# Time Frequency Localization
# of Trigonometric Hermite Operators

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**Abstract.** In this paper, localization properties of trigonometric polynomial Hermite operators are discussed. In particular, time frequency uncertainty and operator norms are compared for the different types of fundamental interpolants which serve as scaling functions for a trigonometric multiresolution analysis.

## §1. Introduction

Recently, several different approaches to periodic multiresolution analyses have been presented. For example, periodic scaling functions and wavelets are discussed by Narcowich and Ward [3] who investigate their time frequency behaviour in terms of an uncertainty principle for periodic functions due to Breitenberger [1]. The periodic basis functions in [3] possess an uncertainty product of $\mathcal{O}(\sqrt{n})$ for increasing dimension $n$ of the corresponding spaces. On the other hand, uniformly bounded uncertainty products are computed by Selig [6] for trigonometric fundamental Lagrange interpolants based on de la Vallée Poussin means. A multiresolution analysis generated by two different types of scaling functions and wavelets, namely fundamental trigonometric Hermite interpolants, is investigated in [5]. One of these scaling functions is a Fejér kernel, while the other is a conjugate Dirichlet kernel. In this paper, we explain the different behaviour of these two types of functions with respect to their localization properties. This includes the computation of the time frequency uncertainty as well as the operator norms. Let us remark here that similar differences also occur for the de la Vallée Poussin and Fourier type Lagrange interpolants which are described in [4].

Approximation Theory VIII
Charles K. Chui and Larry L. Schumaker (eds.), pp. 1–8.


ISBN 0-12-xxxxxx-x

Let $L^p_{2\pi}$ denote the space of real-valued p-integrable $2\pi$-periodic functions for $1 \le p < \infty$, and $C_{2\pi}$ the space of real-valued continuous functions, with their respective norms $\|f\|_p = \left(\frac{1}{2\pi}\int_0^{2\pi} |f(x)|^p\,dx\right)^{1/p}$ and $\|f\|_C = \max_x |f(x)|$. In particular, the $L^2_{2\pi}$-norm is induced by the inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g(x)}\,dx$ for $f, g \in L^2_{2\pi}$.

The following functions are well-known and have been studied in detail, *e.g.*, in [8]. For any $n \in \mathbb{N}$, consider two different kinds of kernels

$$\phi^0_{n,0}(x) := \frac{1}{2n} + \sum_{k=1}^{2n-1} \frac{2n-k}{2n^2}\cos kx = \begin{cases} \frac{1}{4n^2}\frac{\sin^2(nx)}{\sin^2(\frac{x}{2})} & \text{for } x \notin 2\pi\mathbf{Z}, \\ 1 & \text{for } x \in 2\pi\mathbf{Z}, \end{cases}$$

and

$$\phi^1_{n,0}(x) := \frac{1}{2n^2}\left(\sum_{k=1}^{2n-1} \sin kx + \frac{1}{2}\sin 2nx\right) = \phi^0_{n,0}(x)\sin x.$$

For $k = 0, \ldots, 2n-1$, define equally spaced translates by $\phi^0_{n,k}(x) := \phi^0_{n,0}(x - k\pi/n)$ and, analogously, $\phi^1_{n,k}(x) := \phi^1_{n,0}(x - k\pi/n)$.

These functions are fundamental Hermite interpolants, *i.e.*, for each $k, s = 0, \ldots, 2n-1$, it holds that

$$\phi^0_{n,k}(\frac{s\pi}{n}) = \delta_{k,s} \quad \text{and} \quad {\phi^0_{n,k}}'(\frac{s\pi}{n}) = 0$$

and

$$\phi^1_{n,k}(\frac{s\pi}{n}) = 0 \quad \text{and} \quad {\phi^1_{n,k}}'(\frac{s\pi}{n}) = \delta_{k,s}.$$

In particular, $\phi^0_{n,k}$ and $\phi^1_{n,k}$ were used as scaling functions in the Hermite multiresolution analysis studied in [5].

