Polynomial frames on the sphere

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We introduce a class of polynomial frames suitable for analyzing data on the surface of the unit sphere of a Euclidean space. Our frames consist of polynomials, but are well localized, and are stable with respect to all the L^p norms. The frames belonging to higher and higher scale wavelet spaces have more and more vanishing moments.

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1. Introduction

Geophysical or meteorological data collected over the surface of the earth via satellites or ground stations will invariably come from scattered sites. Syn-

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thesizing and analyzing such data has been the motivation for the development of wavelets and frames on S^2.

More specifically, one assumes that data has been sampled at scattered sites on S^2 from some underlying function of a given smoothness class. One next wishes to decompose the underlying function into "high frequency" and "low frequency" components with the hope of further understanding various properties of the function, for example location of singularities. The standard wavelet paradigm of constructing spaces based on an increasing chain of lattices does not apply to this situation. Nevertheless, many authors [1–7,15,19,21,24] have derived various nonstationary constructions of wavelets or frames on the sphere. Such constructions fall roughly into three categories: continuous wavelet transforms (CWT), discrete wavelets or frames and biorthogonal wavelets using lifting techniques. Each approach seems to have its advantages and disadvantages.

The aim of this paper is to derive and analyze a general class of frames in $L_2(S^q)$ consisting of polynomials. Special emphasis is placed on deriving estimates for the localization properties of these frames since there are no a priori bounds on localization for nonstationary wavelets or frames. Our estimates are based on the concepts involved in obtaining the "space-frequency uncertainty principle" [15]. A key feature in our approach is the use of quadrature rules based on scattered sites. Thus, our constructions are "coordinate-free". In particular, they avoid the latitude-longitude grids used in many other constructions, and the resulting requirement of having more data near the poles.

Our work in this paper is influenced by [18]. Our constructions of frames with arbitrarily high vanishing moments are motivated in part by the approach taken in [12]. In theory, these constructions allow for the detection of singularities (of a priori unknown order) of a function sampled at scattered sites. An example illustrating this detection is given in Section 5.

The paper is organized as follows. In Section 2, some pertinent facts concerning spherical harmonics will be given, the role of spherical harmonics in the construction of smooth kernel functions will be detailed, and quadrature rules based on scattered data samples will be discussed. In Section 3, the construction of the polynomial frames will be given, while their localization properties will be presented in Section 4. The paper concludes in Section 5 with an example illustrating the ability of our frames to detect high order singularities.

2. Polynomials on the sphere

2.1. Spherical Harmonics

Let $q \geq 1$ be an integer which will be fixed throughout the rest of this paper, and let $\mathbb{S}^q$ be the (surface of the) unit sphere in the Euclidean space $\mathbb{R}^{q+1}$, with $d\mu_q$ being its usual volume element. We note that the volume element is invariant under arbitrary coordinate changes. The volume of $\mathbb{S}^q$ is

$$\omega_q := \int_{\mathbb{S}^q} d\mu_q = \frac{2\pi^{(q+1)/2}}{\Gamma((q+1)/2)}. \tag{2.1}$$

Corresponding to $d\mu_q$, we have the inner product and $L^p(\mathbb{S}^q)$ norms,

$$\langle f, g\rangle_{\mathbb{S}^q} := \int_{\mathbb{S}^q} f(\mathbf{x})\overline{g(\mathbf{x})}d\mu_q(\mathbf{x}), \tag{2.2}$$

$$\|f\|_{\mathbb{S}^q,p} := \begin{cases} \left\{\int_{\mathbb{S}^q} |f(\mathbf{x})|^p d\mu_q(\mathbf{x})\right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{\mathbf{t}\in\mathbb{S}^q} |f(\mathbf{t})|, & \text{if } p = \infty. \end{cases} \tag{2.3}$$

The class of all measurable functions $f : \mathbb{S}^q \to \mathbb{C}$ for which $\|f\|_{\mathbb{S}^q,p} < \infty$ will be denoted by $L^p(\mathbb{S}^q)$, with the usual understanding that functions that are equal almost everywhere are considered equal as elements of $L^p(\mathbb{S}^q)$. All continuous complex valued functions on $\mathbb{S}^q$ will be denoted by $C(\mathbb{S}^q)$.

