of the sample spaces to obtain multiresolution analyses (MRA's). For the resulting orthogonal wavelet spaces, we proceed as above and find wavelet bases consisting of translates of a single polynomial. Again, interpolatory and orthonormal bases are constructed, which show the same time-frequency behaviour as the sample bases do. The basis transformations can be described analogously by circulant matrices.

Most important for practical reasons are the decomposition of signals in frequency bands, which correspond to the wavelet spaces, and their reconstruction.

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Focusing on basis transformations, the two-scale relations and decomposition formulas are also given in matrix notation suitable for the use of FFT methods. Then the transformations for signal data follow easily (see [19]).

This direct approach to trigonometric wavelets is an alternative to the construction of periodic wavelets by periodization of cardinal wavelets (see [1], [3], [6], [8], [10]). In particular, the de la Vallée Poussin means can be obtained by periodization of cardinal functions which are products of sinus cardinalis functions.

A first constructive approach to trigonometric wavelets was introduced by C. K. Chui and H. N. Mhaskar in [2] taking very simple nested multiresolution spaces and corresponding wavelet spaces. They constructed the scaling functions by means of the partial Fourier sum operator applied to a Haar function with small support in order to get time-localized basis functions. The two disadvantages of this procedure are, first, that the resulting approximation in the sample space is only a quasi-interpolation and, second, that the basis functions are just as localized as the simplest de la Vallée Poussin mean, the modified Dirichlet kernel.

The idea that we followed was initiated by the paper [18] of A. A. Privalov, where he considered the de la Vallée Poussin means in a different content. Namely, he was interested in an orthonormal polynomial Schauder basis for $C_{2\pi}$ of optimal degree. This problem has also been considered by P. Wojtaszczyk and K. Woźniakowski in [23], by D. Offin and K. Oskolkov in [9], and was finally solved by R. Lorentz and A. A. Sahakian in [7] (see also K. Woźniakowski [24]).

Privalov's approach was also the starting point for E. Quak and the first named author to investigate in [14], [15] and [16] particular cases of trigonometric wavelet spaces. A more general concept of periodic wavelets was given by G. Plonka and M. Tasche in [12], where both constructive and periodized versions are included.

The present paper is organized as follows. In Section 2 we introduce sample spaces and discuss different bases and their transformations. Beside the interpolatory basis, its dual basis and the orthonormalized basis, which all consist of translates of a single function, we also consider a frequency basis. We conclude this section by general $L^p_{2\pi}$-stability arguments and asymptotic estimates for the basis functions, which are connected to approximation estimates for the corresponding projections.

In Section 3 we restrict ourselves to certain parameters which allow nested sample spaces and the construction of wavelet spaces. Then the ideas of Section 2 concerning scaling function bases are adapted to the corresponding wavelet bases, their transformations and asymptotics.

Finally, the reconstruction and decomposition formulas are developed in Section 4 for the interpolatory and orthonormal bases, respectively, in order to provide the relevant algorithms for signal analysis.

§2 Sample spaces

We denote by $\mathrm{C}_{2\pi}$ the space of all continuous 2π-periodic functions with the maximum norm, by $\mathrm{L}^2_{2\pi}$ the space of square-integrable 2π-periodic functions and by T_n the set of all trigonometric polynomials of degree at most n. The inner product in $\mathrm{L}^2_{2\pi}$ is given by

$$\langle f\,,g\rangle \;:=\; \frac{1}{2\pi}\int_0^{2\pi} f(x)\,\overline{g(x)}\;dx\,, \qquad \text{for all}\;\; f,g\in \mathrm{L}^2_{2\pi}\,.$$

At first we consider interpolation at equidistant nodal points in the interval $[0,2\pi)$ in its simplest form. This can be achieved by a discretization of the Fourier sum operator S_n with kernel D_n, $n\in\mathbb{N}$,

$$S_n f(x) \;:=\; \frac{a_0}{2}+\sum_{k=1}^{n}(a_k\cos kx+b_k\sin kx) \;=\; \langle f\,,D_n(\,\circ-x)\rangle\,, \tag{2.1}$$

with the coefficients

$$a_0 \;:=\; \frac{1}{\pi}\int_0^{2\pi} f(t)\;dt\,,\quad a_k \;:=\; \frac{1}{\pi}\int_0^{2\pi} f(t)\cos kt\;dt\,,\quad b_k \;:=\; \frac{1}{\pi}\int_0^{2\pi} f(t)\sin kt\;dt\,.$$

The Dirichlet kernel

$$D_n(x) \;\;:=\;\; 1+2\sum_{s=1}^{n}\cos sx \;=\; \begin{cases} \dfrac{\sin(2n+1)\frac{x}{2}}{\sin\frac{x}{2}}\,, & \text{for } x\notin 2\pi\mathbb{Z}\,,\\[2ex] 2n+1\,, & \text{for } x\in 2\pi\mathbb{Z}\end{cases}$$

is vanishing at all nodal points $\frac{2k\pi}{2n+1}$, for $k=1,\ldots,2n$. Taking corresponding translates of D_n as fundamental polynomials, we can define the Lagrange interpolation operator, for any $f\in\mathrm{C}_{2\pi}$, at $2n+1$ equidistant points.

The disadvantages of this approach are, firstly, that periodic multiresolution needs an even (dyadic) number of nodal points, and secondly, that the kernel has no bounded $\mathrm{L}^1_{2\pi}$-norm, for $n\to\infty$, or, in other words, is not sufficiently local. Both of these problems can be resolved by using certain de la Vallée Poussin means of D_n instead of the Dirichlet kernel itself.

2.1 Trigonometric interpolation

We define the de la Vallée Poussin means φ_N^M , for $N, M \in \mathbb{N}$ and $N \geq M$, by

$$\begin{aligned} \varphi_N^M(x) &:= \frac{1}{2M\sqrt{2N}} \sum_{m=N-M}^{N+M-1} D_m(x) \\ &= \frac{1}{\sqrt{2N}} \left(1 + 2\sum_{\ell=1}^{N-M} \cos \ell x + 2 \sum_{\ell=N-M+1}^{N+M-1} \frac{N+M-\ell}{2M} \cos \ell x \right) \qquad (2.1.1) \\ &= \frac{1}{\sqrt{2N}} \left(D_{N-M}(x) + \sum_{k=-M+1}^{M-1} \frac{M-k}{M} \cos(N+k)x \right) . \qquad (2.1.2) \end{aligned}$$

By induction, one easily proves that they can be rewritten in the form

$$\varphi_N^M(x) = \begin{cases} \dfrac{\sin Nx \sin Mx}{2M\sqrt{2N} \sin^2 \frac{x}{2}} , & \text{for} \quad x \notin 2\pi\mathbb{Z} , \\ \sqrt{2N} , & \text{for} \quad x \in 2\pi\mathbb{Z} , \end{cases} \qquad (2.1.3)$$

which guarantees zero values at the nodes $\frac{k\pi}{N}$, $k = 1, \ldots, 2N-1$.

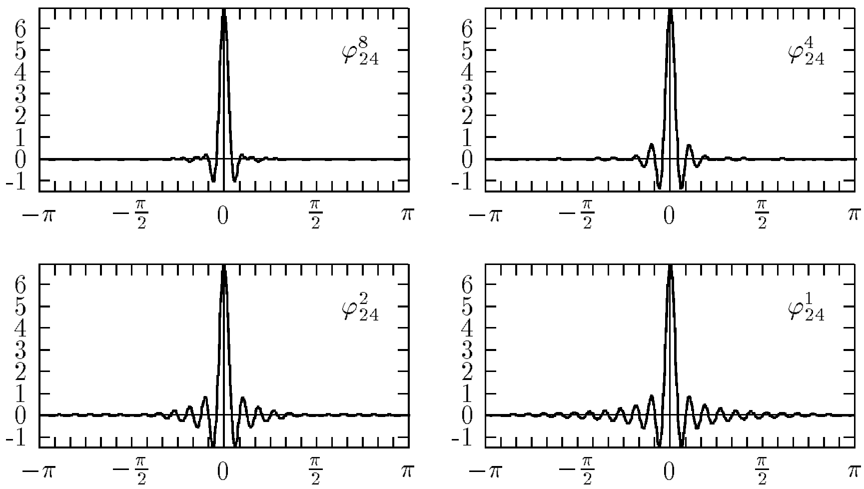


Figure 1. Interpolating sampling functions

The corresponding trigonometric interpolation operator $L_N^M : \mathrm{C}_{2\pi} \to \mathrm{C}_{2\pi}$ is defined by

$$L_N^M f(x) := \frac{1}{\sqrt{2N}} \sum_{s=0}^{2N-1} f\Big(\frac{s\pi}{N}\Big) \; \varphi_N^M \Big(x - \frac{s\pi}{N}\Big) \qquad (2.1.4)$$

satisfying, for all $k \in \mathbb{Z}$,

$$L_N^M f\Big(\frac{k\pi}{N}\Big) \;=\; f\Big(\frac{k\pi}{N}\Big)\,.$$

For $s = 0, \dots, 2N-1$, the translates

$$\varphi_{N,s}^M(x) \;:=\; \varphi_N^M\Big(x - \frac{s\pi}{N}\Big)$$

of the de la Vallée Poussin means are linearly independent since $\varphi_{N,s}^M\left(\frac{k\pi}{N}\right) = \sqrt{2N}\,\delta_{k,s}$, for $k, s = 0, \dots, 2N-1$, but they are not orthogonal (see Proposition 1). Let

$$V_N^M := \operatorname{span}\left\{\varphi_{N,s}^M : s = 0, \dots, 2N-1\right\}\,.$$

Then the dimension of V_N^M is $2N$ independently of M. Being an interpolation operator, $L_N^M : \mathrm{C}_{2\pi} \to V_N^M$ is the identity on V_N^M.

It is instructive to know how single frequencies are mapped by L_N^M.

Lemma 1. *For all $k \in \mathbb{N}_0$, let $k \equiv \ell \bmod 2N$, with $0 \le \ell \le 2N-1$. Then,*

$$L_N^M \cos(k\,\circ)(x) = \begin{cases} \cos \ell x\,, & \text{if } 0 \le \ell \le N-M, \\ \frac{M+(N-\ell)}{2M}\cos \ell x + \frac{M-(N-\ell)}{2M}\cos(2N-\ell)x\,, & \text{if } N-M < \ell < N+M, \\ \cos(2N-\ell)x\,, & \text{if } N+M \le \ell \le 2N-1, \end{cases}$$

$$L_N^M \sin(k\,\circ)(x) = \begin{cases} \sin \ell x\,, & \text{if } 0 < \ell \le N-M, \\ \frac{M+(N-\ell)}{2M}\sin \ell x - \frac{M-(N-\ell)}{2M}\sin(2N-\ell)x\,, & \text{if } N-M < \ell < N+M, \\ -\sin(2N-\ell)x\,, & \text{if } N+M \le \ell \le 2N-1. \end{cases}$$

Proof: We use the well-known fact that, for all $N \in \mathbb{N},\ k \in \mathbb{Z}$,

$$\sum_{s=0}^{2N-1} \cos\left(x + \tfrac{ks\pi}{N}\right) \;=\; \begin{cases} 2N\cos x\,, & \text{if } k \equiv 0 \bmod 2N, \\ 0\,, & \text{if } k \not\equiv 0 \bmod 2N, \end{cases} \tag{2.1.5}$$

$$\sum_{s=0}^{2N-1} \sin\left(x + \tfrac{ks\pi}{N}\right) \;=\; \begin{cases} 2N\sin x\,, & \text{if } k \equiv 0 \bmod 2N, \\ 0\,, & \text{if } k \not\equiv 0 \bmod 2N. \end{cases} \tag{2.1.6}$$

By definition of L_N^M, we have, for any $k \in \mathbb{N}_0$,

$$\begin{aligned}
L_N^M \cos(k\,\circ)(x) &= \frac{1}{\sqrt{2N}} \sum_{s=0}^{2N-1} \cos k(\tfrac{s\pi}{N})\ \varphi_N^M(x - \tfrac{s\pi}{N}) \\
&= \frac{1}{4NM} \sum_{s=0}^{2N-1} \cos \tfrac{ks\pi}{N} \sum_{m=N-M}^{N+M-1} \left(1 + 2\sum_{r=1}^{m} \cos r(x - \tfrac{s\pi}{N})\right) \\
&= \delta_{0,k \bmod 2N} \\
&\quad + \frac{1}{4NM} \sum_{m=N-M}^{N+M-1} \sum_{r=1}^{m} \sum_{s=0}^{2N-1} \left(\cos(rx + \tfrac{(k-r)s\pi}{N}) + \cos(rx - \tfrac{(k+r)s\pi}{N})\right) \\
&= \delta_{0,k \bmod 2N} + \frac{1}{2M} \sum_{m=N-M}^{N+M-1} \sum_{r=1}^{m} (\delta_{0,(k-r) \bmod 2N} + \delta_{0,(k+r) \bmod 2N}) \cos rx.
\end{aligned}$$

Now we have to consider three different cases for ℓ, with $k \equiv \ell \bmod 2N$.
If $0 \le \ell \le N-M$, then

$$L_N^M \cos(k\,\circ)(x) = \delta_{0,\ell} + \frac{1}{2M} \sum_{m=N-M}^{N+M-1} \sum_{r=1}^{m} \delta_{r,\ell} \cos rx = \cos \ell x.$$

If $N-M < \ell < N+M$, then

$$\begin{aligned}
L_N^M \cos(k\,\circ)(x) &= \frac{1}{2M} \sum_{m=N-M}^{N+M-1} \sum_{r=1}^{m} (\delta_{r,\ell} + \delta_{r,2N-\ell}) \cos rx \\
&= \tfrac{(N+M)-\ell}{2M} \cos \ell x + \tfrac{N+M-(2N-\ell)}{2M} \cos(2N-\ell)x \\
&= \left(\tfrac{1}{2} + \tfrac{N-\ell}{2M}\right) \cos \ell x + \left(\tfrac{1}{2} - \tfrac{N-\ell}{2M}\right) \cos(2N-\ell)x.
\end{aligned}$$

If $N+M \le \ell \le 2N-1$, then

$$L_N^M \cos(k\,\circ)(x) = \frac{1}{2M} \sum_{m=N-M}^{N+M-1} \sum_{r=1}^{m} \delta_{r,2N-\ell} \cos rx = \cos(2N-\ell)x.$$

The formula for the sine frequencies is proved analogously. ■

From Lemma 1, it follows immediately that L_N^M is also the identity on T_{N-M}, and

$$T_{N-M} \subset V_N^M \subset T_{N+M-1}, \tag{2.1.7}$$

which is basic for nesting subspaces to form a trigonometric multiresolution analysis.

A more detailed description of V_N^M in terms of frequencies is given by the following result.

Theorem 1. *For $M < N$, the set*

$$\left\{ \varrho_{N,k}^M : \ k = 0, \ldots, 2N-1 \right\} ,$$

with

$$\varrho_{N,0}^M(x) := \sqrt{2}/2 , \quad \varrho_{N,k}^M(x) := \sqrt{2}\cos kx , \quad \varrho_{N,2N-k}^M(x) := \sqrt{2}\sin kx , \qquad (2.1.8)$$

where $k = 1, \ldots, N-M$, and

$$\begin{aligned} \varrho_{N,N}^M(x) &:= \frac{\sqrt{2}}{2}\cos Nx , \\ \varrho_{N,N-k}^M(x) &:= \sqrt{2}\Big(\frac{M+k}{2M}\cos(N-k)x + \frac{M-k}{2M}\cos(N+k)x\Big) , \qquad (2.1.9) \\ \varrho_{N,N+k}^M(x) &:= \sqrt{2}\Big(\frac{M+k}{2M}\sin(N-k)x - \frac{M-k}{2M}\sin(N+k)x\Big) , \end{aligned}$$

where $k = 1, \ldots, M-1$, constitutes an orthogonal basis of V_N^M.

Proof: By (2.1.7), the orthogonal basis (2.1.8) of T_{N-M} is a basis of $V_N^M \cap T_{N-M}$. In order to find the remaining orthogonal basis of $V_N^M \ominus T_{N-M}$, where $\ominus$ denotes the orthogonal difference, we rewrite the high-frequency part of (2.1.2) for the translates $\varphi_{N,\ell}^M$ as follows

$$\begin{aligned} &\sum_{k=-M+1}^{M-1} \tfrac{M-k}{M}\cos(N+k)\left(x - \tfrac{\ell\pi}{N}\right) \\ &= \cos\tfrac{N\ell\pi}{N}\cos Nx \\ &\quad + \sum_{k=1}^{M-1} \tfrac{M-k}{M}\Big(\cos\tfrac{(N+k)\ell\pi}{N}\cos(N+k)x + \sin\tfrac{(N+k)\ell\pi}{N}\sin(N+k)x\Big) \\ &\quad + \sum_{k=1}^{M-1} \tfrac{M+k}{M}\Big(\cos\tfrac{(N-k)\ell\pi}{N}\cos(N-k)x - \sin\tfrac{(N-k)\ell\pi}{N}\sin(N-k)x\Big) \\ &= \cos\tfrac{N\ell\pi}{N}\cos Nx + \sum_{k=1}^{M-1}\cos\tfrac{(N-k)\ell\pi}{N}\left(\tfrac{M+k}{M}\cos(N-k)x + \tfrac{M-k}{M}\cos(N+k)x\right) \\ &\qquad + \sum_{k=1}^{M-1}\sin\tfrac{(N-k)\ell\pi}{N}\left(\tfrac{M+k}{M}\sin(N-k)x - \tfrac{M-k}{M}\sin(N+k)x\right) . \end{aligned}$$

Thus, we see that the functions in (2.1.9) are elements of V_N^M and clearly orthogonal to all functions of T_{N-M}. The mutual orthogonality of these functions is obvious, and their number is $2M-1 \ = \ \dim V_N^M - \dim T_{N-M}$. ■

Let us introduce vector notations for the sequence of basis functions,

$$\underline{\boldsymbol{\varphi}}_N^M \;:=\; \left(\varphi_{N,\ell}^M\right)_{\ell=0}^{2N-1} \quad \text{and} \quad \underline{\boldsymbol{\varrho}}_N^M \;:=\; \left(\varrho_{N,k}^M\right)_{k=0}^{2N-1}.$$

Then the corresponding basis transformations can be described by their transformation matrices.

Theorem 2. *The translates of the de la Vallée Poussin means satisfy the relations*

$$\underline{\boldsymbol{\varphi}}_N^M \;=\; \boldsymbol{U}_N \, \underline{\boldsymbol{\varrho}}_N^M \quad \text{and} \quad \underline{\boldsymbol{\varrho}}_N^M \;=\; \boldsymbol{U}_N^{-1} \, \underline{\boldsymbol{\varphi}}_N^M ,$$

where the matrices $\boldsymbol{U}_N = \left(u_{\ell,k}^N\right)_{\ell,k=0}^{2N-1}$ *and* $\boldsymbol{U}_N^{-1} = \left(\breve{u}_{k,\ell}^N\right)_{k,\ell=0}^{2N-1}$ *have the entries, for all* $\ell = 0, \ldots, 2N-1$,

$$\begin{aligned} u_{\ell,0}^N &= \frac{1}{\sqrt{N}}, & \breve{u}_{0,\ell}^N &= \frac{1}{2\sqrt{N}}, \\ u_{\ell,N}^N &= \frac{(-1)^\ell}{\sqrt{N}}, & \breve{u}_{N,\ell}^N &= \frac{(-1)^\ell}{2\sqrt{N}}, \end{aligned}$$

$$u_{\ell,k}^N \;=\; \breve{u}_{k,\ell}^N \;=\; \begin{cases} \dfrac{1}{\sqrt{N}} \cos \dfrac{\ell k \pi}{N}, & \text{if} \quad 0 < k < N, \\ \dfrac{-1}{\sqrt{N}} \sin \dfrac{\ell k \pi}{N}, & \text{if} \quad N < k < 2N. \end{cases}$$

Proof: From (2.1.2) and the definition of the basis elements $\varrho_{N,k}^M$, it follows immediately that

$$\varphi_{N,\ell}^M(x) \;=\; \frac{\sqrt{2}}{\sqrt{2N}} \Big(\sum_{k=0}^{N} \cos \frac{\ell k \pi}{N} \; \varrho_{N,k}^M(x) + \sum_{k=1}^{N-1} \sin \frac{\ell k \pi}{N} \; \varrho_{N,2N-k}^M(x) \Big),$$

which confirms the entries of $\boldsymbol{U}_N$. The second relation is determined by the inverse transformation. From the interpolation

$$\begin{aligned} \varrho_{N,k}^M(x) &= L_N^M \; \varrho_{N,k}^M(x) \\ &= \frac{1}{\sqrt{2N}} \sum_{\ell=0}^{2N-1} \varrho_{N,k}^M\left(\frac{\ell\pi}{N}\right) \varphi_{N,\ell}^M(x) \end{aligned}$$

and from Lemma 1, we obtain the entries of $\boldsymbol{U}_N^{-1}$. ■

2.2 Gram matrix and dual basis

For a more detailed investigation of the sample spaces, we introduce the Gram matrix

$$\boldsymbol{G}_N^M \quad := \quad \left(\langle \varphi_{N,r}^M \,, \varphi_{N,s}^M \rangle\right)_{r,s=0}^{2N-1} .$$

To simplify the further computations, we will employ the particular structure of this Gram matrix. It is easily seen that $\boldsymbol{G}_N^M$ is symmetric and circulant. The latter fact allows us, according to [4], Chap. 3, to diagonalize $\boldsymbol{G}_N^M$ by means of the $2N$-th Fourier matrix; *i.e.*

$$\boldsymbol{G}_N^M \;=\; \overline{\boldsymbol{F}}_{2N} \; \boldsymbol{D}_N^M \; \boldsymbol{F}_{2N} \,,$$

where

$$\boldsymbol{F}_{2N} \;:=\; \frac{1}{\sqrt{2N}} \left(e^{-\frac{2\pi i r s}{2N}}\right)_{r,s=0}^{2N-1} , \qquad \overline{\boldsymbol{F}}_{2N} \;=\; \left(\boldsymbol{F}_{2N}\right)^{-1} . \tag{2.2.1}$$

Clearly, the diagonal matrix $\boldsymbol{D}_N^M = \operatorname{diag}\left(d_{N,r}^M\right)_{r=0}^{2N-1}$ contains the eigenvalues of $\boldsymbol{G}_N^M$.