For $n \in \mathbb{N}$, the spaces $V^r_n$ are defined for $r = 0, 1$ as

$$V^r_n := \operatorname{span}\{\phi^r_{n,k} : k = 0, \ldots, 2n-1\}.$$

It is well known (see [5,8]) that

$$V_n = V^0_n + V^1_n = \operatorname{span}\{\cos rx, \sin(r+1)x : r = 0, \ldots, 2n-1\}.$$

Thus $V_n \subset V_{n+1}$ and the spaces $V_n$ form a polynomial multiresolution analysis of $L^2_{2\pi}$.

As shown in [5], the corresponding wavelet spaces can be described as

$$W_n = W^0_n + W^1_n,$$

where for $r = 0, 1$

$$W_n^r := \text{span}\{\psi_{n,k}^r : k = 0, \dots, 2n-1\},$$

$$\psi_{n,0}^0(x) = \frac{1}{2n}\cos 2nx + \frac{1}{6n^2}\sum_{\ell=2n+1}^{4n-1}(6n-\ell)\cos \ell x$$

and

$$\psi_{n,0}^1(x) = \frac{1}{6n^2}\sum_{\ell=2n+1}^{4n-1}\sin \ell x + \frac{1}{8n^2}\sin 4nx.$$

As for the scaling functions, let $\psi_{n,k}^0(x) := \psi_{n,0}^0(x - k\pi/n)$ and $\psi_{n,k}^1(x) := \psi_{n,0}^1(x - k\pi/n)$ for $k = 0, \dots, 2n-1$.

These wavelets show the same interpolatory properties as the scaling functions, namely for each $k, s = 0, \dots, 2n-1$:

$$\psi_{n,k}^0(\frac{s\pi}{n}) = \delta_{k,s} \quad \text{and} \quad {\psi_{n,k}^0}'(\frac{s\pi}{n}) = 0$$

and

$$\psi_{n,k}^1(\frac{s\pi}{n}) = 0 \quad \text{and} \quad {\psi_{n,k}^1}'(\frac{s\pi}{n}) = \delta_{k,s}.$$

The following computations are restricted to the scaling functions, but similar investigations yield that both types of wavelet functions behave like $\phi_{n,k}^1$. Let us only briefly mention at this point that by giving up the interpolatory properties, one can find another splitting of the space $W_n$ into two wavelet spaces such that one of the two wavelets possesses the same localization properties as $\phi_{n,k}^0$.

## §2. Localization by interpolation

For any $n \in \mathbb{N}$, a Hermite type interpolation operator can be defined, mapping any real-valued differentiable $2\pi$-periodic function $f$ into the space $V_n$, namely $L_n := L_n^0 + L_n^1$, where for $r = 0, 1$,

$$L_n^r f(x) = \sum_{k=0}^{2n-1} f^{(r)}(\frac{k\pi}{n})\phi_{n,k}^r(x).$$

Hence, each $L_n^r$ is a projection onto $V_n^r$ due to the interpolatory properties.

Note that the decomposition $V_n = V_n^0 + V_n^1$ is direct, but not orthogonal, as can be seen from the following result.

**Lemma 2.1.** *For $n \in \mathbb{N}$, it holds that*

$$\begin{aligned} V_n^0 = \operatorname{span}\{&(n+s)\cos(n-s)x + (n-s)\cos(n+s)x: \quad s=0,\ldots,n,\\ &(n+s)\sin(n-s)x - (n-s)\sin(n+s)x: \quad s=1,\ldots,n-1\} \end{aligned}$$

*and*

$$\begin{aligned} V_n^1 = \operatorname{span}\{&\cos(n-s)x - \cos(n+s)x: \quad s=1,\ldots,n-1,\\ &\sin(n-s)x + \sin(n+s)x: \quad s=0,\ldots,n\}. \end{aligned}$$

**Proof:** To show the first statement, it is sufficient to establish that $V_n^0$ is in the kernel of $L_n^1$. Consider

$$\begin{aligned} &L_n^1((n+s)\cos(n-s)\cdot\ + (n-s)\cos(n+s)\cdot\ )\\ &= \sum_{k=0}^{2n-1} (s^2-n^2)\left(\sin(n-s)\frac{k\pi}{n} + \sin(n+s)\frac{k\pi}{n}\right)\phi_{n,k}^1\\ &= \sum_{k=0}^{2n-1} (2s^2-2n^2)\sin k\pi\ \cos\frac{sk\pi}{n}\ \phi_{n,k}^1 = 0. \end{aligned}$$

The rest follows analogously. ■

A further possibility to describe localization of polynomials is related to a Nikolskii type inequality which can be stated as

$$1 \le \frac{\|p_n\|_p}{\|p_n\|_q} \le C n^{\frac{1}{q}-\frac{1}{p}}$$

for all trigonometric polynomials of degree at most $n$ and $1 \le q \le p \le \infty$ (see [7, Ch.4.9.2]). To obtain equality on the right-hand side can also be regarded as a measure of time localization.

**Lemma 2.2.** *For $n \in \mathbb{N}$, the norms of the fundamental interpolants $\phi_{n,k}^r$, $r=0,1$, $k=0,\ldots,2n-1$, $1<p\le\infty$, are as follows:*

$$\|\phi_{n,k}^r\|_p \sim n^{-\frac{1}{p}}.$$

*Moreover,*

$$\|\phi_{n,k}^0\|_1 = \frac{1}{2n}, \quad \|\phi_{n,k}^0\|_2 = \frac{1}{2n}\sqrt{\frac{8n^2+1}{6n}}, \quad \|\phi_{n,k}^0\|_C = 1,$$

*whereas* $$\|\phi_{n,k}^1\|_1 \sim \frac{\log n}{n^2} \quad \text{and} \quad \|\phi_{n,k}^1\|_2 = \frac{1}{4n^2}\sqrt{4n-\frac{3}{2}}.$$

**Proof:** The asymptotic behaviour of these kernels is well-known (see *e.g.*, [8, Ch.10]).

For later use, the complex Fourier coefficients of $\phi^0_{n,0}$ can be given as $c^0_s = (2n - |s|)/(4n^2)$ for $|s| \leq 2n-1$ and $c^0_s = 0$ for $|s| \geq 2n$. The ones for $\phi^1_{n,0}$ are $c^1_s = \operatorname{sgn} s\ i/(4n^2)$ for $0 < |s| < 2n$, $c^1_s = \operatorname{sgn} s\ i/(8n^2)$ for $|s| = 2n$ and $c^1_s = 0$, otherwise. Hence,

$$\|\phi^0_{n,k}\|_2^2 = \sum_{s=-\infty}^{\infty} |c^0_s|^2 = \frac{1}{4n^2} + \frac{1}{8n^4}\sum_{s=1}^{2n-1} s^2 = \frac{1}{3n} + \frac{1}{24n^3}$$

and

$$\|\phi^1_{n,k}\|_2^2 = \frac{1}{32n^4} + \sum_{s=1}^{2n-1} \frac{1}{8n^4} = \frac{1}{4n^3} - \frac{3}{32n^4}.$$

For the computation of the $L^1$-norm and $C$-norm, the positivity of $\phi^0_{n,k}$ can be applied. ■

## §3. Time frequency uncertainty

According to Breitenberger [1] and Narcowich & Ward [3], it is possible to formulate an uncertainty principle for functions in $L^2_{2\pi}$. For a function $f \in L^2_{2\pi}$, represented by its Fourier series $f(x) = \sum_{s=-\infty}^{\infty} c_s e^{isx}$, the *first trigonometric moment* is defined as

$$\tau(f) := \frac{1}{2\pi}\int_0^{2\pi} e^{ix}|f(x)|^2\,dx = \sum_{s=-\infty}^{\infty} c_s\overline{c_{s+1}}.$$