Lemma 2. *The eigenvalues of the Gram matrix* $\boldsymbol{G}_N^M$ *are*

$$d_{N,r}^M \quad = \quad \begin{cases} \dfrac{M^2 + (N-r)^2}{2M^2} , & \textit{if} \quad N - M < r < N + M \,, \\ 1 \,, & \textit{otherwise.} \end{cases}$$

Proof: We compute the elements of the diagonal matrix $\boldsymbol{D}_N^M = \boldsymbol{F}_{2N} \; \boldsymbol{G}_N^M \; \overline{\boldsymbol{F}}_{2N}$ in the form

$$d_{N,r}^M \quad = \quad \sum_{k=0}^{2N-1} \langle \varphi_{N,0}^M \,, \varphi_{N,k}^M \rangle \; e^{\frac{i k r \pi}{N}} \; .$$

Since $\boldsymbol{G}_N^M$ is symmetric, the eigenvalues are real-valued, and we obtain by the interpolation formula (2.1.4)

$$\begin{aligned} d_{N,r}^M \quad &= \quad \sum_{k=0}^{2N-1} \langle \varphi_{N,0}^M \,, \varphi_{N,k}^M \rangle \; \cos \tfrac{k r \pi}{N} \\ &= \quad \langle \varphi_{N,0}^M \,, \sum_{k=0}^{2N-1} \cos \tfrac{k r \pi}{N} \; \varphi_{N,k}^M \rangle \\ &= \quad \sqrt{2N} \langle \varphi_{N,0}^M \,, L_N^M \cos(r \,\circ) \rangle \, . \end{aligned}$$

Using Lemma 1 and the representation (2.1.2) of φ_N^M , we consider again three cases. If $0 \le r \le N-M$, then

$$d_{N,r}^M \;=\; \langle D_{N-M}\,, \cos(r\,\circ)\rangle \;=\; 1\,,$$

if $N+M \le r \le 2N-1$, then

$$d_{N,r}^M \;=\; \langle D_{N-M}\,, \cos((2N-r)\,\circ)\rangle \;=\; 1\,,$$

and if $N-M < r < N+M$, then

$$\begin{aligned} d_{N,r}^M \;&=\; 2\left\langle \sum_{\ell=N-M+1}^{N+M-1} \tfrac{M+N-\ell}{2M}\cos(\ell\,\circ)\;, \right. \\ &\qquad\qquad \left. \tfrac{M+(N-r)}{2M}\cos(r\,\circ) + \tfrac{M-(N-r)}{2M}\cos((2N-r)\,\circ)\right\rangle \\ &=\; \tfrac{(M+N-r)^2}{4M^2} + \tfrac{(M-(N-r))^2}{4M^2} \;=\; \tfrac{M^2+(N-r)^2}{2M^2}\,. \quad \blacksquare \end{aligned}$$

The functions $\varphi_{N,k}^M$ being non-orthogonal, we are now interested in the dual functions. They are needed in order to find the orthogonal projection of a function $f \in \mathrm{L}_{2\pi}^2$ with respect to the interpolating basis. The dual functions $\tilde{\varphi}_{N,r}^M \in V_N^M$ of the basis functions $\varphi_{N,\ell}^M$ $(r,\ell = 0,\ldots,2N-1)$ are uniquely determined by the orthonormality conditions

$$\left\langle \tilde{\varphi}_{N,r}^M\,, \varphi_{N,\ell}^M \right\rangle \;=\; \delta_{r,\ell}\,.$$

As the duals are also in V_N^M , they possess an expansion in V_N^M . Hence, let

$$\tilde{\varphi}_{N,r}^M(x) \;=\; \sum_{s=0}^{2N-1} \alpha_{N,r,s}^M\; \varphi_{N,s}^M(x)\,. \tag{2.2.2}$$

To simplify the main result on the dual functions, we denote

$$\underline{\tilde{\boldsymbol{\varphi}}}_N^M \;:=\; \left(\tilde{\varphi}_{N,r}^M\right)_{r=0}^{2N-1}\,.$$

Theorem 3. *The dual functions satisfy the transformation equations*

$$\underline{\tilde{\boldsymbol{\varphi}}}_N^M \;=\; \left(\boldsymbol{G}_N^M\right)^{-1} \underline{\boldsymbol{\varphi}}_N^M \quad \text{and} \quad \underline{\boldsymbol{\varphi}}_N^M \;=\; \boldsymbol{G}_N^M\; \underline{\tilde{\boldsymbol{\varphi}}}_N^M\,, \tag{2.2.3}$$

where

$$\left(\boldsymbol{G}_N^M\right)^{-1} = \overline{\boldsymbol{F}}_{2N}\ \left(\boldsymbol{D}_N^M\right)^{-1}\ \boldsymbol{F}_{2N}$$

$$= \left(\delta_{r,s} + \frac{(-1)^{s-r}}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{2M^2}{M^2+k^2} - 1\right) \cos \tfrac{k(s-r)\pi}{N}\right)_{r,s=0}^{2N-1} \qquad (2.2.4)$$

$$\boldsymbol{G}_N^M = \overline{\boldsymbol{F}}_{2N}\ \boldsymbol{D}_N^M\ \boldsymbol{F}_{2N}$$

$$= \left(\delta_{r,s} + \frac{(-1)^{s-r}}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{M^2+k^2}{2M^2} - 1\right) \cos \tfrac{k(s-r)\pi}{N}\right)_{r,s=0}^{2N-1},$$

with

$$\boldsymbol{D}_N^M = \operatorname{diag}\left(d_{N,r}^M\right)_{r=0}^{2N-1}, \qquad \left(\boldsymbol{D}_N^M\right)^{-1} = \operatorname{diag}\left(1/d_{N,r}^M\right)_{r=0}^{2N-1},$$

the entries of which are known from Lemma 2.

Proof: From the definition in (2.2.2), it follows that

$$\langle \tilde{\varphi}_{N,r}^M , \varphi_{N,\ell}^M \rangle = \sum_{s=0}^{2N-1} \alpha_{N,r,s}^M \langle \varphi_{N,s}^M , \varphi_{N,\ell}^M \rangle = \delta_{r,\ell} ,$$

which establishes the coefficient matrix $\left(\alpha_{N,r,s}^M\right)_{r,s=0}^{2N-1} = \left(\boldsymbol{G}_N^M\right)^{-1}$. Furthermore, the inverse of a circulant is again a circulant, and hence,

$$\left(\boldsymbol{G}_N^M\right)^{-1} = \overline{\boldsymbol{F}}_{2N}\ \left(\boldsymbol{D}_N^M\right)^{-1}\ \boldsymbol{F}_{2N} ,$$

with $\left(\boldsymbol{D}_N^M\right)^{-1} = \operatorname{diag}\left(1/d_{N,r}^M\right)_{r=0}^{2N-1}$. Therefrom we calculate

$$\alpha_{N,r,s}^M = \frac{1}{2N} \sum_{\ell=0}^{2N-1} \frac{1}{d_{N,\ell}^M}\ e^{\frac{i\ell(s-r)\pi}{N}}$$

$$= \frac{1}{2N} \left(\sum_{\ell=0}^{2N-1} e^{\frac{i\ell(s-r)\pi}{N}} + \sum_{\ell=N-M+1}^{N+M-1} \left(\tfrac{2M^2}{M^2+(N-\ell)^2} - 1\right) e^{\frac{i\ell(s-r)\pi}{N}} \right)$$

$$= \frac{1}{2N} \left(2N\,\delta_{s,r} + \sum_{k=-M+1}^{M-1} \left(\tfrac{2M^2}{M^2+k^2} - 1\right) e^{\frac{i(N+k)(s-r)\pi}{N}} \right)$$

$$= \delta_{s,r} + \frac{(-1)^{s-r}}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{2M^2}{M^2+k^2} - 1\right) \cos \tfrac{k(s-r)\pi}{N} .$$

Inverting this relation yields the second formula, and we can compute

$$\begin{aligned}
\langle \varphi_{N,r}^M , \varphi_{N,s}^M \rangle &= \frac{1}{2N} \sum_{\ell=0}^{2N-1} d_{N,\ell}^M \, e^{\frac{i\ell(s-r)\pi}{N}} \\
&= \frac{1}{2N} \left(\sum_{\ell=0}^{2N-1} e^{\frac{i\ell(s-r)\pi}{N}} + \sum_{\ell=N-M+1}^{N+M-1} \left(\tfrac{M^2+(N-\ell)^2}{2M^2} - 1 \right) e^{\frac{i\ell(s-r)\pi}{N}} \right) \\
&= \frac{1}{2N} \left(2N \, \delta_{s,r} + \sum_{k=-M+1}^{M-1} \left(\tfrac{M^2+k^2}{2M^2} - 1 \right) e^{\frac{i(N+k)(s-r)\pi}{N}} \right) \\
&= \delta_{s,r} + \frac{(-1)^{s-r}}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{M^2+k^2}{2M^2} - 1 \right) \cos \tfrac{k(s-r)\pi}{N} \, . \quad \blacksquare
\end{aligned}$$

For the entries of $\boldsymbol{G}_N^M$, we also know another form from the paper [18] of A. A. Privalov.

Proposition 1. ([18]) *Suppose $M < N \in \mathbb{N}$. Then*

$$\langle \varphi_{N,r}^M , \varphi_{N,r}^M \rangle = 1 - \frac{M}{3N} + \frac{1}{12NM}$$

and, for $r \neq s$,

$$\langle \varphi_{N,r}^M , \varphi_{N,s}^M \rangle = (-1)^{r-s} \, \frac{2M \cos \frac{M(r-s)\pi}{N} \sin \frac{(r-s)\pi}{2N} - \sin \frac{M(r-s)\pi}{N} \cos \frac{(r-s)\pi}{2N}}{8NM^2 \sin^3 \frac{(r-s)\pi}{2N}} \, .$$

In Section 2.4 we will use the fact that $\boldsymbol{G}_N^M$ is positive definite. In particular, we proved the following inequality.

Proposition 2. ([17]) *For $N, M \in \mathbb{N}$, with $M|N$, and for $r = 0, \ldots , 2N-1$,*

$$\langle \varphi_{N,r}^M , \varphi_{N,r}^M \rangle - \sum_{\substack{s=0 \\ s \neq r}}^{2N-1} \left| \langle \varphi_{N,s}^M , \varphi_{N,r}^M \rangle \right| > \frac{1}{4} \, .$$

Let us end with the representation of the dual functions in the orthogonal basis of frequencies.

Theorem 4. *For the dual functions, we have*

$$\underline{\tilde{\boldsymbol{\varphi}}}_N^M = \boldsymbol{U}_N \left(\boldsymbol{D}_N^M \right)^{-1} \underline{\boldsymbol{\varrho}}_N^M \quad \text{and} \quad \underline{\boldsymbol{\varrho}}_N^M = \boldsymbol{D}_N^M \, \boldsymbol{U}_N^{-1} \, \underline{\tilde{\boldsymbol{\varphi}}}_N^M \, ,$$

with $\boldsymbol{U}_N$, $\boldsymbol{U}_N^{-1}$ from Theorem 2 and $\boldsymbol{D}_N^M$, $\left(\boldsymbol{D}_N^M \right)^{-1}$ as given in Theorem 3.

Proof: We insert the coefficients of (2.2.4) in (2.2.2) and obtain

$$\begin{aligned}
\tilde{\varphi}^M_{N,r}(x) &= \varphi^M_{N,r}(x) + \frac{1}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{2M^2}{M^2+k^2} - 1\right) \sum_{s=0}^{2N-1} \cos \tfrac{(N+k)(s-r)\pi}{N} \; \varphi^M_{N,s}(x) \\
&= \frac{1}{\sqrt{2N}} \Big(D_{N-M}\left(x - \tfrac{r\pi}{N}\right) + \sum_{k=-M+1}^{M-1} \tfrac{M-k}{M} \cos(N+k)\left(x - \tfrac{r\pi}{N}\right) \\
&\quad + \sum_{k=-M+1}^{M-1} \left(\tfrac{2M^2}{M^2+k^2} - 1\right) \\
&\qquad \times \left(\tfrac{M-k}{2M} \cos(N+k)\left(x - \tfrac{r\pi}{N}\right) + \tfrac{M+k}{2M} \cos(N-k)\left(x - \tfrac{r\pi}{N}\right)\right) \Big) .
\end{aligned}$$

Thus, the dual functions are

$$\begin{aligned}
\tilde{\varphi}^M_{N,r}(x) &= \tfrac{1}{\sqrt{2N}} \Big(D_{N-M}\left(x - \tfrac{r\pi}{N}\right) \\
&\quad + 2M \sum_{k=-M+1}^{M-1} \tfrac{M-k}{M^2+k^2} \cos(N+k)\left(x - \tfrac{r\pi}{N}\right) \Big) .
\end{aligned} \tag{2.2.5}$$

By the definition of $\varrho^M_{N,k}$, we can conclude that

$$\tilde{\varphi}^M_{N,r} = \frac{\sqrt{2}}{\sqrt{2N}} \left(\sum_{k=0}^{N} \frac{1}{d^M_{N,k}} \cos \frac{rk\pi}{N} \, \varrho^M_{N,k} + \sum_{k=1}^{N-1} \frac{1}{d^M_{N,2N-k}} \sin \frac{rk\pi}{N} \, \varrho^M_{N,2N-k} \right) .$$

Writing this in matrix notation yields the first equation. Then the inverse transformation works by means of the identity $(\boldsymbol{U}_N (\boldsymbol{D}^M_N)^{-1})^{-1} = \boldsymbol{D}^M_N \boldsymbol{U}^{-1}_N$.
■

From (2.2.5), it follows that the dual functions are shifts of $\tilde{\varphi}^M_{N,0}$ again, which is an even function like $\varphi^M_{N,0}$. Note also that the coefficients $\alpha^M_{N,r,s}$ are equal to the function values

$$\alpha^M_{N,r,s} = (2N)^{-1/2} \, \tilde{\varphi}^M_{N,0}\left(\tfrac{(s-r)\pi}{N}\right) , \tag{2.2.6}$$

which follows from the interpolation property of the basis in the expansion (2.2.2).

2.3 Orthonormal basis of translates

We also seek a basis of V^M_N consisting of orthonormal translates $\mathcal{O}\varphi^M_{N,r}(x) := \mathcal{O}\varphi^M_N\left(x - \frac{r\pi}{N}\right)$ of a function $\mathcal{O}\varphi^M_N \in V^M_N$. Writing

$$\mathcal{O}\varphi^M_{N,r}(x) = \sum_{s=0}^{2N-1} \gamma^M_{N,r,s} \, \varphi^M_{N,s}(x) , \tag{2.3.1}$$

for all $r = 0, \ldots, 2N-1$, we have to determine the coefficients $\gamma^M_{N,r,s}$. The coefficient matrix

$$\mathbf{\Gamma}^M_N \;:=\; \left(\gamma^M_{N,r,s}\right)_{r,s=0}^{2N-1} \;=\; (2N)^{-1/2}\left(\mathcal{O}\varphi^M_N\left(\frac{(s-r)\pi}{N}\right)\right)_{r,s=0}^{2N-1}$$

is obviously circulant. Hence, we can write $\mathbf{\Gamma}^M_N \;=\; \overline{\boldsymbol{F}}_{2N}\;\mathbf{\Delta}^M_N\;\boldsymbol{F}_{2N}$, with a diagonal matrix $\mathbf{\Delta}^M_N$. If we require the orthonormality

$$\left\langle \mathcal{O}\varphi^M_{N,k}\,,\mathcal{O}\varphi^M_{N,r}\right\rangle \;=\; \sum_{\ell=0}^{2N-1}\sum_{s=0}^{2N-1}\gamma^M_{N,k,\ell}\gamma^M_{N,r,s}\left\langle \varphi^M_{N,\ell}\,,\varphi^M_{N,s}\right\rangle \;=\; \delta_{k,r}\,, \tag{2.3.2}$$

and denote

$$\underline{\mathcal{O}\varphi}^M_N \;:=\; \left(\mathcal{O}\varphi^M_{N,r}\right)_{r=0}^{2N-1}\,,$$

then we can summarize the representation as follows.

Theorem 5. *In V^M_N, there exists a basis of orthonormal translates of an even function $\mathcal{O}\varphi^M_N$ satisfying the relations*

$$\underline{\mathcal{O}\varphi}^M_N \;=\; \mathbf{\Gamma}^M_N\;\underline{\varphi}^M_N\,, \qquad\qquad \underline{\varphi}^M_N \;=\; \left(\mathbf{\Gamma}^M_N\right)^{-1}\underline{\mathcal{O}\varphi}^M_N\,, \tag{2.3.3}$$

$$\underline{\mathcal{O}\varphi}^M_N \;=\; \left(\mathbf{\Gamma}^M_N\right)^{-1}\underline{\tilde{\varphi}}^M_N \qquad \text{and} \qquad \underline{\tilde{\varphi}}^M_N \;=\; \mathbf{\Gamma}^M_N\;\underline{\mathcal{O}\varphi}^M_N\,, \tag{2.3.4}$$

where

$$\begin{aligned}\mathbf{\Gamma}^M_N \;&=\; \overline{\boldsymbol{F}}_{2N}\;\mathbf{\Delta}^M_N\;\boldsymbol{F}_{2N}\\ &=\; \left(\delta_{r,s} + \frac{(-1)^{s-r}}{2N}\sum_{k=-M+1}^{M-1}\left(\tfrac{2M}{\sqrt{2M^2+2k^2}}-1\right)\cos\tfrac{k(s-r)\pi}{N}\right)_{r,s=0}^{2N-1}\end{aligned} \tag{2.3.5}$$

and

$$\begin{aligned}\left(\mathbf{\Gamma}^M_N\right)^{-1} \;&=\; \overline{\boldsymbol{F}}_{2N}\;\left(\mathbf{\Delta}^M_N\right)^{-1}\;\boldsymbol{F}_{2N}\\ &=\; \left(\delta_{r,s} + \frac{(-1)^{s-r}}{2N}\sum_{k=-M+1}^{M-1}\left(\tfrac{\sqrt{2M^2+2k^2}}{2M}-1\right)\cos\tfrac{k(s-r)\pi}{N}\right)_{r,s=0}^{2N-1},\end{aligned}$$

with

$$\mathbf{\Delta}^M_N \;=\; \operatorname{diag}\left(1/\sqrt{d^M_{N,\ell}}\right)_{\ell=0}^{2N-1}, \qquad \left(\mathbf{\Delta}^M_N\right)^{-1} \;=\; \operatorname{diag}\left(\sqrt{d^M_{N,\ell}}\right)_{\ell=0}^{2N-1}$$

and $d^M_{N,\ell}$ from Lemma 2.

Proof: With $\boldsymbol{\Delta}_N^M = \operatorname{diag}\left(1/\sqrt{d_{N,\ell}^M}\right)_{\ell=0}^{2N-1}$, we are able to compute the coefficients $\gamma_{N,r,s}^M$, for $r, s = 0, \ldots, 2N-1$, from the equation $\boldsymbol{\Gamma}_N^M = \overline{\boldsymbol{F}}_{2N}\, \boldsymbol{\Delta}_N^M\, \boldsymbol{F}_{2N}$ and obtain

$$\begin{aligned}
\gamma_{N,r,s}^M &= \frac{1}{2N} \sum_{\ell=0}^{2N-1} \frac{1}{\sqrt{d_{N,\ell}^M}}\, e^{\frac{i\ell(s-r)\pi}{N}} \\
&= \frac{1}{2N} \left(\sum_{\ell=0}^{2N-1} e^{\frac{i\ell(s-r)\pi}{N}} + \sum_{\ell=N-M+1}^{N+M-1} \left(\frac{1}{\sqrt{d_{N,\ell}^M}} - 1 \right) e^{\frac{i\ell(s-r)\pi}{N}} \right) \\
&= \frac{1}{2N} \left(2N\, \delta_{s,r} + \sum_{k=-M+1}^{M-1} \left(\frac{\sqrt{2M^2}}{\sqrt{M^2+k^2}} - 1 \right) e^{\frac{i(N+k)(s-r)\pi}{N}} \right) \\
&= \delta_{s,r} + \frac{(-1)^{s-r}}{2N} \sum_{k=-M+1}^{M-1} \left(\frac{2M}{\sqrt{2M^2+2k^2}} - 1 \right) \cos \frac{k(s-r)\pi}{N}\,.
\end{aligned}$$

The entries of $\left(\boldsymbol{\Gamma}_N^M\right)^{-1} = \left(\breve{\gamma}_{N,s,r}^M\right)_{s,r=0}^{2N-1} = \overline{\boldsymbol{F}}_{2N} \left(\boldsymbol{\Delta}_N^M\right)^{-1} \boldsymbol{F}_{2N}$ follow from

$$\begin{aligned}
\breve{\gamma}_{N,s,r}^M &= \frac{1}{2N} \sum_{\ell=0}^{2N-1} \sqrt{d_{N,\ell}^M}\, e^{\frac{i\ell(s-r)\pi}{N}} \\
&= \frac{1}{2N} \left(\sum_{\ell=0}^{2N-1} e^{\frac{i\ell(s-r)\pi}{N}} + \sum_{k=-M+1}^{M-1} \left(\frac{\sqrt{M^2+k^2}}{\sqrt{2M^2}} - 1 \right) e^{\frac{i(N+k)(s-r)\pi}{N}} \right) \\
&= \delta_{s,r} + \frac{(-1)^{s-r}}{2N} \sum_{k=-M+1}^{M-1} \left(\frac{\sqrt{2M^2+2k^2}}{2M} - 1 \right) \cos \frac{k(s-r)\pi}{N}\,.
\end{aligned}$$

Now we prove the orthonormality of the translates for the representations in (2.3.3). From (2.3.5), we derive that $\gamma_{N,r,s}^M = \gamma_{N,s,r}^M$. Hence, $\mathcal{O}\varphi_N^M$ is even, and we have

$$\boldsymbol{\Gamma}_N^M\, \boldsymbol{G}_N^M \left(\boldsymbol{\Gamma}_N^M\right)^T = \boldsymbol{\Gamma}_N^M\, \boldsymbol{G}_N^M\, \boldsymbol{\Gamma}_N^M = \overline{\boldsymbol{F}}_{2N}\, \boldsymbol{\Delta}_N^M\, \boldsymbol{D}_N^M\, \boldsymbol{\Delta}_N^M\, \boldsymbol{F}_{2N} = \boldsymbol{I}_{2N}$$

being the $2N$-th identity matrix, which proves (2.3.2). The relations in (2.3.4) are easy to derive from (2.3.3) and (2.2.3). ■

Finally, we want to represent the orthonormal translates in the basis of frequencies.

Theorem 6. *For the orthonormal translates, we have the relations*

$$\underline{\mathcal{O}\boldsymbol{\varphi}}_N^M = \boldsymbol{U}_N\, \boldsymbol{\Delta}_N^M\, \underline{\boldsymbol{\varrho}}_N^M \quad \textit{and} \quad \underline{\boldsymbol{\varrho}}_N^M = \left(\boldsymbol{\Delta}_N^M\right)^{-1} \boldsymbol{U}_N^{-1}\, \underline{\mathcal{O}\boldsymbol{\varphi}}_N^M\,, \qquad (2.3.6)$$

with $\boldsymbol{U}_N$, $\boldsymbol{U}_N^{-1}$ *from Theorem 2 and* $\boldsymbol{\Delta}_N^M$, $\left(\boldsymbol{\Delta}_N^M\right)^{-1}$ *as given in Theorem 5.*

Proof: We obtain an explicit form for the orthogonal translates of $\mathcal{O}\varphi_N^M$ by evaluating

$$\begin{aligned}
\mathcal{O}\varphi_{N,r}^M(x) &= \sum_{s=0}^{2N-1} \gamma_{N,r,s}^M \varphi_{N,s}^M(x) \\
&= \varphi_{N,r}^M + \sum_{s=0}^{2N-1} \frac{1}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{2M}{\sqrt{2M^2+2k^2}} - 1\right) \cos \tfrac{(N+k)(s-r)\pi}{N} \; \varphi_{N,s}^M(x) \\
&= \varphi_{N,r}^M + \frac{1}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{2M}{\sqrt{2M^2+2k^2}} - 1\right) \sum_{s=0}^{2N-1} \cos \tfrac{(N+k)(s-r)\pi}{N} \; \varphi_{N,s}^M(x)
\end{aligned}$$

with

$$\begin{aligned}
\sum_{s=0}^{2N-1} \cos \tfrac{(N+k)(s-r)\pi}{N} \; \varphi_{N,s}^M(x) &= L_N^M \cos((N+k)\circ)\left(x - \tfrac{r\pi}{N}\right) \\
&= \tfrac{M-k}{2M} \cos(N+k)\left(x - \tfrac{r\pi}{N}\right) + \tfrac{M+k}{2M} \cos(N-k)\left(x - \tfrac{r\pi}{N}\right).
\end{aligned}$$

Thus,

$$\mathcal{O}\varphi_{N,r}^M(x) = \frac{1}{\sqrt{2N}}\left(D_{N-M}(x - \tfrac{r\pi}{N}) + 2 \sum_{k=-M+1}^{M-1} \tfrac{M-k}{\sqrt{2M^2+2k^2}} \cos(N+k)(x - \tfrac{r\pi}{N})\right).$$

This can be rewritten:

$$\mathcal{O}\varphi_{N,r}^M = \frac{1}{\sqrt{2N}}\left(\sum_{k=0}^{N} \frac{1}{\sqrt{d_{N,k}^M}} \cos \frac{kr\pi}{N} \varrho_{N,k}^M + \sum_{k=1}^{N-1} \frac{1}{\sqrt{d_{N,2N-k}^M}} \sin \frac{kr\pi}{N} \varrho_{N,2N-k}^M\right),$$

which yields the matrices in (2.3.6). ■

2.4 Time-frequency-localization and projection estimates

$L_{2\pi}^p$-Norms and stability

Our first aim is to describe the time-localization in terms of the $L_{2\pi}^p$-norms of the scaling functions. Therefore let us define, for $1 \le p < \infty$,

$$\|f\|_p := \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p dx\right)^{\frac{1}{p}} \quad \text{and} \quad \|f\|_\infty := \operatorname*{ess\,sup}_{x\in[0,2\pi]} |f(x)|,$$

and analogously for vectors $\{\alpha_k\} \in \mathbb{C}^{2N}$

$$\|\{\alpha_k\}\|_{\ell^p} := \left(\sum_{k=0}^{2N-1} |\alpha_k|^p\right)^{\frac{1}{p}} \quad \text{and} \quad \|\{\alpha_k\}\|_{\ell^\infty} := \max_k |\alpha_k| .$$

From the well-known inequalities of Hölder and Nikolskii (see *e.g.* A.F. Timan [22], Chap. 4.9), one knows that

$$1 \le \frac{\|t_n\|_q}{\|t_n\|_p} \le C\, n^{\frac{1}{p}-\frac{1}{q}} ,$$

for all polynomials $t_n \in T_n$ and $1 \le p \le q \le \infty$. We want to have basis functions with the order $n^{\frac{1}{p}-\frac{1}{q}}$ on the right-hand side, which ensures the best possible time-localization with respect to a polynomial MRA. In particular, we are looking for polynomials satisfying

$$\|t_n\|_\infty \sim n^{\frac{1}{p}} \|t_n\|_p \sim n\|t_n\|_1 .$$

The notation $A_n \sim B_n$ means $|A_n^{-1}B_n| \le C$ and $|A_n B_n^{-1}| \le C$. Here and in the following, C denotes positive constants depending on fixed parameters involved, but their values may be different at each occurrence.

Let us recall the basic result of an $L^1_{2\pi}$-estimate for the interpolating functions established by S.B. Stečkin [20]

$$\|\varphi^M_{N,0}\|_1 = \tfrac{2}{\pi^2\sqrt{2N}} \log \tfrac{N+M}{2M} + C + \mathcal{O}\left(\tfrac{M}{N+M}\right) .$$

In order to obtain explicit constants, we proceed with the following simple derivation. For $2M \le N$,

$$\begin{aligned}
\|\varphi^M_{N,0}\|_1 &= \frac{2}{\pi\, 2M\sqrt{2N}} \int_0^{\frac{\pi}{2}} \left|\frac{\sin(2Mt)\sin(2Nt)}{\sin^2 t}\right| dt \\
&< \frac{2}{\pi\sqrt{2N}} \left(\frac{2N\pi}{2N} + \int_{\frac{\pi}{2N}}^{\frac{\pi}{4M}} \left|\frac{\sin(2Nt)}{\sin t}\right| dt + \frac{1}{2M}\int_{\frac{\pi}{4M}}^{\frac{\pi}{2}} \frac{1}{\sin^2 t} dt\right) \\
&< \frac{1}{\pi\sqrt{2N}} \left(2\pi + \pi \ln \frac{N}{2M} + \pi\right) = \frac{1}{\sqrt{2N}}\left(3 + \ln\frac{N}{2M}\right) . \qquad (2.4.1)
\end{aligned}$$

This computation contains some information about the time-localization of $\varphi^M_{N,0}$ in the three different intervals of integration. Moreover, we know the maximum value of $\varphi^M_{N,0}$, and thus

$$\|\varphi^M_{N,0}\|_\infty = \sqrt{2N} . \qquad (2.4.2)$$

In Section 3.1 we will define N_j and M_j in such a way that $3M_j \le N_j$ and that N_j/M_j is bounded from above by a constant. Assuming $N/2M$ to be constant, we conclude from (2.4.1) and (2.4.2) by means of Nikolskii's inequality, for arbitrary $1 \le p \le \infty$, that

$$\|\varphi_{N,0}^M\|_p \;\sim\; N^{\frac{1}{2}-\frac{1}{p}} \,. \tag{2.4.3}$$

As a further step, we formulate the Riesz stability in the more general $L^p_{2\pi}$-setting.

Theorem 7. *For arbitrary sequences* $\{\alpha_k\} \in \mathbb{C}^{2N}$ *and* $1 \le p \le \infty$, *the inequalities*

$$\frac{3}{4\pi+3}\,\|\{\alpha_k\}\|_{\ell^p} \;\le\; (2N)^{\frac{1}{p}-\frac{1}{2}} \left\| \sum_{k=0}^{2N-1} \alpha_k\, \varphi_{N,k}^M \right\|_p \;\le\; A\, \|\{\alpha_k\}\|_{\ell^p} \tag{2.4.4}$$

are satisfied, with

$$A \;=\; \frac{4\pi+3}{3}\,\sqrt{2N}\,\|\varphi_{N,0}^M\|_1 \,,$$

independent of p, *or*

$$A \;=\; \begin{cases} \dfrac{C}{p-1}\,, & \text{if} \quad 1 < p < 2\,, \\ C\,p\,, & \text{if} \quad 2 \le p < \infty\,. \end{cases} \tag{2.4.5}$$

As noted above, for simplicity, we assume

$$3 \;\le\; \frac{N}{M} \;\le\; K\,, \tag{2.4.6}$$

where K is a constant independent of N. This implies the uniform boundedness of A in (2.4.4), for $1 \le p \le \infty$, because of

$$A \;\le\; \frac{4\pi+3}{3}(3+\ln\frac{N}{2M}) \;<\; 12+5.2\,\ln K\,.$$

Proof: a) We start with the inequality on the left-hand side, which can be proved by using a well-known inequality for trigonometric polynomials $t_n \in T_n$ (see *e.g.* [22], Chap. 4.9), namely:

$$\sup_x \left(\frac{1}{2N} \sum_{\ell=0}^{2N-1} \left| t_n\left(x - \frac{\ell\pi}{N} \right) \right|^p \right)^{\frac{1}{p}} \;\le\; \left(1+\frac{n\pi}{N}\right) \|t_n\|_p\,, \tag{2.4.7}$$

for $1 \le p \le \infty$ and $N \in \mathbb{N}$. Then, by applying the interpolatory properties of $\varphi_{N,k}^M$, we conclude that

$$\begin{aligned} \left(\frac{1}{2N} \sum_{\ell=0}^{2N-1} |\alpha_\ell|^p \right)^{\frac{1}{p}} &= \left(\frac{1}{2N} \sum_{\ell=0}^{2N-1} \left| \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \alpha_k \, \varphi_{N,k}^M \left(\tfrac{\ell\pi}{N}\right) \right|^p \right)^{\frac{1}{p}} \\ &\le \left(1 + \tfrac{(N+M-1)\pi}{N}\right) \left\| \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \alpha_k \, \varphi_{N,k}^M \right\|_p \\ &\le \left(1 + \tfrac{4\pi}{3}\right) \left\| \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \alpha_k \, \varphi_{N,k}^M \right\|_p . \end{aligned}$$

b) In order to prove the inequality on the right-hand side, we use, for $1 \le p < \infty$, the function g from the space $L_{2\pi}^q$, where $1/p + 1/q = 1$, with $\|g\|_q = 1$ such that

$$\left\| \sum_{k=0}^{2N-1} \alpha_k \, \varphi_{N,k}^M \right\|_p = \frac{1}{2\pi} \int_0^{2\pi} g(x) \sum_{k=0}^{2N-1} \alpha_k \, \varphi_{N,k}^M(x) \, dx .$$

Then Hölder's inequality gives

$$\begin{aligned} &\sum_{k=0}^{2N-1} \alpha_k \frac{1}{2\pi} \int_0^{2\pi} g(x) \, \varphi_{N,k}^M(x) \, dx \\ &\qquad \le 2N \left(\frac{1}{2N} \sum_{k=0}^{2N-1} |\alpha_k|^p \right)^{\frac{1}{p}} \left(\frac{1}{2N} \sum_{k=0}^{2N-1} \left| (g * \varphi_{N,0}^M) \left(\tfrac{k\pi}{N}\right) \right|^q \right)^{\frac{1}{q}} , \end{aligned}$$

where $*$ denotes the usual 2π-periodic convolution,

$$(g * \varphi_{N,0}^M)(t) := \frac{1}{2\pi} \int_0^{2\pi} g(x) \, \varphi_{N,0}^M(t - x) \, dx .$$

Now we apply (2.4.7) and obtain two different estimates,

$$\begin{aligned} \left(\frac{1}{2N} \sum_{k=0}^{2N-1} \left| (g * \varphi_{N,0}^M) \left(\tfrac{k\pi}{N}\right) \right|^q \right)^{\frac{1}{q}} &\le \left(1 + \tfrac{4\pi}{3}\right) \|g * \varphi_{N,0}^M\|_q \\ \le \left(1 + \frac{4\pi}{3}\right) \times &\begin{cases} \|g\|_q \, \|\varphi_{N,0}^M\|_1 , & \text{if} \quad 1 \le q < \infty , \\ \dfrac{1}{2M\sqrt{2N}} \displaystyle\sum_{m=N-M}^{N+M-1} \|S_m g\|_q , & \text{if} \quad 1 < q < \infty . \end{cases} \end{aligned}$$

The second estimate in (2.4.5) follows from the boundedness of the Fourier sum operator S_m in $L^q_{2\pi}$, for $1 < q < \infty$ (see *e.g.* [25]). The proof is completed by a similar derivation for $p = \infty$. Applying (2.4.7) again, we find that

$$\begin{aligned}\left\| \frac{1}{\sqrt{2N}} \sum_{k=0}^{2N-1} \alpha_k \, \varphi^M_{N,k} \right\|_\infty &\leq \sqrt{2N} \max_k |\alpha_k| \sup_x \left(\frac{1}{2N} \sum_{k=0}^{2N-1} \left| \varphi^M_{N,0} \left(x - \tfrac{k\pi}{N} \right) \right| \right) \\ &\leq \|\{\alpha_k\}\|_{\ell^\infty} \, \tfrac{4\pi+3}{3} \, \sqrt{2N} \, \|\varphi^M_{N,0}\|_1 . \quad \blacksquare\end{aligned}$$

Asymptotics for the dual and orthogonal functions

Our next goal is to achieve the same time localization results for the dual functions $\tilde{\varphi}^M_N$ and for the orthogonal functions $\mathcal{O}\varphi^M_N$ as we have in (2.4.3) for the interpolatory functions. At first we determine estimates for the coefficients $\alpha^M_{N,r,s}$ and $\gamma^M_{N,r,s}$.

Lemma 3. *The coefficients in the basis representations* (2.2.2) *and* (2.3.1) *satisfy the decay conditions, for* $r, s = 0, \ldots, 2N-1$,

$$|\alpha^M_{N,r,s}|, \, |\gamma^M_{N,r,s}| \; \leq \; C \max\left\{ (|r-s|+1)^{-2}, \; (2N - |r+s|)^{-2} \right\} . \tag{2.4.8}$$

Proof: For $r = s$, both of the coefficients given in (2.2.4) and (2.3.5) are less than 2, since $M < N$. For $r \neq s$, let us focus on

$$|\gamma^M_{N,r,s}| \; = \; \left| \frac{1}{2N} \sum_{k=-M+1}^{M-1} \left(\tfrac{2M}{\sqrt{2M^2+2k^2}} - 1 \right) \cos \tfrac{k(r-s)\pi}{N} \right| .$$

We define the 2π-periodic even function f_γ by

$$f_\gamma(x) \; := \; \begin{cases} \frac{M\pi\sqrt{2}}{\sqrt{(\pi M)^2 + N^2 x^2}} - 1 , & \text{for} \quad |x| \leq \frac{\pi M}{N} , \\ 0 , & \text{for} \quad \frac{\pi M}{N} < |x| \leq \pi , \end{cases}$$

The derivative of f_γ is of bounded variation and possesses two jumps in $[-\pi, \pi]$. Hence, for the Fourier coefficients a_{r-s}, with $r \geq s$, we have (see [25], Chap. 10)

$$\left| a_{|r-s|} \right| \; = \; \left| \frac{1}{\pi} \int_{-\pi}^{\pi} f_\gamma(x) \cos(r-s)x \, dx \right| \; \leq \; C(|r-s|+1)^{-2} ,$$

and by the aliasing formula for discrete Fourier coefficients,

$$\begin{aligned} |\gamma^M_{N,r,s}| &= \left| \frac{1}{2N} \sum_{k=-N}^{N-1} f_\gamma(\tfrac{k\pi}{N}) \cos(r-s) \tfrac{k\pi}{N} \right| \\ &= \frac{1}{4} \left| \sum_{\ell=-\infty}^{\infty} a_{|2N\ell+r-s|} + a_{|2N\ell-r+s|} \right| \\ &\leq C \max\left\{ (|r-s|)^{-2}, \; |2N-r+s|^{-2} \right\} . \end{aligned}$$

Thus, (2.4.8) is proved for $\gamma^M_{N,r,s}$. The same idea works for $\alpha^M_{N,r,s}$ (see Theorem 3), if one replaces f_γ by

$$f_\alpha(x) \;:=\; \begin{cases} \frac{2M^2\pi^2}{M^2\pi^2+N^2x^2} - 1\,, & \text{for} \quad |x| \le \frac{\pi M}{N}, \\ 0\,, & \text{for} \quad \frac{\pi M}{N} < |x| \le \pi. \end{cases} \quad \blacksquare$$

Now it is easy to obtain the same time-localization result for the dual and orthonormal functions as we have in (2.4.3) for φ^M_N .