Then, the *angle variance* for $f \in L^2_{2\pi}$, $f \not\equiv 0$, is defined by

$$\operatorname{var}_A(f) \ := \ \frac{\|f\|_2^4 - |\tau(f)|^2}{|\tau(f)|^2} = \left|\frac{\sum_{s=-\infty}^{\infty}|c_s|^2}{\sum_{s=-\infty}^{\infty} c_s\overline{c_{s+1}}}\right|^2 - 1,$$

and the *frequency variance* for real-valued functions by

$$\operatorname{var}_F(f) \ := \ \sum_{s=-\infty}^{\infty} s^2|c_s|^2.$$

The *Uncertainty Principle for real-valued functions* $f \in L^2_{2\pi}$, $f \not\equiv 0$, (see [1,3]) states that

$$\frac{\sqrt{\operatorname{var}_A(f)\operatorname{var}_F(f)}}{\|f\|_2} \geq \frac{1}{2}.$$

We now proceed to compute the uncertainty products of the two trigonometric Hermite interpolants $\phi^0_{n,k}$ and $\phi^1_{n,k}$.

**Theorem 3.1.** *For $n \in \mathbb{N}$, the angle variances of $\phi^0_{n,k}$ and $\phi^1_{n,k}$ are*

$$\operatorname{var}_A(\phi^0_{n,k}) = \frac{3}{4}\frac{16n^2-1}{(4n^2-1)^2} \quad \text{and} \quad \operatorname{var}_A(\phi^1_{n,k}) = \frac{1}{4}\frac{48n-27}{(4n-3)^2},$$

*while*

$$\operatorname{var}_F(\phi^0_{n,k}) = \frac{16n^4-1}{120n^3} \quad \text{and} \quad \operatorname{var}_F(\phi^1_{n,k}) = \frac{8n^2-3n+1}{24n^3}.$$

*Consequently, the uncertainty products are*

$$\frac{\sqrt{\operatorname{var}_A(\phi^0_{n,k})\operatorname{var}_F(\phi^0_{n,k})}}{\|\phi^0_{n,k}\|_2} = \sqrt{\frac{3}{10}\left(1+\frac{20n^2+1}{2(8n^2+1)(4n^2-1)}\right)}$$

*and*

$$\frac{\sqrt{\operatorname{var}_A(\phi^1_{n,k})\operatorname{var}_F(\phi^1_{n,k})}}{\|\phi^1_{n,k}\|_2} = \sqrt{n\left(1+\frac{120n^2-101n+18}{(8n-3)(4n-3)^2}\right)}.$$

**Proof:** As the variances are translation invariant, it suffices to deal with the case $k=0$.

One computes

$$\sum_{s=-\infty}^{\infty} c^0_s \overline{c^0_{s+1}} = 2\sum_{s=0}^{2n-2} c^0_s c^0_{s+1} = \frac{1}{8n^4}\sum_{s=0}^{2n-2}(2n-s)(2n-s-1)$$
$$= \frac{1}{8n^4}\left(\sum_{s=1}^{2n} s(s-1)\right) = \frac{4n^2-1}{12n^3},$$

and similarly all the other necessary terms. ■

**Discussion:** According to the above theorem, $\phi^1_{n,k}$ displays an $\mathcal{O}(\sqrt{n})$ product, while in the limit, one obtains for $\phi^0_{n,k}$ a value of $\sqrt{0.3} = 0.5477..$, relatively close to the optimal value of $1/2$. This is due to the fact that the function $\phi^0_{n,0}$ can be interpreted as an approximation of the theta functions

$$\sum_{s=-\infty}^{\infty} e^{-cs^2+isx},$$

for which the time frequency localization tends to the optimal value of $1/2$ if the positive parameter $c$ tends to 0. Details will be discussed in a forthcoming paper.