Theorem 8. *For $1 \le p \le \infty$, we have the asymptotic behaviour*

$$\|\tilde{\varphi}^M_{N,0}\|_p \;\sim\; \|\mathcal{O}\varphi^M_{N,0}\|_p \;\sim\; N^{\frac{1}{2}-\frac{1}{p}}.$$

Proof: The proof simply consists of the application of Lemma 3 to (2.4.4). E.g., we obtain

$$\|\mathcal{O}\varphi^M_{N,0}\|_p \;=\; \left\| \sum_{k=0}^{2N-1} \gamma^M_{N,0,k}\varphi^M_{N,k} \right\|_p \;\sim\; \sqrt{N}\left(\frac{1}{2N}\sum_{k=0}^{2N-1} \left|\gamma^M_{N,0,k}\right|^p \right)^{\frac{1}{p}} \;\sim\; N^{\frac{1}{2}-\frac{1}{p}}. \quad \blacksquare$$

Let us end this subsection with a quite different description of time localization (see also [16]). In $[0, 2\pi]$, we define, for $k = 1, \dots, 2N$, the intervals

$$I_{N,k} \;:=\; \left[\frac{(k-1)\pi}{N}, \frac{k\pi}{N} \right]$$

and describe the decay in terms of the distance from the peak. Again, we start with the interpolating function $\varphi^M_{N,0}(x)$. From (2.1.3), one obtains by standard estimates and (2.4.6) that

$$\max_{x\in I_{N,k}} |\varphi^M_{N,0}(x)| \;\sim\; \begin{cases} \dfrac{\sqrt{N}}{k^2}\,, & \text{if} \quad 0 < k \le N\,, \\ \dfrac{\sqrt{N}}{(2N-k)^2}\,, & \text{if} \quad N < k < 2N\,. \end{cases} \tag{2.4.9}$$

Now, for simplicity, we restrict ourselves to $0 < k \le N$. The same time localization results can be shown for the dual and orthonormal functions.

Theorem 9. *Let $0 < k \le N$. Then*

$$\|\tilde{\varphi}^M_{N,0}\|_\infty \;\sim\; \|\mathcal{O}\varphi^M_N\|_\infty \;\sim\; \sqrt{N} \tag{2.4.10}$$

and

$$\max_{x\in I_{N,k}} \left\{ |\tilde{\varphi}^M_{N,0}(x)|, |\mathcal{O}\varphi^M_{N,0}(x)| \right\} \;\le\; C\,\frac{\sqrt{N}}{k^2}. \tag{2.4.11}$$

Proof: Formula (2.4.10) is already included in Theorem 8. Here we only show the estimate in (2.4.11) for $\mathcal{O}\varphi_N^M$. With Lemma 3, (2.3.1) and (2.4.9), we obtain

$$\begin{aligned}
&\max_{x\in I_{N,k}} |\mathcal{O}\varphi_{N,0}^M(x)| \\
&\leq \; C \max_{x\in I_{N,k}} \Big(\sum_{r=0}^{N} \frac{1}{(r+1)^2} |\varphi_{N,r}^M(x)| + \sum_{r=N+1}^{2N-1} \frac{1}{(2N-r)^2} |\varphi_{N,r}^M(x)| \Big) \\
&\leq \; C\sqrt{N} \Big(\sum_{r=0}^{N} \frac{1}{((r+1)(|r-k|+1))^2} \\
&\qquad + \sum_{r=N+1}^{2N-1} \frac{1}{(2N-r)^2 (\min\{(r-k),(2N-r+k)\})^2} \Big) .
\end{aligned}$$

Then (2.4.11) follows directly by splitting the sums on the right-hand side into

$$\sum_{r=0}^{2N-1} = \sum_{r=0}^{k/2} + \sum_{r=1+k/2}^{3k/2} + \sum_{r=1+3k/2}^{2N-1} . \qquad \blacksquare$$

Error estimates

Here we want to summarize approximation properties of the interpolatory and orthogonal projections onto the spaces V_N^M. For simplicity, we only deal with the most interesting case of continuous functions and estimates in the sup-norm. Roughly speaking, the same results are available for $L_{2\pi}^1$ and especially for $L_{2\pi}^p$, $1 < p < \infty$, avoiding the logarithmic term in the estimates.
In particular, we want to compare the approximation order with the best approximation by trigonometric polynomials of corresponding degree, as A. A. Privalov mentioned in [18]. Further results connecting the convergence order of a given function f with smoothness conditions on f can be easily deduced. Denote, for $f \in C_{2\pi}$,

$$E_n(f) \; := \; \inf\{\|f - t_n\|_\infty \; : \; t_n \in T_n\} \; .$$

In this subsection we do not necessarily assume the uniform boundedness of N/M as done in (2.4.6). But, for further use, we restrict ourselves to $N/M = 3, 4, 5, \ldots$. However, in the case of (2.4.6), we obtain error estimates in the order of the best approximation.

Theorem 10. *Let $f \in C_{2\pi}$. For the interpolatory projection L_N^M onto V_N^M, we have the estimates*

$$E_{N+M-1}(f) \; \leq \; \|f - L_N^M f\|_\infty \; \leq \; (A+1) \; E_{N-M}(f) \, , \tag{2.4.12}$$

with A as defined in (2.4.5).

Proof: The inequality on the left-hand side follows immediately from $L_N^M f \in T_{N+M-1}$. For the inequality on the right-hand side of (2.4.12), we write

$$E_{N-M}(f) \;=\; \|f - t_{N-M}\|_\infty$$

to conclude with the help of (2.4.4) that

$$\begin{aligned}\|f - L_N^M f\|_\infty \;&\leq\; E_{N-M}(f) + \|t_{N-M} - L_N^M f\|_\infty \\ &\leq\; E_{N-M}(f) + A\,\|t_{N-M} - f\|_\infty\,. \quad \blacksquare\end{aligned}$$

Now we are looking for error estimates for the orthogonal projection. The proof of Theorem 10 shows that we only have to deal with the operator norm of the orthogonal projection operator $P_N^M : \mathrm{L}^2_{2\pi} \to V_N^M$,

$$P_N^M f \;:=\; \sum_{k=0}^{2N-1} \langle f\,, \tilde{\varphi}_{N,k}^M\rangle\, \varphi_{N,k}^M \;=\; \sum_{k=0}^{2N-1} \langle f\,, \mathcal{O}\varphi_{N,k}^M\rangle\, \mathcal{O}\varphi_{N,k}^M\,.$$

Lemma 4. *The orthogonal projection P_N^M onto V_N^M satisfies*

$$\|P_N^M\|_{C_{2\pi}\to C_{2\pi}} \;\leq\; C\,\log^2\frac{N}{M}\,.$$

Proof: Writing

$$P_N^M\, f \;=\; \sum_{k=0}^{2N-1} \epsilon_k\; \varphi_{N,k}^M\,, \tag{2.4.13}$$

we obtain by (2.4.4)

$$\|P_N^M f\|_\infty \;\leq\; C\,N\,\|\varphi_{N,0}^M\|_1 \max_{0\leq k<2N} |\epsilon_k| \;=:\; C\,N\,\|\varphi_{N,0}^M\|_1\,|\epsilon_\ell|\,.$$

Taking the inner product with $\varphi_{N,\ell}^M$ in (2.4.13), we estimate

$$\begin{aligned}|\langle P_N^M f\,, \varphi_{N,\ell}^M\rangle| \;&=\; \left|\sum_{k=0}^{2N-1} \epsilon_k \langle \varphi_{N,k}^M\,, \varphi_{N,\ell}^M\rangle\right| \\ &\geq\; |\epsilon_\ell| \Big(\langle \varphi_{N,\ell}^M\,, \varphi_{N,\ell}^M\rangle - \sum_{\substack{k=0\\ k\neq\ell}}^{2N-1} |\langle \varphi_{N,k}^M\,, \varphi_{N,\ell}^M\rangle|\Big)\,.\end{aligned}$$

Applying Proposition 2 yields

$$|\epsilon_\ell| \;\leq\; 4\,|\langle P_N^M f\,, \varphi_{N,\ell}^M\rangle| \;=\; 4\,|\langle f\,, \varphi_{N,\ell}^M\rangle| \;\leq\; 4\,\|f\|_\infty\,\|\varphi_{N,\ell}^M\|_1\,.$$

Hence,

$$\|P_N^M f\|_\infty \;\leq\; C\,N\,\|\varphi_{N,0}^M\|_1^2\,\|f\|_\infty\,,$$

which, together with (2.4.1), proves the lemma. ■

Using Lemma 4 and standard projection estimates, we arrive at the following final result.

Theorem 11. *Let $f \in \mathrm{L}^2_{2\pi}$. For the orthogonal projection P_N^M onto V_N^M, we have the estimates*

$$E_{N+M-1}(f) \;\le\; \|f - P_N^M f\|_\infty \;\le\; C \log^2 \frac{N}{M}\, E_{N-M}(f)\,. \tag{2.4.14}$$

It remains an open question whether one can replace the factor $\log^2 \frac{N}{M}$ in (2.4.14) by $\log \frac{N}{M}$, the best possible constant in (2.4.12).

§3 Wavelet spaces

3.1 Trigonometric MRA

For periodic polynomial multiresolution analyses, it is natural to consider a dyadic number of nodal points (see [10]). To allow more generality for practical applications, we additionally allow any (odd) factor $c \in \mathbb{N}$ and define

$$N_j \;:=\; c\, 2^j\,, \qquad M_j \;:=\; \begin{cases} 2^{j-\lambda}\,, & \text{if } j \ge \lambda\,, \\ 1\,, & \text{if } j < \lambda\,, \end{cases} \tag{3.1.1}$$

for all $j \in \mathbb{N}_0$ and any constant $\lambda \in \mathbb{N}_0$. For fixed c and λ, we define a trigonometric MRA $\{V_j\}_{j=0}^\infty$ of $\mathrm{L}^2_{2\pi}$ by

$$V_j \;:=\; V_{N_j}^{M_j}\,.$$

Theorem 12. *If $c\, 2^\lambda \ge 3$, then all properties of a periodic MRA are satisfied, namely:*

(i) $\quad V_j \subset V_{j+1}\,,\quad$ for all $j \in \mathbb{N}_0\,,\quad \dim V_j = 2N_j\,,$

(ii) $\quad \mathrm{clos}_{L^2_{2\pi}} \left(\bigcup_{j=0}^{\infty} V_j \right) = \mathrm{L}^2_{2\pi}\,,$

(iii) $\quad f \in V_j \;\Rightarrow\; f\Big(\circ - \dfrac{k\pi}{N_j}\Big) \in V_j\,,\quad$ for all $k \in \mathbb{Z},\ j \in \mathbb{N}_0\,,$

(iv) $\quad \exists\, \phi_j \in V_j :\quad V_j = \mathrm{span}\Big\{\phi_{j,k} := \phi_j\Big(\circ - \dfrac{k\pi}{N_j}\Big) :\ k = 0, \ldots, 2N_j - 1\Big\}\,.$

Taking $\phi_j := \varphi_{N_j}^{M_j}$ as the generating function, for any $j \in \mathbb{N}_0$ and $\{\alpha_k\} \in \mathbb{C}^{2N_j}$, we have the stability condition

$$A\, \|\{\alpha_k\}\|_{\ell^2} \;\le\; \left\| \sum_{k=0}^{2N_j-1} \alpha_k\, \phi_{j,k} \right\|_2 \;\le\; B\, \|\{\alpha_k\}\|_{\ell^2}\,, \tag{3.1.2}$$

with best possible constants $A = 1/\sqrt{2}$ and $B = 1$.

Note that, if we replace $\phi_{j,k}$ by the orthonormal translates $\mathcal{O}\phi_{j,k} := \mathcal{O}\varphi_{N_j,k}^{M_j}$, then we obtain best stability; *i.e.* $A = B = 1$.

Proof: The inclusion relation $V_j \subset V_{j+1}$, for all $j \in \mathbb{N}_0$, follows from (2.1.7), if

$$N_j + M_j - 1 \;\leq\; N_{j+1} - M_{j+1} \,.$$

For $j < \lambda$, this is obviously true, and for $j \geq \lambda$, it means that

$$M_j + M_{j+1} \;\leq\; 3 \cdot 2^{j-\lambda} \;\leq\; c\, 2^j + 1 \,.$$

This yields the condition

$$3 \;\leq\; c\, 2^\lambda\,; \quad i.e. \quad N_j \;\geq\; 3M_j\,, \qquad \text{for all } \; j \geq \lambda \,.$$

Then we have $N_j - M_j \geq 2^{j+1-\lambda}$, which, by (2.1.7), implies property (ii). The third and fourth properties follow directly from the construction in Section 2.1. It remains to establish the stability constants in (3.1.2).

Let us denote by $\underline{\boldsymbol{v}}_{j,r}$, for $r = 0, \dots, 2N_j - 1$, the eigenvectors of the Gram matrix $\boldsymbol{G}_j = (\langle \phi_{j,k}\,, \phi_{j,\ell}\rangle)_{k,\ell=0}^{2N_j-1}$ such that $\boldsymbol{G}_j\, \underline{\boldsymbol{v}}_{j,r} = d_{j,r}\, \underline{\boldsymbol{v}}_{j,r}$, where $d_{j,r} = d_{N_j,r}^{M_j}$ are the eigenvalues known from Lemma 2. The eigenvectors form an orthogonal basis of $\mathbb{C}^{2N_j}$. Therefore we can write any $2N_j$-dimensional vector $\underline{\boldsymbol{\alpha}}$ as a linear combination $\underline{\boldsymbol{\alpha}} = \sum_{r=0}^{2N_j-1} a_r\, \underline{\boldsymbol{v}}_{j,r}$, with complex coefficients a_r. Let $\langle \underline{\boldsymbol{\alpha}}\,, \underline{\boldsymbol{\beta}}\rangle_E = \sum_{k=0}^{2N_j-1} \alpha_k \overline{\beta}_k$ denote the Euclidian inner product of $\underline{\boldsymbol{\alpha}}\,, \underline{\boldsymbol{\beta}} \in \mathbb{C}^{2N_j}$. Then we have

$$\begin{aligned}\left\| \sum_{k=0}^{2N_j-1} \alpha_k\, \phi_{j,k} \right\|_2^2 &= \left\langle \sum_{k=0}^{2N_j-1} \alpha_k\, \phi_{j,k}\,, \sum_{l=0}^{2N_j-1} \alpha_l\, \phi_{j,l} \right\rangle \\ &= \langle \boldsymbol{G}_j\, \underline{\boldsymbol{\alpha}}\,, \underline{\boldsymbol{\alpha}}\rangle_E = \sum_{r=0}^{2N_j-1} d_{j,r}\, |a_r|^2\, \langle \underline{\boldsymbol{v}}_{j,r}\,, \underline{\boldsymbol{v}}_{j,r}\rangle_E\,,\end{aligned}$$

and since

$$\sum_{k=0}^{2N_j-1} |\alpha_k|^2 = \langle \underline{\boldsymbol{\alpha}}\,, \underline{\boldsymbol{\alpha}}\rangle_E = \sum_{r=0}^{2N_j-1} |a_r|^2\, \langle \underline{\boldsymbol{v}}_{j,r}\,, \underline{\boldsymbol{v}}_{j,r}\rangle_E\,,$$

the quotient $\left\| \sum_{k=0}^{2N_j-1} \alpha_k\, \phi_{j,k} \right\|_2^2 \Big/ \left(\sum_{k=0}^{2N_j-1} |\alpha_k|^2 \right)$ is bounded by the minimal and maximal eigenvalue of $\boldsymbol{G}_j$, respectively. ■

Let $L_j := L_{N_j}^{M_j}$ denote the interpolation operator mapping $C_{2\pi}$ onto V_j. We can prove the following two-scale relations for the shifted scaling functions.

Theorem 13. *For* $r = 0, \dots, 2N_j - 1$, *we have the refinement equations*

$$\phi_{j,r}(x) = \frac{1}{\sqrt{2}}\phi_{j+1,2r}(x) + \frac{1}{2\sqrt{N_j}} \sum_{s=0}^{2N_j-1} \phi_{j,r}\Big(\frac{(2s+1)\pi}{2N_j}\Big)\, \phi_{j+1,2s+1}(x) \qquad (3.1.3)$$

and, for $j \geq \lambda$, *the dilation relation*

$$\phi_{j+1}(x) \;=\; \sqrt{2}\;\phi_j(2x)\;\frac{1+\cos x}{2}\,. \qquad (3.1.4)$$

Note that in (3.1.4) the function $(1 + \cos x)/2$ maintains the values of $\phi_j(2\circ)$ around $2k\pi$ and suppresses the interposed oscillation of $\phi_j(2\circ)$ around $(2k{+}1)\pi$, where $k \in \mathbb{Z}$.

Proof: By the interpolation property of $\phi_{j,r}$, we have

$$\begin{aligned}
\phi_{j,r}(x) &= L_{j+1}\,\phi_{j,r}(x) \\
&= \frac{1}{\sqrt{2N_{j+1}}}\left(\sum_{k=0}^{4N_j-1} \phi_{j,r}\left(\tfrac{k\pi}{2N_j}\right)\;\phi_{j+1,k}(x)\right) \\
&= \frac{1}{\sqrt{2}}\left(\phi_{j+1,2r}(x) + \frac{1}{\sqrt{2N_j}}\sum_{s=0}^{2N_j-1}\phi_{j,r}\left(\tfrac{(2s+1)\pi}{2N_j}\right)\;\phi_{j+1,2s+1}(x)\right).
\end{aligned}$$

For all levels $j \geq \lambda$, we have $N_{j+1} = 2N_j$ and $M_{j+1} = 2M_j$, which yields

$$\begin{aligned}
\phi_{j+1}(x) &= \frac{\sin N_j(2x)\sin M_j(2x)}{M_j\sqrt{4N_j}\,\sin^2 x}\cos^2\frac{x}{2} \\
&= \sqrt{2}\,\phi_j(2x)\;\frac{1+\cos x}{2}\,. \quad \blacksquare
\end{aligned}$$

Furthermore, we need the following two-scale inner product.

Lemma 5. *For* $\ell = 0, \dots, 2N_j - 1$, *and* $m = 0, \dots, 2N_{j+1} - 1$, *we have*

$$\langle \phi_{j,\ell}\,, \phi_{j+1,m}\rangle \;=\; \frac{1}{\sqrt{2N_{j+1}}}\;\phi_{j,\ell}\left(\frac{m\pi}{2N_j}\right). \qquad (3.1.5)$$

Proof: By (2.1), (2.1.2) and $\langle\cos(m\,\circ)\,,\cos(n\,\circ)\rangle = \frac{1}{2}\,\delta_{m,n}$, we can write

$$\begin{aligned}
\langle \phi_{j,\ell}\,, \phi_{j+1,m}\rangle &= \tfrac{1}{\sqrt{2N_{j+1}}}\;\Big\langle \phi_{j,\ell}\,, D_{N_j+M_j-1}\left(\circ - \tfrac{m\pi}{N_{j+1}}\right)\Big\rangle \\
&= \frac{1}{\sqrt{2N_{j+1}}}\;\phi_{j,\ell}\left(\frac{m\pi}{2N_j}\right). \quad \blacksquare
\end{aligned}$$

3.2 Interpolating wavelets

The wavelet spaces are defined as the relevant orthogonal complements of V_j in V_{j+1}; *i.e.* $V_{j+1} = V_j \oplus W_j$, for all $j \in \mathbb{N}_0$, where $\oplus$ denotes the orthogonal sum. From Theorem 1, we derive a simple orthogonal basis of the wavelet spaces.

Theorem 14. *An orthogonal basis of* W_j *is given by the set*

$$\{\sigma_{j,\ell} : \ \ell = 0, \ldots, 2N_j - 1\} ,$$

with the functions

$$\begin{aligned} \sigma_{j,0}(x) &:= \tfrac{\sqrt{2}}{2} \cos 2N_j x , & \sigma_{j,N_j}(x) &:= \tfrac{\sqrt{2}}{2} \sin N_j x , \\ \sigma_{j,2N_j-k}(x) &:= \sqrt{2} \cos kx , & \sigma_{j,k}(x) &:= \sqrt{2} \sin kx , \end{aligned} \tag{3.2.1}$$

where $k = N_j + M_j, \ldots, N_{j+1} - M_{j+1}$*;*

$$\begin{aligned} \sigma_{j,k}(x) &:= \sqrt{2}\Big(\tfrac{M_{j+1}+k}{2M_{j+1}} \cos(2N_j - k)x + \tfrac{M_{j+1}-k}{2M_{j+1}} \cos(2N_j + k)x \Big), \\ \sigma_{j,2N_j-k}(x) &:= \sqrt{2}\Big(\tfrac{M_{j+1}+k}{2M_{j+1}} \sin(2N_j - k)x - \tfrac{M_{j+1}-k}{2M_{j+1}} \sin(2N_j + k)x \Big), \end{aligned} \tag{3.2.2}$$

where $k = 1, \ldots, M_{j+1} - 1$*; and*

$$\begin{aligned} \sigma_{j,N_j-k}(x) &:= \sqrt{2}\Big(\tfrac{M_j+k}{2M_j} \cos(N_j + k)x - \tfrac{M_j-k}{2M_j} \cos(N_j - k)x \Big), \\ \sigma_{j,N_j+k}(x) &:= \sqrt{2}\Big(\tfrac{M_j+k}{2M_j} \sin(N_j + k)x + \tfrac{M_j-k}{2M_j} \sin(N_j - k)x \Big), \end{aligned} \tag{3.2.3}$$

where $k = 1, \ldots, M_j - 1$.

Proof: First, we check that all functions $\sigma_{j,\ell}$, for $\ell = 0, \ldots, 2N_j - 1$, belong to V_{j+1} . By Theorem 1, this is obvious for the polynomials in (3.2.1) and (3.2.3). Let $\rho_{j,\ell} := \varrho^{M_j}_{N_j,\ell}$. In (3.2.2), we have, for $k = 1, \ldots, M_{j+1} - 1$,

$$\sigma_{j,k} \ = \ \rho_{j+1,2N_j-k} , \quad \sigma_{j,2N_j-k} \ = \ \rho_{j+1,2N_j+k} .$$

Second, we verify that $\langle \sigma_{j,\ell} , \rho_{j,k} \rangle = 0$, for all $k, \ell = 0, \ldots, 2N_j - 1$. This is evident for all $\sigma_{j,\ell}$ in (3.2.1) and (3.2.2). For (3.2.3), it follows by simple calculations. ■

Regarding shift-invariant spaces, we are now looking for a localized wavelet function $\psi_j \in V_{j+1}$ that generates the wavelet space W_j ; *i.e.*

$$W_j \ = \ \operatorname{span} \left\{ \psi_j\Big(\circ - \frac{n\pi}{N_j} \Big) : \quad n = 0, \ldots, 2N_j - 1 \right\} .$$

To determine the wavelet ψ_j uniquely, we may additionally require interpolation properties for the translates $\psi_{j,n}$, in this case at the $2N_j$ interpolation nodes of V_{j+1} which are not interpolation nodes of the translates of ϕ_j. Hence, we demand that $\psi_j \in V_{j+1}$,

$$\langle \phi_{j,k}\,, \psi_j \rangle \;=\; 0\,, \qquad \text{for all} \quad k = 0, \ldots, 2N_j - 1\,, \tag{3.2.4}$$

and, in analogy to the scaling function, that

$$\psi_j\Big(\frac{(2m+1)\pi}{2N_j}\Big) \;=\; \sqrt{2N_j}\;\delta_{m,0}\,, \qquad \text{for all} \quad m = 0, \ldots, 2N_j - 1\,. \tag{3.2.5}$$

Theorem 15. *There exists a uniquely determined function $\psi_j \in V_{j+1}$ satisfying (3.2.4) and (3.2.5), namely:*

$$\psi_j(x) \;:=\; \sqrt{2}\,\phi_{j+1}\Big(x - \frac{\pi}{2N_j}\Big) - \phi_j\Big(x - \frac{\pi}{2N_j}\Big)\,. \tag{3.2.6}$$

For its translates $\psi_{j,n}(x) := \psi_j(x - \frac{n\pi}{N_j})$, $n = 0, \ldots, 2N_j - 1$, the refinement equations are given by

$$\psi_{j,n}(x) = \frac{1}{\sqrt{2}}\phi_{j+1,2n+1}(x) - \frac{1}{2\sqrt{N_j}} \sum_{s=0}^{2N_j-1} \phi_{j,s}\Big(\frac{(2n+1)\pi}{2N_j}\Big)\;\phi_{j+1,2s}(x). \tag{3.2.7}$$

For $j \geq \lambda$, we have the special dilation equation

$$\psi_{j+1}(x) \;=\; \sqrt{2}\,\psi_j(2x)\;\frac{1 + \cos(x - \frac{\pi}{2N_{j+1}})}{2}\,. \tag{3.2.8}$$

Proof: For an arbitrary $f \in V_{j+1}$, there is an expansion

$$f \;=\; L_{j+1} f \;=\; \frac{1}{\sqrt{2N_{j+1}}} \sum_{s=0}^{4N_j-1} \mu_s\;\phi_{j+1,s}\,,$$

with $4N_j$ coefficients $\mu_s = f(\frac{s\pi}{N_{j+1}})$. By means of (3.1.5), the orthogonality condition in (3.2.4) is equivalent to

$$\begin{aligned} 0 \;&=\; \langle \phi_{j,k}\,, f\rangle \\ &=\; \frac{1}{\sqrt{2N_{j+1}}} \sum_{s=0}^{4N_j-1} \mu_s\;\langle \phi_{j,k}\,, \phi_{j+1,s}\rangle \\ &=\; \frac{1}{2N_{j+1}} \sum_{s=0}^{4N_j-1} \mu_s\;\phi_{j,k}\Big(\frac{s\pi}{2N_j}\Big)\,, \end{aligned}$$

for all $k = 0, \ldots, 2N_j{-}1$. The interpolation property in (3.2.5) holds if and only if

$$f\Big(\frac{(2m+1)\pi}{2N_j}\Big) \;=\; \mu_{2m+1} \;=\; \sqrt{2N_j}\;\delta_{m,0}\,,$$

for all $m = 0, \ldots, 2N_j{-}1$. Hence,

$$f \;=\; \frac{1}{\sqrt{2}}\Bigg(\phi_{j+1,1} - \frac{1}{\sqrt{2N_j}}\sum_{k=0}^{2N_j-1}\phi_{j,k}\Big(\frac{\pi}{2N_j}\Big)\;\phi_{j+1,2k}\Bigg)$$

is the only function in V_{j+1} satisfying (3.2.4) and (3.2.5). By (3.1.3), it follows $f = \psi_j$, where ψ_j is defined in (3.2.6). For its translates, (3.2.7) is obtained, using that ϕ_j is an even function. Finally, by means of (3.1.4), we can write, for $j \geq \lambda$,

$$\begin{aligned}\psi_{j+1}(x) \;&=\; \sqrt{2}\,\phi_{j+2}\Big(x - \frac{\pi}{2N_{j+1}}\Big) - \phi_{j+1}\Big(x - \frac{\pi}{2N_{j+1}}\Big)\\ &=\; \Bigg(2\,\phi_{j+1}\Big(2x - \frac{\pi}{2N_j}\Big) - \sqrt{2}\,\phi_j\Big(2x - \frac{\pi}{2N_j}\Big)\Bigg)\,\frac{1+\cos(x - \frac{\pi}{2N_{j+1}})}{2}\,,\end{aligned}$$

which proves (3.2.8). ■

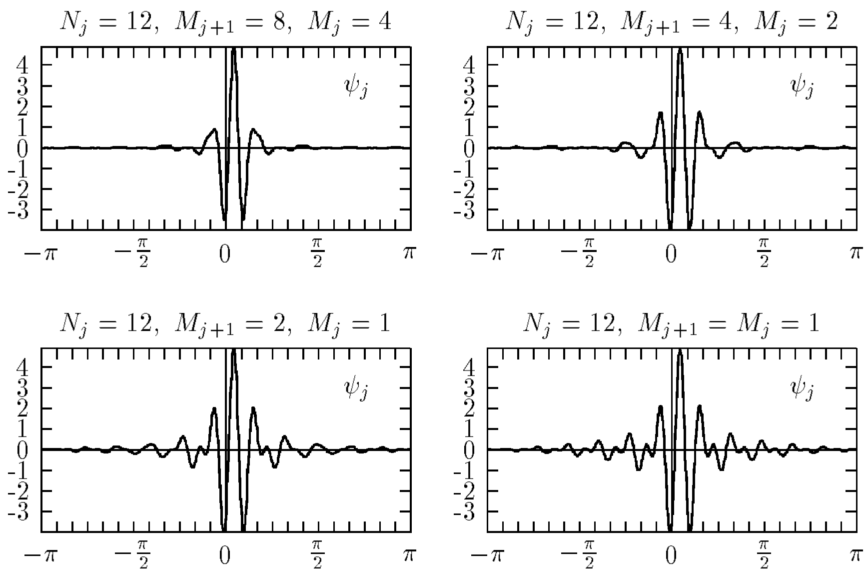


Figure 2. Interpolating wavelet functions

Now we can define the interpolation operator $R_j : \mathrm{C}_{2\pi} \to W_j$,

$$R_j f \;:=\; \frac{1}{\sqrt{2N_j}} \sum_{k=0}^{2N_j-1} f\Big(\frac{(2k+1)\pi}{2N_j}\Big)\; \psi_{j,k}\,, \tag{3.2.9}$$

which is the identity in W_j and fulfils, for $k \in \mathbb{Z}$,

$$R_j f\left(\tfrac{(2k+1)\pi}{2N_j}\right) \;=\; f\left(\tfrac{(2k+1)\pi}{2N_j}\right).$$

Further, we are interested in the expansion of the localized basis in the orthogonal basis from Theorem 14. Let us introduce the notations for the vectors of the functions

$$\underline{\boldsymbol{\psi}}_j \;:=\; (\psi_{j,r})_{r=0}^{2N_j-1} \quad \text{and} \quad \underline{\boldsymbol{\sigma}}_j \;:=\; (\sigma_{j,k})_{k=0}^{2N_j-1}\,.$$

Theorem 16. *The translates of the wavelet ψ_j defined in* (3.2.6) *satisfy the relations*

$$\underline{\boldsymbol{\psi}}_j \;=\; \boldsymbol{O}_j\; \underline{\boldsymbol{\sigma}}_j \quad \text{and} \quad \underline{\boldsymbol{\sigma}}_j \;=\; \boldsymbol{O}_j^{-1}\; \underline{\boldsymbol{\psi}}_j\,,$$

where the matrices $\boldsymbol{O}_j = (o_{j,r,s})_{r,s=0}^{2N_j-1}$ and $\boldsymbol{O}_j^{-1} = (\breve{o}_{j,s,r})_{s,r=0}^{2N_j-1}$ have, for $r = 0,\ldots,2N_j-1$, the entries

$$o_{j,r,0} \;=\; \frac{-1}{\sqrt{N_j}}\,, \qquad \breve{o}_{j,0,r} \;=\; \frac{-1}{2\sqrt{N_j}}\,,$$
$$o_{j,r,N_j} \;=\; \frac{(-1)^r}{\sqrt{N_j}}\,, \qquad \breve{o}_{j,N_j,r} \;=\; \frac{(-1)^r}{2\sqrt{N_j}}\,,$$

$$o_{j,r,s} \;=\; \breve{o}_{j,s,r} \;=\; \begin{cases} \dfrac{-1}{\sqrt{N_j}}\,\cos\dfrac{(2r+1)s\pi}{2N_j}\,, & \text{if} \quad 0 < s < N_j\,, \\[2ex] \dfrac{1}{\sqrt{N_j}}\,\sin\dfrac{(2r+1)s\pi}{2N_j}\,, & \text{if} \quad N_j < s < 2N_j\,. \end{cases}$$

Proof: Similar to (2.1.1), we can write

$$\begin{aligned}
\psi_{j,r}(x) &= \frac{1}{\sqrt{2N_j}} \Bigg(D_{N_{j+1}-M_{j+1}} \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) - D_{N_j - M_j} \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
&\quad + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}} \cos(N_{j+1}+k) \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
&\quad - \sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j-k}{M_j} \cos(N_j+k) \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \Bigg) \\
&= \frac{1}{\sqrt{2N_j}} \Bigg(\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{M_j} \cos(N_j+k) \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
&\quad + 2 \sum_{\ell=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos \ell \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
&\quad + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}} \cos(N_{j+1}+k) \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \Bigg).
\end{aligned}$$

Rewriting this in the basis defined in Theorem 14 gives

$$\begin{aligned}
\psi_{j,r}(x) &= \frac{\sqrt{2}}{\sqrt{2N_j}} \Big(\sum_{k=0}^{N_j-1} \cos \frac{(2N_j-k)(2r+1)\pi}{2N_j} \; \sigma_{j,k}(x) \\
&\qquad + \sum_{k=1}^{N_j} \sin \frac{(2N_j-k)(2r+1)\pi}{2N_j} \; \sigma_{j,2N_j-k}(x) \Big) ,
\end{aligned}$$

which yields the entries of $\boldsymbol{O}_j$. The inverse transformation matrix $\boldsymbol{O}_j^{-1}$ is easily obtained by the interpolation

$$\begin{aligned}
\sigma_{j,s}(x) &= R_j \; \sigma_{j,s}(x) \\
&= \frac{1}{\sqrt{2N_j}} \sum_{r=0}^{2N_j-1} \sigma_{j,s}\Big(\frac{(2r+1)\pi}{2N_j}\Big) \; \psi_{j,r}(x) . \quad \blacksquare
\end{aligned}$$

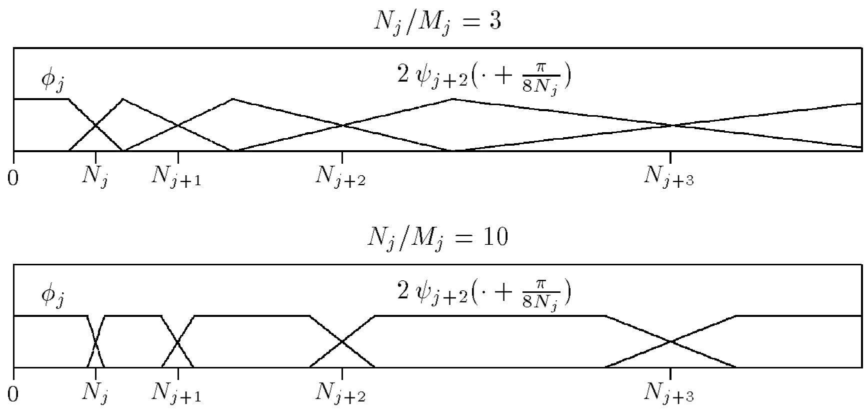


Figure 3. Frequency spectrum of ϕ_j and $2^{(k-j)/2}\,\psi_k(\circ + \frac{\pi}{2N_k}),\ k = 0, 1, 2, \ldots$

3.3 Gram matrix and dual wavelets

In the sequel, the index j of a matrix indicates that its dimension is $2N_j \times 2N_j$. In particular, let $\boldsymbol{I}_j$ be the $2N_j$-th identity matrix and $\boldsymbol{F}_j$ be the $2N_j$-th Fourier matrix defined in (2.2.1). We denote the Gram matrix of the translates of the wavelet function $\psi_j \in W_j$ by

$$\boldsymbol{H}_j \;:=\; (\langle \psi_{j,r}\,,\,\psi_{j,s}\rangle)_{r,s=0}^{2N_j-1}\,.$$

The Gram matrix is symmetric and circulant, and so we can diagonalize it as well by means of the $2N_j$-th Fourier matrix $\boldsymbol{F}_j$ such that

$$\boldsymbol{H}_j \;=:\; \overline{\boldsymbol{F}}_j\ \boldsymbol{E}_j\ \boldsymbol{F}_j\,, \tag{3.3.1}$$

with a diagonal matrix $\boldsymbol{E}_j =: \operatorname{diag}(e_{j,k})_{k=0}^{2N_j-1}$.

Lemma 6. *The eigenvalues of the Gram matrix $\boldsymbol{H}_j$ are*

$$e_{j,k} \;=\; \begin{cases} \frac{1}{2} + \frac{k^2}{2M_{j+1}^2}\,, & \textit{if} \quad 0 \le k < M_{j+1}\,, \\ 1\,, & \textit{if} \quad M_{j+1} \le k \le N_j - M_{j+1}\,, \\ \frac{1}{2} + \frac{(N_j-k)^2}{2M_j^2}\,, & \textit{if} \quad N_j - M_j < k < N_j + M_j\,, \\ 1\,, & \textit{if} \quad N_j + M_j \le k \le 2N_j - M_{j+1}\,, \\ \frac{1}{2} + \frac{(2N_j-k)^2}{2M_{j+1}^2}\,, & \textit{if} \quad 2N_j - M_{j+1} < k \le 2N_j - 1\,. \end{cases}$$

Proof: By means of (3.2.6) and (3.3.1), it is possible to compute the eigenvalues as elements of the diagonal matrix $\boldsymbol{E}_j = \boldsymbol{F}_j \boldsymbol{H}_j \overline{\boldsymbol{F}}_j$, using that

$$\begin{aligned}
&\langle \psi_{j,r}, \psi_{j,s} \rangle \\
&= \left\langle \sqrt{2}\phi_{j+1,2r} - \phi_{j,r}, \sqrt{2}\phi_{j+1,2s} - \phi_{j,s} \right\rangle \\
&= 2\langle \phi_{j+1,2r}, \phi_{j+1,2s} \rangle + \langle \phi_{j,r}, \phi_{j,s} \rangle - 2\sqrt{2} \left\langle \phi_{j+1}, \phi_j \left(\circ - \tfrac{(s-r)\pi}{N_j} \right) \right\rangle \\
&= 2\langle \phi_{j+1,2r}, \phi_{j+1,2s} \rangle + \langle \phi_{j,r}, \phi_{j,s} \rangle - 2\,\delta_{r,s}.
\end{aligned}$$

Hence,

$$\boldsymbol{H}_j = 2\,\breve{\boldsymbol{G}}_j + \boldsymbol{G}_j - 2\,\boldsymbol{I}_j,$$

where $\boldsymbol{G}_j = (\langle \phi_{j,r}, \phi_{j,s} \rangle)_{r,s=0}^{2N_j-1} = \boldsymbol{G}_{N_j}^{M_j}$ and $\breve{\boldsymbol{G}}_j := (\langle \phi_{j+1,2r}, \phi_{j+1,2s} \rangle)_{r,s=0}^{2N_j-1}$ are known to be circulant. By the distributive law for matrices, we have

$$\boldsymbol{H}_j = \overline{\boldsymbol{F}}_j \boldsymbol{E}_j \boldsymbol{F}_j = \overline{\boldsymbol{F}}_j \left(2\,\breve{\boldsymbol{D}}_j + \boldsymbol{D}_j - 2\,\boldsymbol{I}_j \right) \boldsymbol{F}_j,$$

with $\boldsymbol{D}_j = \operatorname{diag}\left(d_{N_j,r}^{M_j} \right)_{r=0}^{2N_j-1}$ and $\breve{\boldsymbol{D}}_j := \operatorname{diag}(\eta_{j,\ell})_{\ell=0}^{2N_j-1}$. To compute the elements of $\breve{\boldsymbol{D}}_j$, we rewrite the elements of the circulant $\breve{\boldsymbol{G}}_j$,

$$\begin{aligned}
\frac{1}{2N_j} \sum_{\ell=0}^{2N_j-1} \eta_{j,\ell}\, e^{\frac{i\ell(s-r)\pi}{N_j}} &= \langle \phi_{j+1,2r}, \phi_{j+1,2s} \rangle \\
&= \frac{1}{2N_{j+1}} \sum_{\ell=0}^{2N_{j+1}-1} d_{j+1,\ell}\, e^{\frac{i\ell(2s-2r)\pi}{N_{j+1}}} \\
&= \frac{1}{4N_j} \sum_{\ell=0}^{2N_{j+1}-1} d_{j+1,\ell}\, e^{\frac{i\ell(s-r)\pi}{N_j}} \\
&= \frac{1}{2N_j} \sum_{\ell=0}^{2N_j-1} \frac{d_{j+1,\ell} + d_{j+1,2N_j+\ell}}{2}\, e^{\frac{i\ell(s-r)\pi}{N_j}}.
\end{aligned}$$

Then, by Lemma 2, we obtain

$$\begin{aligned}
\eta_{j,\ell} &= \frac{d_{j+1,\ell} + d_{j+1,2N_j+\ell}}{2} \\
&= \begin{cases} \frac{3M_{j+1}^2 + \ell^2}{4M_{j+1}^2}, & \text{if} \quad 0 \le \ell < M_{j+1}, \\ 1, & \text{if} \quad M_{j+1} \le \ell \le 2N_j - M_{j+1}, \\ \frac{3M_{j+1}^2 + (2N_j - \ell)^2}{4M_{j+1}^2}, & \text{if} \quad 2N_j - M_{j+1} < \ell \le 2N_j - 1. \end{cases}
\end{aligned}$$

The eigenvalues of $\boldsymbol{H}_j$ follow immediately, for $k = 0, \ldots, 2N_j - 1$, from

$$e_{j,k} \;=\; 2\eta_{j,k} + d_{j,k} - 2\,. \quad \blacksquare$$

Now, we define the dual functions $\tilde{\psi}_{j,r} \in W_j$ of the interpolating translates $\psi_{j,\ell}$, for $r, \ell = 0, \ldots, 2N_j - 1$, by the orthogonality conditions

$$\left\langle \tilde{\psi}_{j,r}\,, \psi_{j,\ell} \right\rangle \;=\; \delta_{r,\ell}\,. \tag{3.3.2}$$

As elements of W_j, they have a representation

$$\tilde{\psi}_{j,r}(x) \;=\; \sum_{s=0}^{2N_j-1} \beta_{j,r,s}\;\psi_{j,s}(x)\,. \tag{3.3.3}$$

We denote the function vector

$$\underline{\tilde{\boldsymbol{\psi}}}_j \;:=\; \left(\tilde{\psi}_{j,r}\right)_{r=0}^{2N_j-1}\,.$$

Theorem 17. *For the dual functions, we have the relations*

$$\underline{\tilde{\boldsymbol{\psi}}}_j \;=\; \boldsymbol{H}_j^{-1}\,\underline{\boldsymbol{\psi}}_j \quad \text{and} \quad \underline{\boldsymbol{\psi}}_j \;=\; \boldsymbol{H}_j\,\underline{\tilde{\boldsymbol{\psi}}}_j\,,$$

with

$$\begin{aligned}\boldsymbol{H}_j \;&=\; \overline{\boldsymbol{F}}_j\,\boldsymbol{E}_j\,\boldsymbol{F}_j \\ &=\; \left(\delta_{r,s} - \frac{1}{2N_j}\Bigg(\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j^2-k^2}{2M_j^2}\cos\tfrac{(N_j+k)(s-r)\pi}{N_j} \right. \\ &\qquad\left. + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}^2-k^2}{2M_{j+1}^2}\cos\tfrac{k(s-r)\pi}{N_j}\Bigg)\right)_{r,s=0}^{2N_j-1}\end{aligned}$$

and

$$\begin{aligned}\boldsymbol{H}_j^{-1} \;&=\; \overline{\boldsymbol{F}}_j\,\boldsymbol{E}_j^{-1}\,\boldsymbol{F}_j \\ &=\; \left(\delta_{r,s} + \frac{1}{2N_j}\Bigg(\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j^2-k^2}{M_j^2+k^2}\cos\tfrac{(N_j+k)(s-r)\pi}{N_j} \right. \\ &\qquad\left. + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}^2-k^2}{M_{j+1}^2+k^2}\cos\tfrac{k(s-r)\pi}{N_j}\Bigg)\right)_{r,s=0}^{2N_j-1}\end{aligned} \tag{3.3.4}$$

where

$$\boldsymbol{E}_j \;=\; \operatorname{diag}\left(e_{j,r}\right)_{r=0}^{2N_j-1}\,, \qquad \boldsymbol{E}_j^{-1} \;=\; \operatorname{diag}\left(1/e_{j,r}\right)_{r=0}^{2N_j-1}\,,$$

the entries of which are known from Lemma 6.