## §4. Approximation Properties

Our goal in this section is to describe time-frequency localization in terms of operator norms. Let $X$ be one of the spaces $L^p_{2\pi}$ or $C_{2\pi}$. We consider the orthoprojections $P^r_n$ from $X$ onto $V^r_n$.

**Theorem 4.1.** *Let $n \in \mathbb{N}$. For the operator norms of the orthoprojections, it holds that*

$$\|P^0_n\|_{X\to X} < 3$$

*and*

$$\|P^1_n\|_{X\to X} \sim \begin{cases} \log n, & \text{if } X = C_{2\pi} \text{ or } X = L^1_{2\pi}, \\ C_p, & \text{if } X = L^p_{2\pi}, 1 < p < \infty. \end{cases}$$

**Proof:** Let $f \in C_{2\pi}$ be given such that $P^0_n f = \sum_{k=0}^{2n-1} \gamma_k \phi^0_{n,k} \not\equiv 0$. By the positivity of $\phi^0_{n,k}$, we obtain

$$\begin{aligned} \|P^0_n f\|_C &= \|\sum_{k=0}^{2n-1} \gamma_k \phi^0_{n,k}\|_C \le \left(\max_{0\le k<2n} |\gamma_k|\right) \|\sum_{k=0}^{2n-1} \phi^0_{n,k}\|_C \\ &= \max_{0\le k<2n} |\gamma_k| =: |\gamma_l|. \end{aligned}$$

On the other hand, by Hölder's inequality,

$$\begin{aligned} \|f\|_C \|\phi^0_{n,l}\|_1 &\ge |\langle f, \phi^0_{n,l}\rangle| = |\langle P^0_n f, \phi^0_{n,l}\rangle| = |\sum_{k=0}^{2n-1} \gamma_k \langle \phi^0_{n,k}, \phi^0_{n,l}\rangle| \\ &\ge |\gamma_l| \left(2\langle \phi^0_{n,l}, \phi^0_{n,l}\rangle - \sum_{k=0}^{2n-1} \frac{|\gamma_k|}{|\gamma_l|} \langle \phi^0_{n,k}, \phi^0_{n,l}\rangle\right) \\ &\ge |\gamma_l| \left(2\|\phi^0_{n,l}\|_2^2 - \|\phi^0_{n,l}\|_1\right). \end{aligned}$$

Using the norm equalities from Lemma 2.2, we summarize

$$\|P^0_n\|_{C_{2\pi}\to C_{2\pi}} \le \frac{|\gamma_l|}{\|f\|_C} \le \left(\frac{2\|\phi^0_{n,l}\|_2^2}{\|\phi^0_{n,l}\|_1} - 1\right)^{-1} = \frac{6n^2}{2n^2+1} < 3.$$

The uniform boundedness of $\|P^0_n\|_{C_{2\pi}\to C_{2\pi}}$ implies the one for $X = L^p_{2\pi}$, $1 \le p < \infty$, (see [2, Ch.1, §4, Th.9]) .

To estimate $\|P^1_n\|_{X\to X}$, we use the similarity of $P^1_n$ to the Fourier sum projection.

Namely, from Lemma 2.1, we conclude that $V^1_n$ has the orthogonal basis $\{\sin nx \cdot \cos kx : k = 0, \dots, n;\ \sin nx \cdot \sin kx : k = 1, \dots, n-1\}$.

Hence, in terms of the classical Fourier sum operator $S_n$ with respect to span$\{\cos kx : k = 0, \ldots, n;\ \sin kx : k = 1, \ldots, n-1\}$, we write

$$P_n^1 f(x) \sim S_n(f \sin n\cdot)(x)\ \sin nx,$$

which leads to an asymptotic behaviour as stated in the theorem. ■

**Acknowledgments.** The second author was supported in part by the Department of the Air Force, Wright-Patterson AFB, OH, contract F33600-94-M-2603.

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