Proof: By (3.3.2) and (3.3.3), we have

$$\left\langle \tilde{\psi}_{j,r}\,, \psi_{j,\ell} \right\rangle \;=\; \sum_{s=0}^{2N_j-1} \beta_{j,r,s} \left\langle \psi_{j,s}\,, \psi_{j,\ell} \right\rangle \;=\; \delta_{r,\ell}\,,$$

which yields

$$\left(\beta_{j,r,s}\right)_{r,s=0}^{2N_j-1} \;=\; \boldsymbol{H}_j^{-1} \;=\; \overline{\boldsymbol{F}}_j\, \boldsymbol{E}_j^{-1}\, \boldsymbol{F}_j\,.$$

Due to the circulant structure of the inverse $\boldsymbol{H}_j^{-1} = \overline{\boldsymbol{F}}_j\, \boldsymbol{E}_j^{-1}\, \boldsymbol{F}_j$, we can compute its elements as

$$\begin{aligned}
\beta_{j,r,s} &= \frac{1}{2N_j} \sum_{k=0}^{2N_j-1} \frac{1}{e_{j,k}}\, e^{\frac{ik(s-r)\pi}{N_j}} \\
&= \frac{1}{2N_j} \left(\sum_{k=0}^{2N_j-1} e^{\frac{ik(s-r)\pi}{N_j}} + \sum_{k=-M_j+1}^{M_j-1} \left(\tfrac{2M_j^2}{M_j^2+k^2} - 1\right) e^{\frac{i(N_j+k)(s-r)\pi}{N_j}} \right. \\
&\qquad \left. + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\tfrac{2M_{j+1}^2}{M_{j+1}^2+k^2} - 1\right) e^{\frac{ik(s-r)\pi}{N_j}} \right) \\
&= \delta_{s,r} + \frac{1}{2N_j} \left(\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j^2-k^2}{M_j^2+k^2} \cos \tfrac{(N_j+k)(s-r)\pi}{N_j} \right. \\
&\qquad \left. + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}^2-k^2}{M_{j+1}^2+k^2} \cos \tfrac{k(s-r)\pi}{N_j} \right)\,.
\end{aligned}$$

Finally, we need the entries of $\boldsymbol{H}_j = \overline{\boldsymbol{F}}_j\, \boldsymbol{E}_j\, \boldsymbol{F}_j$, which are

$$\begin{aligned}
\left\langle \psi_{j,r}\,, \psi_{j,s} \right\rangle &= \frac{1}{2N_j} \sum_{k=0}^{2N_j-1} e_{j,k}\, e^{\frac{ik(s-r)\pi}{N_j}} \\
&= \frac{1}{2N_j} \left(\sum_{k=0}^{2N_j-1} e^{\frac{ik(s-r)\pi}{N_j}} + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{k^2-M_{j+1}^2}{2M_{j+1}^2}\, e^{\frac{ik(s-r)\pi}{N_j}} \right. \\
&\qquad \left. + \sum_{k=-M_j+1}^{M_j-1} \tfrac{k^2-M_j^2}{2M_j^2}\, e^{\frac{i(N_j+k)(s-r)\pi}{N_j}} \right) \\
&= \delta_{s,r} + \frac{1}{2N_j} \left(\sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{k^2-M_{j+1}^2}{2M_{j+1}^2} \cos \tfrac{k(s-r)\pi}{N_j} \right. \\
&\qquad \left. + \sum_{k=-M_j+1}^{M_j-1} \tfrac{k^2-M_j^2}{2M_j^2} \cos \tfrac{(N_j+k)(s-r)\pi}{N_j} \right)\,. \quad \blacksquare
\end{aligned}$$

A. A. Privalov computed the entries of $\boldsymbol{H}_j$ as follows.

Proposition 3. (see [18]) *If* $M_{j+1} = 2M_j$; *i.e.* $j \geq \lambda$, *and if* $3M_j \leq N_j$, *then*

$$\langle \psi_{j,r} \, , \psi_{j,r} \rangle \quad = \quad 1 - \frac{M_j}{N_j} + \frac{1}{8N_j M_j}$$

and, for $r \neq s$,

$$\begin{aligned} \langle \psi_{j,r} \, , \psi_{j,s} \rangle \;=\; & \frac{1}{32 N_j M_j^2 \sin^3 \frac{(r-s)\pi}{2N_j}} \\ & \times \left(4M_j \sin \frac{(r-s)\pi}{2N_j} \left(\cos \frac{2M_j(r-s)\pi}{N_j} + 2(-1)^{r-s} \cos \frac{M_j(r-s)\pi}{N_j} \right) \right. \\ & \qquad \left. - \cos \frac{(r-s)\pi}{2N_j} \left(\sin \frac{2M_j(r-s)\pi}{N_j} + 4(-1)^{r-s} \sin \frac{M_j(r-s)\pi}{N_j} \right) \right) . \end{aligned}$$

For completeness, we deduce the representation of the dual translates in terms of the frequencies.

Theorem 18. *We can transform*

$$\underline{\tilde{\boldsymbol{\psi}}}_j \; = \; \boldsymbol{O}_j \; \boldsymbol{E}_j^{-1} \; \underline{\boldsymbol{\sigma}}_j \qquad \text{and} \qquad \underline{\boldsymbol{\sigma}}_j \; = \; \boldsymbol{E}_j \; \boldsymbol{O}_j^{-1} \; \underline{\tilde{\boldsymbol{\psi}}}_j \, ,$$

with $\boldsymbol{O}_j$, $\boldsymbol{O}_j^{-1}$ *from Theorem* 16 *and* $\boldsymbol{E}_j$, $\boldsymbol{E}_j^{-1}$ *as given in Theorem* 17*.*

Proof: First, we compute the dual wavelets from (3.3.3),

$$\begin{aligned} & \tilde{\psi}_{j,r}(x) \\ & = \psi_{j,r}(x) + \frac{1}{2N_j} \left(\sum_{k=-M_j+1}^{M_j-1} \left(\tfrac{2M_j^2}{M_j^2+k^2} - 1 \right) \sum_{s=0}^{2N_j-1} \cos \tfrac{(N_j+k)(s-r)\pi}{N_j} \;\, \psi_{j,s}(x) \right. \\ & \qquad \left. + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\tfrac{2M_{j+1}^2}{M_{j+1}^2+k^2} - 1 \right) \sum_{s=0}^{2N_j-1} \cos \tfrac{k(s-r)\pi}{N_j} \;\, \psi_{j,s}(x) \right) \end{aligned}$$

$$\begin{aligned}
= \; & \frac{1}{\sqrt{2N_j}} \Bigg(\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{M_j} \cos(N_j+k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
& \quad + 2 \sum_{\ell=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos \ell \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
& \quad + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}} \cos(N_{j+1}+k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \Bigg) \\
& + \frac{1}{\sqrt{2N_j}} \Bigg(\sum_{k=-M_j+1}^{M_j-1} \left(\tfrac{2M_j^2}{M_j^2+k^2} - 1\right) \left(\tfrac{M_j+k}{2M_j} \cos(N_j+k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \right. \\
& \qquad \left. + \tfrac{M_j-k}{2M_j} \cos(N_j-k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right)\right) \\
& \quad + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\tfrac{2M_{j+1}^2}{M_{j+1}^2+k^2} - 1\right) \left(\tfrac{M_{j+1}-k}{2M_{j+1}} \cos(N_{j+1}+k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \right. \\
& \qquad \left. + \tfrac{M_{j+1}+k}{2M_{j+1}} \cos(N_{j+1}-k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right)\right) \Bigg),
\end{aligned}$$

and therefore

$$\begin{aligned}
\tilde{\psi}_{j,r}(x) \; = \; & \frac{1}{\sqrt{2N_j}} \Bigg(2M_j \sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{M_j^2+k^2} \cos(N_j+k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
& + 2 \sum_{\ell=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos \ell \left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \\
& + 2M_{j+1} \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}^2+k^2} \cos(N_{j+1}+k)\left(x - \tfrac{(2r+1)\pi}{2N_j}\right) \Bigg).
\end{aligned}$$

Therefrom the transformation matrices follow easily. ■

Note that $\tilde{\psi}_{j,r}(x) = \tilde{\psi}_{j,0}(x - \frac{r\pi}{N_j})$, for $r = 0, \ldots, 2N_j - 1$, are translates of one function, again. Moreover, since $V_j \perp W_j$, we have

$$\left\langle \tilde{\phi}_{j,r}\,, \psi_{j,s} \right\rangle \;=\; \left\langle \tilde{\psi}_{j,r}\,, \phi_{j,s} \right\rangle \;=\; 0\,,$$

for all $r, s = 0, \ldots, 2N_j - 1$ and $j \in \mathbb{N}_0$.

3.4 Orthonormal translates

In order to determine wavelets with orthonormal translates, we proceed in the same way as for the scaling functions; *i.e.* via the Gram matrix. We need orthonormal translates $\mathcal{O}\psi_{j,r}(x) := \mathcal{O}\psi_j\left(x - \frac{r\pi}{N_j}\right)$ of a function $\mathcal{O}\psi_j \in W_j$; *i.e.* coefficients $\nu_{j,r,s}$, such that

$$\mathcal{O}\psi_{j,r}(x) = \sum_{s=0}^{2N_j-1} \nu_{j,r,s}\ \psi_{j,s}(x)$$

and

$$\langle \mathcal{O}\psi_{j,r}\, , \mathcal{O}\psi_{j,k} \rangle = \delta_{r,k}\ , \tag{3.4.1}$$

for all $r, k = 0, \ldots, 2N_j - 1$. As in Section 2.2, we use the matrix notation and introduce the coefficient matrix $\boldsymbol{\mathcal{V}}_j := (\nu_{j,r,s})_{r,s=0}^{2N_j-1}$ and the function vector

$$\underline{\mathcal{O}\psi}_j := (\mathcal{O}\psi_{j,r})_{r=0}^{2N_j-1}\ .$$

Theorem 19. *There exists an orthonormal wavelet basis of* W_j *with the relations*

$$\underline{\mathcal{O}\psi}_j = \boldsymbol{\mathcal{V}}_j\ \underline{\psi}_j\ , \qquad \underline{\psi}_j = \boldsymbol{\mathcal{V}}_j^{-1}\ \underline{\mathcal{O}\psi}_j\ , \tag{3.4.2}$$

$$\underline{\mathcal{O}\psi}_j = \boldsymbol{\mathcal{V}}_j^{-1}\ \underline{\tilde{\psi}}_j \quad \text{and} \quad \underline{\tilde{\psi}}_j = \boldsymbol{\mathcal{V}}_j\ \underline{\mathcal{O}\psi}_j\ , \tag{3.4.3}$$

where

$$\begin{aligned}\boldsymbol{\mathcal{V}}_j &= \overline{\boldsymbol{F}}_j\ \boldsymbol{\Lambda}_j\ \boldsymbol{F}_j \\ &= \Bigg(\delta_{r,s} + \frac{1}{2N_j} \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\frac{2M_{j+1}}{\sqrt{2M_{j+1}^2+2k^2}} - 1\right) \cos\frac{k(s-r)\pi}{N_j} \\ &\qquad + \frac{1}{2N_j} \sum_{k=-M_j+1}^{M_j-1} \left(\frac{2M_j}{\sqrt{2M_j^2+2k^2}} - 1\right) \cos\frac{(N_j+k)(s-r)\pi}{N_j}\Bigg)_{r,s=0}^{2N_j-1}\end{aligned} \tag{3.4.4}$$

$$\begin{aligned}\boldsymbol{\mathcal{V}}_j^{-1} &= \overline{\boldsymbol{F}}_j\ \boldsymbol{\Lambda}_j^{-1}\ \boldsymbol{F}_j \\ &= \Bigg(\delta_{r,s} + \frac{1}{2N_j} \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\frac{\sqrt{2M_{j+1}^2+2k^2}}{2M_{j+1}} - 1\right) \cos\frac{k(s-r)\pi}{N_j} \\ &\qquad + \frac{(-1)^{s-r}}{2N_j} \sum_{k=-M_j+1}^{M_j-1} \left(\frac{\sqrt{2M_j^2+2k^2}}{2M_j} - 1\right) \cos\frac{k(s-r)\pi}{N_j}\Bigg)_{r,s=0}^{2N_j-1}\end{aligned} \tag{3.4.5}$$

with

$$\boldsymbol{\Lambda}_j \;:=\; \operatorname{diag}\left(1/\sqrt{e_{j,\ell}}\right)_{\ell=0}^{2N_j-1} \quad \text{and} \quad \boldsymbol{\Lambda}_j^{-1} \;=\; \operatorname{diag}\left(\sqrt{e_{j,\ell}}\right)_{\ell=0}^{2N_j-1} ,$$

the entries of which are given in Lemma 6.

Proof: Here we follow the same line as in the proof of Theorem 5. At first, we show (3.4.4) and (3.4.5). With $\boldsymbol{\mathcal{V}}_j \;=\; \overline{\boldsymbol{F}}_j\,\boldsymbol{\Lambda}_j\,\boldsymbol{F}_j \;=\; (\nu_{j,r,s})_{r,s=0}^{2N_j-1}$, we compute

$$\begin{aligned}
\nu_{j,r,s} &= \frac{1}{2N_j}\sum_{k=0}^{2N_j-1}\frac{1}{\sqrt{e_{j,k}}}\,e^{\frac{ik(s-r)\pi}{N_j}} \\
&= \frac{1}{2N_j}\left(\sum_{k=0}^{2N_j-1} e^{\frac{ik(s-r)\pi}{N_j}} + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1}\left(\frac{\sqrt{2M_{j+1}^2}}{\sqrt{M_{j+1}^2+k^2}}-1\right)e^{\frac{ik(s-r)\pi}{N_j}}\right. \\
&\qquad\left. + \sum_{k=-M_j+1}^{M_j-1}\left(\frac{\sqrt{2M_j^2}}{\sqrt{M_j^2+k^2}}-1\right)e^{\frac{i(N_j+k)(s-r)\pi}{N_j}}\right) \\
&= \delta_{s,r} + \frac{1}{2N_j}\left(\sum_{k=-M_{j+1}+1}^{M_{j+1}-1}\left(\frac{2M_{j+1}}{\sqrt{2M_{j+1}^2+2k^2}}-1\right)\cos\frac{k(s-r)\pi}{N_j}\right. \\
&\qquad\left. + \sum_{k=-M_j+1}^{M_j-1}\left(\frac{2M_j}{\sqrt{2M_j^2+2k^2}}-1\right)\cos\frac{(N_j+k)(s-r)\pi}{N_j}\right) .
\end{aligned}$$

Then, for the entries of

$$\boldsymbol{\mathcal{V}}_j^{-1} \;=\; \overline{\boldsymbol{F}}_j\,\boldsymbol{\Lambda}_j^{-1}\,\boldsymbol{F}_j \;=:\; (\breve{\nu}_{j,s,r})_{s,r=0}^{2N_j-1} ,$$

we have

$$\begin{aligned}
\breve{\nu}_{j,s,r} &= \frac{1}{2N_j}\sum_{k=0}^{2N_j-1}\sqrt{e_{j,k}}\,e^{\frac{ik(s-r)\pi}{N_j}} \\
&= \frac{1}{2N_j}\left(\sum_{k=0}^{2N_j-1} e^{\frac{ik(s-r)\pi}{N_j}} + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1}\left(\frac{\sqrt{M_{j+1}^2+k^2}}{\sqrt{2M_{j+1}^2}}-1\right)e^{\frac{ik(s-r)\pi}{N_j}}\right. \\
&\qquad\left. + \sum_{k=-M_j+1}^{M_j-1}\left(\frac{\sqrt{M_j^2+k^2}}{\sqrt{2M_j^2}}-1\right)e^{\frac{i(N_j+k)(s-r)\pi}{N_j}}\right) \\
&= \delta_{s,r} + \frac{1}{2N_j}\sum_{k=-M_{j+1}+1}^{M_{j+1}-1}\left(\frac{\sqrt{2M_{j+1}^2+2k^2}}{2M_{j+1}}-1\right)\cos\frac{k(s-r)\pi}{N_j} \\
&\qquad + \frac{(-1)^{s-r}}{2N_j}\sum_{k=-M_j+1}^{M_j-1}\left(\frac{\sqrt{2M_j^2+2k^2}}{2M_j}-1\right)\cos\frac{k(s-r)\pi}{N_j} .
\end{aligned}$$

So, $\boldsymbol{\mathcal{V}}_j$ and $\boldsymbol{\mathcal{V}}_j^{-1}$ are symmetric. Hence, t he orthonormality of the translates $\mathcal{O}\psi_{j,r}$, required in (3.4.1), follows easily from

$$\boldsymbol{\mathcal{V}}_j\, \boldsymbol{H}_j\, \boldsymbol{\mathcal{V}}_j^T \;=\; \overline{\boldsymbol{F}}_j\, \boldsymbol{\Lambda}_j\, \boldsymbol{E}_j\, \boldsymbol{\Lambda}_j\, \boldsymbol{F}_j \;=\; \boldsymbol{I}_j\,.$$

This, together with (3.4.2) and Theorem 17, proves (3.4.3), too. ■

Last but not least, we give the representation of the orthonormal translates in the basis of frequencies.

Theorem 20. *We have*

$$\underline{\mathcal{O}\psi}_j \;=\; \boldsymbol{O}_j\, \boldsymbol{\Lambda}_j\, \underline{\boldsymbol{\sigma}}_j \quad \textit{and} \quad \underline{\boldsymbol{\sigma}}_j \;=\; \boldsymbol{\Lambda}_j^{-1}\, \boldsymbol{O}_j^{-1}\, \underline{\mathcal{O}\psi}_j\,, \tag{3.4.6}$$

with $\boldsymbol{O}_j\,, \boldsymbol{O}_j^{-1}$ *from Theorem* 16 *and* $\boldsymbol{\Lambda}_j\,,\ \boldsymbol{\Lambda}_j^{-1}$ *as given in Theorem* 19.

Proof: First, we need an explicit formula for

$$\begin{aligned}
\mathcal{O}\psi_j(x) \;&=\; \sum_{s=0}^{2N_j-1} \nu_{j,0,s}\; \psi_{j,s}(x) \\
&=\; \psi_{j,0}(x) + \frac{1}{2N_j}\left(\sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\frac{2M_{j+1}}{\sqrt{2M_{j+1}^2+2k^2}} - 1 \right) \sum_{s=0}^{2N_j-1} \cos\tfrac{ks\pi}{N_j}\; \psi_{j,s}(x) \right. \\
&\qquad \left. + \sum_{k=-M_j+1}^{M_j-1} \left(\frac{2M_j}{\sqrt{2M_j^2+2k^2}} - 1 \right) \sum_{s=0}^{2N_j-1} \cos\tfrac{(N_j+k)s\pi}{N_j}\; \psi_{j,s}(x) \right).
\end{aligned}$$

Therefore we consider, for $-M_{j+1} < k < M_{j+1}$ and $N_j - M_j < k < N_j + M_j$, the sum

$$\begin{aligned}
\sum_{s=0}^{2N_j-1} \cos\tfrac{ks\pi}{N_j}\, \psi_{j,s}(x) \;&=\; \sqrt{2} \sum_{s=0}^{2N_j-1} \cos\tfrac{ks\pi}{N_j}\, \phi_{j+1,2s}\left(x - \tfrac{\pi}{2N_j}\right) \\
&\qquad - \sum_{s=0}^{2N_j-1} \cos\tfrac{ks\pi}{N_j}\, \phi_{j,s}\left(x - \tfrac{\pi}{2N_j}\right)\,.
\end{aligned}$$

By means of (2.1.5) and (2.1.6), we calculate the first part,

$$
\begin{aligned}
&\sqrt{2}\sum_{s=0}^{2N_j-1} \cos\tfrac{ks\pi}{N_j}\ \phi_{j+1,2s}(x) \\
&= \frac{\sqrt{2}}{\sqrt{2N_{j+1}}}\sum_{s=0}^{2N_j-1}\cos\tfrac{ks\pi}{N_j}\left(1+2\sum_{\ell=1}^{N_{j+1}-M_{j+1}}\cos\ell\left(x-\tfrac{2s\pi}{N_{j+1}}\right)\right.\\
&\qquad\left.+\ 2\sum_{\ell=-M_{j+1}+1}^{M_{j+1}-1}\tfrac{M_{j+1}-\ell}{2M_{j+1}}\cos(N_{j+1}+\ell)\left(x-\tfrac{2s\pi}{N_{j+1}}\right)\right)\\
&= \sqrt{2N_j}\,\delta_{k,0}+\frac{1}{\sqrt{2N_j}}\left(\sum_{\ell=1}^{N_{j+1}-M_{j+1}}\sum_{s=0}^{2N_j-1}\left(\cos\left(\ell x+\tfrac{(k-\ell)s\pi}{N_j}\right)\right.\right.\\
&\quad\left.+\ \cos\left(\ell x-\frac{(k+\ell)s\pi}{N_j}\right)\right)+\sum_{\ell=-M_{j+1}+1}^{M_{j+1}-1}\tfrac{M_{j+1}-\ell}{2M_{j+1}}\\
&\quad\left.\times\sum_{s=0}^{2N_j-1}\left(\cos\left((N_{j+1}+\ell)x+\tfrac{(k-\ell)s\pi}{N_j}\right)+\cos\left((N_{j+1}+\ell)x-\tfrac{(k+\ell)s\pi}{N_j}\right)\right)\right)\\
&= \sqrt{2N_j}\left(\delta_{k,0}+\sum_{\ell=1}^{N_{j+1}-M_{j+1}}\left(\delta_{0,(k-\ell)\bmod 2N_j}+\delta_{0,(k+\ell)\bmod 2N_j}\right)\cos\ell x\right.\\
&\quad\left.+\sum_{\ell=-M_{j+1}+1}^{M_{j+1}-1}\tfrac{M_{j+1}-\ell}{2M_{j+1}}\left(\delta_{0,(k-\ell)\bmod 2N_j}+\delta_{0,(k+\ell)\bmod 2N_j}\right)\cos(2N_j+\ell)x\right)\\
&= \begin{cases}\sqrt{2N_j}\left(\cos kx+\tfrac{M_{j+1}-k}{2M_{j+1}}\cos(2N_j+k)x\right), & \text{if}\quad |k|<M_{j+1},\\ \sqrt{2N_j}\left(\cos kx+\cos(2N_j-k)x\right), & \text{if}\quad |k-N_j|<M_j.\end{cases}
\end{aligned}
$$

The second part is known from Lemma 1. Hence,

$$
\begin{aligned}
&\frac{1}{\sqrt{2N_j}}\sum_{s=0}^{2N_j-1}\cos\frac{ks\pi}{N_j}\ \psi_{j,s}(x)\\
&= \begin{cases}\tfrac{M_{j+1}-k}{2M_{j+1}}\cos(2N_j+k)\left(x-\tfrac{\pi}{N_{j+1}}\right)+\tfrac{M_{j+1}+k}{2M_{j+1}}\cos(2N_j-k)\left(x-\tfrac{\pi}{N_{j+1}}\right),\\ \qquad\qquad \text{if}\quad -M_{j+1}<k<M_{j+1},\\ \tfrac{M_j-(N_j-k)}{2M_j}\cos k\left(x-\tfrac{\pi}{2N_j}\right)+\tfrac{M_j+(N_j-k)}{2M_j}\cos(2N_j-k)\left(x-\tfrac{\pi}{2N_j}\right),\\ \qquad\qquad \text{if}\quad N_j-M_j<k<N_j+M_j.\end{cases}
\end{aligned}
$$

Consequently,

$$\begin{aligned}
\mathcal{O}\psi_j(x) \;=\; & \frac{1}{\sqrt{2N_j}}\left(\sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{M_j}\cos(N_j+k)\left(x-\tfrac{\pi}{2N_j}\right)\right.\\
&+ 2\sum_{\ell=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos\ell\left(x-\tfrac{\pi}{2N_j}\right)\\
&+ \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}}\cos(N_{j+1}+k)\left(x-\tfrac{\pi}{2N_j}\right)\\
&+ \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \left(\tfrac{2M_{j+1}}{\sqrt{2M_{j+1}^2+2k^2}}-1\right)\left(\tfrac{M_{j+1}-k}{2M_{j+1}}\cos(N_{j+1}+k)\left(x-\tfrac{\pi}{2N_j}\right)\right.\\
&\qquad\qquad \left.+\ \tfrac{M_{j+1}+k}{2M_{j+1}}\cos(N_{j+1}-k)\left(x-\tfrac{\pi}{2N_j}\right)\right)\\
&+ \sum_{k=-M_j+1}^{M_j-1} \left(\tfrac{2M_{j+1}}{\sqrt{2M_j^2+2k^2}}-1\right)\left(\tfrac{M_{j+1}-(N_j-k)}{2M_{j+1}}\cos k\left(x-\tfrac{\pi}{2N_j}\right)\right.\\
&\qquad\qquad \left.\left.+\ \tfrac{M_j+(N_j-k)}{2M_j}\cos(2N_j-k)\left(x-\tfrac{\pi}{2N_j}\right)\right)\right),
\end{aligned}$$

and finally,

$$\begin{aligned}
\mathcal{O}\psi_j(x) \;=\; & \frac{1}{\sqrt{2N_j}}\left(\sum_{k=-M_j+1}^{M_j-1} \tfrac{2M_j+2k}{\sqrt{2M_j^2+2k^2}}\cos(N_j+k)\left(x-\tfrac{\pi}{2N_j}\right)\right.\\
&+ 2\sum_{\ell=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos\ell\left(x-\tfrac{\pi}{2N_j}\right)\\
&\left.+ \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{2M_{j+1}-2k}{\sqrt{2M_{j+1}^2+2k^2}}\cos(N_{j+1}+k)\left(x-\tfrac{\pi}{2N_j}\right)\right).
\end{aligned}$$

From this, the matrices in (3.4.6) can be easily derived. ■

3.5 Localization and projection estimates

We present results for the wavelet spaces, which are analogous to the ones of Section 2.4. Here, we can be brief because most of the proofs of that section can be adapted easily.

$L^p_{2\pi}$-Norms and stability

As in (2.4.6) in Section 2.4, we assume N_j/M_j to be bounded from above by a constant. Then the $L^p_{2\pi}$-stability of the wavelets $\psi_j \in W_j$, for $1 \le p \le \infty$, is given by

$$\left\| \sum_{k=0}^{2N_j-1} \alpha_k \, \psi_{j,k} \right\|_p \sim (2N_j)^{\frac{1}{2}-\frac{1}{p}} \, \|\{\alpha_k\}\|_{\ell^p} .$$

This can be proved by applying (3.2.6) and (2.4.4).
For the special case $\alpha_k = \delta_{k,0}$, we obtain immediately the same asymptotics as we have in (2.4.4), namely:

$$\|\psi_{j,0}\|_p \sim (2N_j)^{\frac{1}{2}-\frac{1}{p}}, \qquad \text{for any } 1 \le p \le \infty .$$

In order to determine the asymptotics of the dual and orthonormal wavelets, we follow the same line as in Section 2.4. Thus, we have to estimate the coefficients $\beta_{j,r,s}$ and $\nu_{j,r,s}$.

Lemma 7. *For the entries of the circulant matrices $\boldsymbol{H}_j^{-1}$ and $\boldsymbol{\mathcal{V}}_j$, the inequalities*

$$|\beta_{j,r,s}|, \, |\nu_{j,r,s}| \le C \max\left\{(|r-s|+1)^{-2}, \; (2N_j - |r-s|)^{-2}\right\}$$

hold, for $r, s = 0, \ldots, 2N_j - 1$.

Proof: Again, we have to estimate discrete Fourier coefficients. But this time, with respect to (3.3.4) and (3.4.4), we consider the 2π-periodic functions

$$f_\beta(x) := \begin{cases} \frac{M_{j+1}^2\pi^2 - N_j^2x^2}{M_{j+1}^2\pi^2 + N_j^2x^2} + (-1)^{s-r}\frac{M_j^2\pi^2 - N_j^2x^2}{M_j^2\pi^2 + N_j^2x^2}, & \text{for } |x| \le \frac{\pi M_j}{N_j}, \\ \frac{M_{j+1}^2\pi^2 - N_j^2x^2}{M_{j+1}^2\pi^2 + N_j^2x^2}, & \text{for } \frac{\pi M_j}{N_j} < |x| \le \frac{\pi M_{j+1}}{N_j}, \\ 0, & \text{for } \frac{\pi M_{j+1}}{N_j} < |x| \le \pi, \end{cases}$$

and

$$f_\nu(x) := \begin{cases} \frac{\sqrt{2}\pi M_{j+1}}{\sqrt{\pi^2M_{j+1}^2 + N_j^2x^2}} - 1 + \frac{\sqrt{2}(-1)^{s-r}\pi M_j}{\sqrt{\pi^2M_j^2 + N_j^2x^2}} - (-1)^{s-r}, & \text{for } |x| \le \frac{\pi M_j}{N_j}, \\ \frac{\sqrt{2}\pi M_{j+1}}{\sqrt{\pi^2M_{j+1}^2 + N_j^2x^2}} - 1, & \text{for } \frac{\pi M_j}{N_j} < |x| \le \frac{\pi M_{j+1}}{N_j}, \\ 0, & \text{for } \frac{\pi M_{j+1}}{N_j} < |x| \le \pi, \end{cases}$$

which have a piecewise continuous first derivative. ■

The application of Lemma 7 gives the time-localization result for the dual and orthonormal wavelets.

Theorem 21. *For* $1 \le p \le \infty$,

$$\|\tilde{\psi}_{j,r}\|_p \;\sim\; \|\mathcal{O}\psi_{j,r}\|_p \;\sim\; (2N_j)^{\frac{1}{2}-\frac{1}{p}} .$$

The pointwise estimates, which correspond to (2.4.9) and (2.4.11), can be deduced using (3.2.6).

Theorem 22. *For simplicity, we restrict ourselves to* $0 < k \le N_j$. *Then*

$$\max_{x \in I_{N_j,k}} \left\{ |\psi_{j,0}(x)| \,,\, |\mathcal{O}\psi_{j,0}(x)| \,,\, |\tilde{\psi}_{j,0}(x)| \right\} \;\le\; \frac{C\sqrt{N_j}}{k^2} .$$

Error estimates

In Section 2.4 we investigated the approximation of continuous functions by elements from the sample space V_j . Now we consider the corresponding results for approximation processes in W_j . Of particular interest are the interpolation operator R_j defined in (3.2.9) and the orthogonal projection Q_j that can be easily handled by

$$Q_j \;=\; P_{N_{j+1}}^{M_{j+1}} - P_{N_j}^{M_j} .$$

Here, we only assume (3.1.1) for N_j and M_j , with arbitrary $\lambda \in \mathbb{N}$.

Theorem 23. *For the interpolatory and orthogonal projections, we have*

$$\|R_j\|_{C_{2\pi} \to C_{2\pi}} \;\le\; C\,\log\frac{N_j}{M_j} \quad \textit{and} \quad \|Q_j\|_{C_{2\pi} \to C_{2\pi}} \;\le\; C\,\log^2\frac{N_j}{M_j} .$$

Note that this Theorem is a key to find polynomial bases of $C_{2\pi}$. We discuss this in the paper [17].

§4 Decomposition and reconstruction

The wavelet analysis of functions is based on the transformations between a sufficiently large level sample space and the wavelet spaces of lower levels; *i.e.* the iterative decomposition of V_{j+1} into the orthogonal sum $V_j \oplus W_j$. Starting from an approximation of a given function in an specific sample space, either by interpolation or by orthogonal projection, we only have to know the corresponding basis transformations to calculate the wavelet coefficients of the function. Conversely, we can reconstruct the projection of the function from the wavelet coefficients.

4.1 Interpolating bases

For the interpolatory scaling and wavelet functions, we have already determined the refinement relations in the Theorems 13 and 15. Now we present the reverse basis transformation.

Theorem 24. *The decomposition formulas, for $\ell = 0, \ldots, 2N_j - 1$, are*

$$\phi_{j+1,2\ell}(x) = \frac{1}{\sqrt{2N_{j+1}}}\Bigg(\sum_{s=0}^{2N_j-1} \tilde{\phi}_{j,s}\Big(\frac{\ell\pi}{N_j}\Big)\phi_{j,s}(x) - \sum_{s=0}^{2N_j-1} \tilde{\phi}_{j,\ell}\Big(\frac{(2s+1)\pi}{2N_j}\Big)\psi_{j,s}(x)\Bigg) \tag{4.1.1}$$

and

$$\phi_{j+1,2\ell+1}(x) = \frac{1}{\sqrt{2N_{j+1}}}\Bigg(\sum_{s=0}^{2N_j-1} \tilde{\phi}_{j,s}\Big(\frac{(2\ell+1)\pi}{2N_j}\Big)\phi_{j,s}(x) + \sum_{s=0}^{2N_j-1} \tilde{\phi}_{j,\ell}\Big(\frac{s\pi}{N_j}\Big)\psi_{j,s}(x)\Bigg). \tag{4.1.2}$$

Proof: We only have to calculate the coefficients $a_{j,\ell,s}$ and $b_{j,\ell,s}$, such that

$$\phi_{j+1,\ell}(x) = \sum_{s=0}^{2N_j-1} a_{j,\ell,s}\ \phi_{j,s}(x) + \sum_{s=0}^{2N_j-1} b_{j,\ell,s}\ \psi_{j,s}(x)$$

for $\ell = 0, \ldots, 4N_j - 1$. These coefficients are the inner products

$$\begin{aligned} a_{j,\ell,s} &= \left\langle \phi_{j+1,\ell}, \tilde{\phi}_{j,s} \right\rangle \\ &= \frac{1}{\sqrt{2N_{j+1}}}\left\langle D_{N_{j+1}-M_{j+1}}\left(\circ - \frac{\ell\pi}{N_{j+1}}\right), \tilde{\phi}_{j,s} \right\rangle \\ &= \frac{1}{\sqrt{2N_{j+1}}}\ \tilde{\phi}_{j,s}\left(\frac{\ell\pi}{2N_j}\right) \end{aligned}$$

and

$$
\begin{aligned}
b_{j,\ell,s} \;&=\; \left\langle \phi_{j+1,\ell}\,,\,\tilde{\psi}_{j,s} \right\rangle \\
&= \frac{1}{\sqrt{2}\;N_{j+1}} \Bigg\langle D_{N_{j+1}-M_{j+1}} \left(\circ - \tfrac{\ell\pi}{N_{j+1}} \right) \\
&\qquad + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}} \cos(N_{j+1}+k) \left(\circ - \tfrac{\ell\pi}{N_{j+1}} \right), \\
&\quad 2M_j \sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{M_j^2+k^2} \cos(N_j+k) \left(\circ - \tfrac{(2s+1)\pi}{2N_j} \right) \\
&\qquad + 2 \sum_{k=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos k \left(\circ - \tfrac{(2s+1)\pi}{2N_j} \right) \\
&\qquad + \; 2M_{j+1} \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{M_{j+1}-k}{M_{j+1}^2+k^2} \cos(N_{j+1}+k) \left(\circ - \tfrac{(2s+1)\pi}{2N_j} \right) \Bigg\rangle \\
&= \frac{1}{\sqrt{2}\;N_{j+1}} \Bigg(2M_j \sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j+k}{M_j^2+k^2} \cos \tfrac{(N_j+k)(\ell-1-2s)\pi}{2N_j} \\
&\qquad + 2 \sum_{k=N_j+M_j}^{2N_j-M_{j+1}} \cos \tfrac{k(\ell-1-2s)\pi}{2N_j} \\
&\qquad + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \tfrac{(M_{j+1}-k)^2}{M_{j+1}^2+k^2} \cos \tfrac{(2N_j+k)(\ell-1-2s)\pi}{2N_j} \Bigg) \\
&= \frac{(-1)^{\ell-1}}{\sqrt{2}\;N_{j+1}} \Bigg(1 + 2 \sum_{k=1}^{M_{j+1}} \cos \tfrac{k(\ell-1-2s)\pi}{2N_j} + 2 \sum_{k=M_{j+1}}^{N_j-M_j} \cos \tfrac{k(\ell-1-2s)\pi}{2N_j} \\
&\qquad + 2M_j \sum_{k=-M_j+1}^{M_j-1} \tfrac{M_j-k}{M_j^2+k^2} \cos \tfrac{(N_j+k)(\ell-1-2s)\pi}{2N_j} \Bigg) \\
&= \tfrac{(-1)^{\ell-1}}{\sqrt{2N_{j+1}}}\; \tilde{\phi}_{j,s} \left(\tfrac{(\ell-1)\pi}{2N_j} \right) \;=\; \tfrac{(-1)^{\ell-1}}{\sqrt{2N_{j+1}}}\; \tilde{\phi}_{j,0} \left(\tfrac{(2s-\ell-1)\pi}{2N_j} \right) \\
&= \frac{1}{\sqrt{2N_{j+1}}} \times \begin{cases} \tilde{\phi}_{j,\frac{\ell-1}{2}} \left(\frac{s\pi}{N_j} \right), & \text{for odd } \ell\,, \\ -\,\tilde{\phi}_{j,\frac{\ell}{2}} \left(\frac{(2s+1)\pi}{2N_j} \right), & \text{for even } \ell\,. \quad \blacksquare \end{cases}
\end{aligned}
$$

Motivated by the different structures for translates $\phi_{j+1,\ell}$ with even and odd index ℓ, respectively, we introduce a permutation matrix

$$\boldsymbol{P}_{j+1} := \left(\begin{array}{cccc} 1\,0\,0\,0 & \cdots & 0\,0\,0\,0 \\ 0\,0\,1\,0 & \cdots & 0\,0\,0\,0 \\ \vdots\,\vdots\,\vdots\,\vdots & \ddots & \vdots\,\vdots\,\vdots\,\vdots \\ 0\,0\,0\,0 & \cdots & 1\,0\,0\,0 \\ 0\,0\,0\,0 & \cdots & 0\,0\,1\,0 \\ \hline 0\,1\,0\,0 & \cdots & 0\,0\,0\,0 \\ 0\,0\,0\,1 & \cdots & 0\,0\,0\,0 \\ \vdots\,\vdots\,\vdots\,\vdots & \ddots & \vdots\,\vdots\,\vdots\,\vdots \\ 0\,0\,0\,0 & \cdots & 0\,1\,0\,0 \\ 0\,0\,0\,0 & \cdots & 0\,0\,0\,1 \end{array}\right) = \left(\delta_{2r,s} + \delta_{2r,2N_{j+1}+s-1}\right)_{r,s=0}^{2N_{j+1}-1},$$

that yields

$$\begin{aligned} &\boldsymbol{P}_{j+1}\, \underline{\boldsymbol{\phi}}_{j+1} \\ &= \left(\phi_{j+1,0}, \phi_{j+1,2}, \ldots, \phi_{j+1,4N_j-2}, \phi_{j+1,1}, \phi_{j+1,3}, \ldots, \phi_{j+1,4N_j-1}\right)^T . \end{aligned} \tag{4.1.3}$$

In order to write the basis transformations in matrix-vector-notations, analogous to Sections 2 and 3, we have to investigate the following matrix of function values.

Lemma 8. *The singular matrix*

$$\boldsymbol{K}_j \;:=\; \left(\frac{1}{\sqrt{2N_j}}\,\phi_{j,r}\left(\frac{(2s+1)\pi}{2N_j}\right)\right)_{r,s=0}^{2N_j-1} \;=\; \overline{\boldsymbol{F}}_j\, \boldsymbol{Q}_j\, \boldsymbol{F}_j\,,$$

with the diagonal matrix $\boldsymbol{Q}_j := \operatorname{diag}\left(q_{j,r}\right)_{r=0}^{2N_j-1}$*, has the eigenvalues*

$$q_{j,r} \;=\; e^{-\frac{ir\pi}{2N_j}} \times \begin{cases} 1\,, & \text{if} \quad 0 \le r \le N_j - M_j\,, \\ \frac{N_j - r}{M_j}\,, & \text{if} \quad N_j - M_j < r < N_j + M_j\,, \\ -1\,, & \text{if} \quad N_j + M_j \le r \le 2N_j - 1\,, \end{cases}$$

and the transpose $\boldsymbol{K}_j^T = \overline{\boldsymbol{F}}_j\, \overline{\boldsymbol{Q}}_j\, \boldsymbol{F}_j$ *.*

Proof: From $\boldsymbol{K}_j = \left(\frac{1}{\sqrt{2N_j}}\,\phi_{j,0}\left(\frac{(2(s-r)+1)\pi}{2N_j}\right)\right)_{r,s=0}^{2N_j-1}$, we conclude that the matrix is circulant and can be written as $\boldsymbol{K}_j = \overline{\boldsymbol{F}}_j\, \boldsymbol{Q}_j\, \boldsymbol{F}_j$, where $\boldsymbol{Q}_j$ is a diagonal matrix. The entries of $\boldsymbol{Q}_j$ follow from

$$\begin{aligned} q_{j,r} &= \frac{1}{\sqrt{2N_j}} \sum_{s=0}^{2N_j-1} \phi_{j,0}\left(\tfrac{(2s+1)\pi}{2N_j}\right) e^{\frac{irs\pi}{N_j}} \\ &= \frac{1}{\sqrt{2N_j}} \sum_{s=0}^{2N_j-1} \cos r\tfrac{s\pi}{N_j}\ \phi_{j,s}\left(-\tfrac{\pi}{2N_j}\right) + i \sum_{s=0}^{2N_j-1} \sin r\tfrac{s\pi}{N_j}\ \phi_{j,s}\left(-\tfrac{\pi}{2N_j}\right) \\ &= L_j \cos(r\,\circ)\left(-\tfrac{\pi}{2N_j}\right) + i\, L_j \sin(r\,\circ)\left(-\tfrac{\pi}{2N_j}\right) . \end{aligned}$$

and the application of Lemma 1. Since $\boldsymbol{K}_j$ is real-valued,

$$\boldsymbol{K}_j^T = \overline{\boldsymbol{K}_j^T} = \overline{\boldsymbol{F}}_j\, \overline{\boldsymbol{Q}}_j\, \boldsymbol{F}_j . \quad \blacksquare$$

Now we can formulate the decomposition and reconstruction in matrix notation to be used in numerical algorithms.

Theorem 25. *Let $\boldsymbol{A}_{j+1}$ and $\boldsymbol{B}_{j+1}$ be the transformation matrices in the decomposition and the reconstruction equations such that*

$$\underline{\boldsymbol{\phi}}_{j+1} = \boldsymbol{P}_{j+1}^T\, \boldsymbol{B}_{j+1} \begin{pmatrix} \underline{\boldsymbol{\phi}}_j \\ \underline{\boldsymbol{\psi}}_j \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \underline{\boldsymbol{\phi}}_j \\ \underline{\boldsymbol{\psi}}_j \end{pmatrix} = \boldsymbol{A}_{j+1}\, \boldsymbol{P}_{j+1}\, \underline{\boldsymbol{\phi}}_{j+1} .$$

Then $\boldsymbol{A}_{j+1}$ and $\boldsymbol{B}_{j+1}$ have circulant blocks, namely:

$$\boldsymbol{A}_{j+1} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \boldsymbol{I}_j & \overline{\boldsymbol{F}}_j\, \boldsymbol{Q}_j\, \boldsymbol{F}_j \\ \hline -\, \overline{\boldsymbol{F}}_j\, \overline{\boldsymbol{Q}}_j\, \boldsymbol{F}_j & \boldsymbol{I}_j \end{array}\right) ,$$

$$\boldsymbol{B}_{j+1} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \overline{\boldsymbol{F}}_j\, \boldsymbol{D}_j^{-1}\, \boldsymbol{F}_j & -\, \overline{\boldsymbol{F}}_j\, \boldsymbol{D}_j^{-1}\, \boldsymbol{Q}_j\, \boldsymbol{F}_j \\ \hline \overline{\boldsymbol{F}}_j\, \boldsymbol{D}_j^{-1}\, \overline{\boldsymbol{Q}}_j\, \boldsymbol{F}_j & \overline{\boldsymbol{F}}_j\, \boldsymbol{D}_j^{-1}\, \boldsymbol{F}_j \end{array}\right) ,$$

with $\boldsymbol{Q}_j$ known from Lemma 8 and $\boldsymbol{D}_j = \boldsymbol{D}_{N_j}^{M_j}$ as given in Lemma 2.

Proof: From the relations (3.1.3) and (3.2.7), together with (4.1.3), we derive the matrix

$$\begin{aligned} \boldsymbol{A}_{j+1} &= \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \left(\frac{1}{\sqrt{2N_j}}\ \phi_{j,r}\left(\frac{s\pi}{N_j}\right)\right)_{r,s=0}^{2N_j-1} & \left(\frac{1}{\sqrt{2N_j}}\ \phi_{j,r}\left(\frac{(2s+1)\pi}{2N_j}\right)\right)_{r,s=0}^{2N_j-1} \\ \hline -\left(\frac{1}{\sqrt{2N_j}}\ \phi_{j,s}\left(\frac{(2r+1)\pi}{2N_j}\right)\right)_{r,s=0}^{2N_j-1} & \left(\frac{1}{\sqrt{2N_j}}\ \phi_{j,r}\left(\frac{s\pi}{2N_j}\right)\right)_{r,s=0}^{2N_j-1} \end{array}\right) \\ &= \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \boldsymbol{I}_j & \boldsymbol{K}_j \\ \hline -\, \boldsymbol{K}_j^T & \boldsymbol{I}_j \end{array}\right) , \end{aligned}$$

where $\boldsymbol{K}_j$ is the matrix described in Lemma 8. Analogously, we deduce from (4.1.1) and (4.1.2) that

$$\boldsymbol{B}_{j+1} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \left(\frac{1}{\sqrt{2N_j}} \, \tilde{\phi}_{j,r} \left(\frac{s\pi}{N_j} \right) \right)_{r,s=0}^{2N_j-1} & - \left(\frac{1}{\sqrt{2N_j}} \, \tilde{\phi}_{j,r} \left(\frac{(2s+1)\pi}{2N_j} \right) \right)_{r,s=0}^{2N_j-1} \\ \hline \left(\frac{1}{\sqrt{2N_j}} \, \tilde{\phi}_{j,s} \left(\frac{(2r+1)\pi}{2N_j} \right) \right)_{r,s=0}^{2N_j-1} & \left(\frac{1}{\sqrt{2N_j}} \, \tilde{\phi}_{j,r} \left(\frac{s\pi}{2N_j} \right) \right)_{r,s=0}^{2N_j-1} \end{array} \right)$$

$$= \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \boldsymbol{G}_j^{-1} & - \boldsymbol{G}_j^{-1} \, \boldsymbol{K}_j \\ \hline \left(\boldsymbol{G}_j^{-1} \, \boldsymbol{K}_j \right)^T & \boldsymbol{G}_j^{-1} \end{array} \right) ,$$

using (2.2.6) and Theorem 3. ■

4.2 Orthonormal bases

For orthonormal bases, the transformation matrices are unitary. The decomposition and reconstruction relations for the orthonormal translates are given, for $\ell = 0, \ldots, 4N_j - 1$, by

$$\mathcal{O}\phi_{j+1,\ell} = \sum_{r=0}^{2N_j-1} \langle \mathcal{O}\phi_{j+1,\ell} \, , \mathcal{O}\phi_{j,r} \rangle \, \mathcal{O}\phi_{j,r} + \sum_{r=0}^{2N_j-1} \langle \mathcal{O}\phi_{j+1,\ell} \, , \mathcal{O}\psi_{j,r} \rangle \, \mathcal{O}\psi_{j,r} \, , \qquad (4.2.1)$$

from which we deduce the transformation matrices.

Theorem 26. *For the matrices* $\mathcal{O}\boldsymbol{A}_{j+1}$ *and* $\mathcal{O}\boldsymbol{B}_{j+1}$ *in the relations*

$$\underline{\mathcal{O}\phi}_{j+1} \;=\; \boldsymbol{P}_{j+1}^T \; \mathcal{O}\boldsymbol{B}_{j+1} \left(\begin{array}{c} \underline{\mathcal{O}\phi}_j \\ \underline{\mathcal{O}\psi}_j \end{array} \right) \qquad (4.2.2)$$

and

$$\left(\begin{array}{c} \underline{\mathcal{O}\phi}_j \\ \underline{\mathcal{O}\psi}_j \end{array} \right) \;=\; \mathcal{O}\boldsymbol{A}_{j+1} \; \boldsymbol{P}_{j+1} \; \underline{\mathcal{O}\phi}_{j+1} \, , \qquad (4.2.3)$$

we have

$$\mathcal{O}\boldsymbol{A}_{j+1} \;=\; \mathcal{O}\boldsymbol{B}_{j+1}^T \;=\; \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \overline{\boldsymbol{F}}_j \; \boldsymbol{\Delta}_j \; \boldsymbol{F}_j & \overline{\boldsymbol{F}}_j \; \boldsymbol{\Delta}_j \; \boldsymbol{Q}_j \; \boldsymbol{F}_j \\ \hline -\boldsymbol{F}_j \; \boldsymbol{\Delta}_j \; \boldsymbol{Q}_j \; \overline{\boldsymbol{F}}_j & \overline{\boldsymbol{F}}_j \; \boldsymbol{\Delta}_j \; \boldsymbol{F}_j \end{array} \right) ,$$

with $\boldsymbol{\Delta}_j = \boldsymbol{\Delta}_{N_j}^{M_j}$ *as given in Theorem* 5 *and* $\boldsymbol{Q}_j$ *from Lemma* 8.

Proof: Analogous to the proof of Theorem 25, we distinguish between the translates $\mathcal{O}\phi_{j+1,\ell}$ with even and odd index ℓ. Then formula (4.2.2) is relation (4.2.1) in matrix notation, with

$$\mathcal{OB}_{j+1} = \left(\begin{array}{c|c} \left(\langle \mathcal{O}\phi_{j+1,2s}\,,\mathcal{O}\phi_{j,r}\rangle\right)_{s,r=0}^{2N_j-1} & -\left(\langle \mathcal{O}\phi_{j+1,2s}\,,\mathcal{O}\psi_{j,r}\rangle\right)_{s,r=0}^{2N_j-1} \\ \hline \left(\langle \mathcal{O}\phi_{j+1,2s+1}\,,\mathcal{O}\phi_{j,r}\rangle\right)_{s,r=0}^{2N_j-1} & \left(\langle \mathcal{O}\phi_{j+1,2s+1}\,,\mathcal{O}\psi_{j,r}\rangle\right)_{s,r=0}^{2N_j-1} \end{array}\right) .$$

Similarly, we obtain (4.2.3) with $\mathcal{OA}_{j+1} = \mathcal{OB}_{j+1}^T$.

We compute, for $\ell = 0, \dots, 4N_j - 1$ and $r = 0, \dots, 2N_j - 1$,

$$\begin{aligned} &\langle \mathcal{O}\phi_{j+1,\ell}\,,\mathcal{O}\phi_{j,r}\rangle \\ &= \frac{1}{2\sqrt{N_jN_{j+1}}}\Big\langle D_{N_{j+1}-M_{j+1}}\left(\circ - \tfrac{\ell\pi}{N_{j+1}}\right), \\ &\qquad D_{N_j-M_j}\left(\circ - \tfrac{r\pi}{N_j}\right) + \sum_{k=-M_j+1}^{M_j-1} \frac{2M_j-2k}{\sqrt{2M_j^2+2k^2}}\cos(N_j+k)\left(\circ - \tfrac{r\pi}{N_j}\right)\Big\rangle \\ &= \frac{1}{2\sqrt{N_jN_{j+1}}}\, D_{N_j-M_j}\left(\tfrac{(\ell-2r)\pi}{2N_j}\right) + 2\sum_{k=-M_j+1}^{M_j-1}\frac{M_j-k}{\sqrt{2M_j^2+2k^2}}\cos\frac{(N_j+k)(\ell-2r)\pi}{2N_j} \\ &= \frac{1}{\sqrt{2N_{j+1}}}\,\mathcal{O}\phi_{j,r}\left(\tfrac{\ell\pi}{2N_j}\right) \end{aligned}$$

and

$$\begin{aligned} \langle \mathcal{O}\phi_{j+1,\ell}\,,\mathcal{O}\psi_{j,r}\rangle = &\frac{1}{2\sqrt{N_jN_{j+1}}}\Big\langle D_{N_{j+1}-M_{j+1}}\left(\circ - \tfrac{\ell\pi}{N_{j+1}}\right) \\ &+ \sum_{k=-M_{j+1}+1}^{M_{j+1}-1}\frac{2M_{j+1}-2k}{\sqrt{2M_{j+1}^2+2k^2}}\cos(N_{j+1}+k)\left(\circ - \tfrac{\ell\pi}{N_{j+1}}\right), \\ &\sum_{k=-M_j+1}^{M_j-1}\frac{2M_j+2k}{\sqrt{2M_j^2+2k^2}}\cos(N_j+k)\left(\circ - \tfrac{(2r+1)\pi}{2N_j}\right) \\ &+ 2\sum_{k=N_j+M_j}^{N_{j+1}-M_{j+1}}\cos k\left(\circ - \tfrac{(2r+1)\pi}{2N_j}\right) \\ &+ \sum_{k=-M_{j+1}+1}^{M_{j+1}-1}\frac{2M_{j+1}-2k}{\sqrt{2M_{j+1}^2+2k^2}}\cos(N_{j+1}+k)\left(\circ - \tfrac{(2r+1)\pi}{2N_j}\right)\Big\rangle \end{aligned}$$

$$= \frac{1}{2\sqrt{N_j N_{j+1}}} \left(\sum_{k=-M_j+1}^{M_j-1} \frac{2M_j+2k}{\sqrt{2M_j^2+2k^2}} \cos \frac{(N_j+k)(\ell-2r-1)\pi}{2N_j} \right.$$

$$+ 2 \sum_{k=N_j+M_j}^{N_{j+1}-M_{j+1}} \cos \frac{k(\ell-2r-1)\pi}{2N_j}$$

$$\left. + \sum_{k=-M_{j+1}+1}^{M_{j+1}-1} \frac{(M_{j+1}-k)^2}{M_{j+1}^2+k^2} \cos \frac{(N_{j+1}+k)(\ell-2r-1)\pi}{2N_j} \right)$$

$$= \frac{1}{\sqrt{2N_{j+1}}} \times \begin{cases} \mathcal{O}\phi_{j,r}\left(\frac{\ell\pi}{2N_j}\right), & \text{for odd } \ell\,, \\ -\,\mathcal{O}\phi_{j,\frac{\ell}{2}}\left(\frac{(2r+1)\pi}{2N_j}\right), & \text{for even } \ell\,. \end{cases}$$

Hence, from Theorem 5 and Theorem 25 we conclude that

$$\mathcal{O}\boldsymbol{A}_{j+1} = \frac{1}{\sqrt{2}} \left(\begin{array}{c|c} \boldsymbol{\Gamma}_j & \boldsymbol{\Gamma}_j \ \boldsymbol{K}_j \\ \hline -(\boldsymbol{\Gamma}_j \ \boldsymbol{K}_j)^T & \boldsymbol{\Gamma}_j \end{array} \right),$$

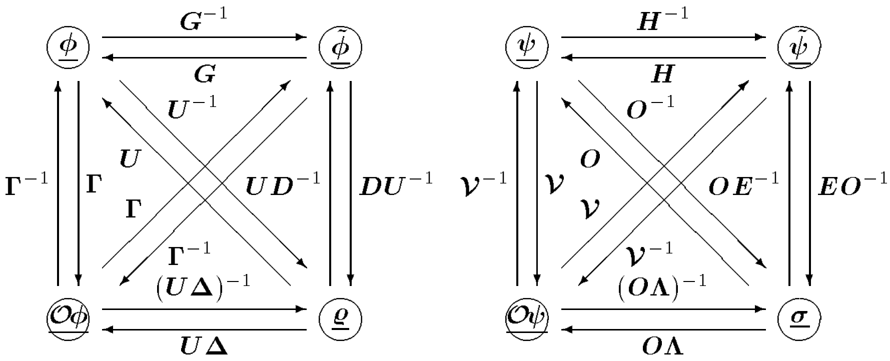
which finally proves the desired form of the transformation matrices. ■

§5 Conclusion

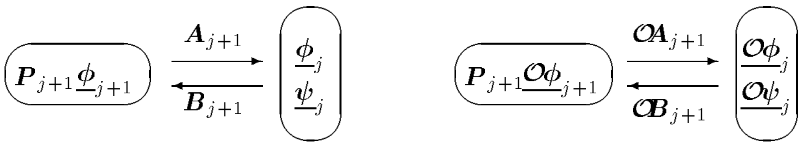
In this paper we have studied nested subspaces $V_j = V_{N_j}^{M_j}$ of trigonometric polynomials constructed from translates of de la Vallée Poussin kernels $\varphi_{N_j}^{M_j}$ based on averaging over $2M_j$ Dirichlet kernels. Moreover, we considered orthogonal polynomial wavelet spaces $W_j = V_{j+1} \ominus V_j$ and their biorthogonal wavelet bases. The greater the number M_j for fixed N_j, the better time-localized those basis functions behave, as it was investigated in Sections 2.4 and 3.5. On the other hand, the best frequency-localized translates as well as the best frequency splittings between the wavelet spaces W_j are obtained for $M_j = 1$; *i.e.* $\lambda = \infty$, which is the original Fourier case at an even number of nodes.

Our considerations have been focused not only on the interpolatory bases and their dual bases, but also on two orthogonal bases in each space. Among all orthogonal bases functions in V_j and W_j, respectively, $\mathcal{O}\phi_{j,r}$ and $\mathcal{O}\psi_{j,s}$ are the most time-localized (and translation invariant) bases functions, and $\rho_{j,k}$ and $\sigma_{j,\ell}$ are the most frequency-localized ones. The main part of the paper has been concerned with the algorithms for the basis transformations being described by their transformation matrices with respect to fast numerical implementations. For the circulant matrices in *i)* and *ii)* below, the knowledge of the eigenvalues reduces the computations to the application of the FFT. In *ii)*, we also need to permute the vector entries.

Let us summarize the relationship between the bases by the relevant matrices in the following schemes
i) for the sample spaces V_N^M or V_j and the wavelet spaces W_j on one level:



ii) for the decomposition and reconstruction $V_{j+1} = V_j \oplus W_j$:



Error estimates for the initial approximation of a given function by the interpolatory or the orthogonal projection onto a sample space of a sufficiently large level are included in Section 2.4.

Let us end with the remark that $\phi_{j,r}(t) + \phi_{j,-r}(t)$ and $\psi_{j,r}(t) + \psi_{j,-r-1}(t)$ are even functions. Therefore, this approach can be transformed by $x = \cos t$, for $t \in [0, \pi]$, to the algebraic case which yields polynomial wavelet spaces on $[-1, 1]$ orthogonal with respect to the Chebyshev weight $(1 - x^2)^{-1/2}$. For this see [5], [11] and [21].

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