For a fixed integer $\ell \ge 0$, the restriction to $\mathbb{S}^q$ of a homogeneous harmonic polynomial of degree ℓ is called a spherical harmonic of degree ℓ. Most of the following information is based on [14] and [22, §IV.2], although we use a different notation. The class of all spherical harmonics of degree ℓ will be denoted by $\mathbf{H}^q_\ell$, and the class of all spherical harmonics of degree $\ell \le n$ will be denoted by Π^q_n. The spaces $\mathbf{H}^q_\ell$'s are mutually orthogonal relative to (2.2). Of course, $\Pi^q_n = \bigoplus_{\ell=0}^n \mathbf{H}^q_\ell$, and it comprises the restriction to $\mathbb{S}^q$ of all algebraic polynomials in $q+1$ variables of total degree not exceeding n. The dimension of $\mathbf{H}^q_\ell$ is given by

$$d^{\,q}_{\,\ell} := \dim \mathbf{H}^q_\ell = \begin{cases} \dfrac{2\ell+q-1}{\ell+q-1}\dbinom{\ell+q-1}{\ell}, & \text{if } \ell \ge 1, \\ 1, & \text{if } \ell = 0. \end{cases} \tag{2.4}$$

and that of Π^q_n is $\sum_{\ell=0}^n d^{\,q}_{\,\ell}$. Furthermore, $L^2(\mathbb{S}^q) = L^2\text{-closure}\{\bigoplus_\ell \mathbf{H}^q_\ell\}$. Hence, if we choose an orthonormal basis $\{Y_{\ell,k} : k = 1, \cdots, d^q_\ell\}$ for each $\mathbf{H}^q_\ell$, then the set $\{Y_{\ell,k} : \ell = 0, 1, \ldots \text{ and } k = 1, \cdots, d^{\,q}_{\,\ell}\}$ is an orthonormal basis for $L^2(\mathbb{S}^q)$. One has the well-known addition formula [14]:

$$\sum_{k=1}^{d^{\,q}_{\,\ell}} Y_{\ell,k}(\mathbf{x})\overline{Y_{\ell,k}(\mathbf{y})} = \frac{d^{\,q}_{\,\ell}}{\omega_q}\mathcal{P}_\ell(q+1; \mathbf{x}\cdot\mathbf{y}), \qquad \ell = 0, 1, \cdots, \tag{2.5}$$

where $\mathcal{P}_\ell(q+1; x)$ is the degree-ℓ Legendre polynomial in $q+1$-dimensions.

The Legendre polynomials are normalized so that $\mathcal{P}_\ell(q+1; 1) = 1$, and satisfy the orthogonality relations [14, Lemma 10]

$$\int_{-1}^1 \mathcal{P}_\ell(q+1; x)\mathcal{P}_k(q+1; x)(1-x^2)^{\frac{q}{2}-1}dx = \frac{\omega_q}{\omega_{q-1}d^{\,q}_{\,\ell}}\delta_{\ell,k}. \tag{2.6}$$

They are related to the ultraspherical (Gegenbauer) polynomials $P_\ell^{(\frac{q-1}{2})}$ (cf. [23], [14, p. 33]), and the Jacobi polynomials, $P_\ell^{(\alpha,\beta)}$, with $\alpha = \beta = \frac{q}{2} - 1$, via

$$P_\ell^{(\frac{q-1}{2})}(x) = \binom{\ell + q - 2}{\ell} \mathcal{P}_\ell(q+1; x) \quad (q \geq 2), \tag{2.7}$$

$$P_\ell^{(\frac{q}{2}-1,\frac{q}{2}-1)}(x) = \binom{\ell + \frac{q}{2} - 1}{\ell} \mathcal{P}_\ell(q+1; x). \tag{2.8}$$

When $q = 1$, the Legendre polynomials $\mathcal{P}_\ell(2; x)$ coincide with the Chebyshev polynomials $T_\ell(x)$; the ultraspherical polynomials $P_\ell^{(0)}(x) = (2/\ell)T_\ell(x)$, if $\ell \geq 1$. For $\ell = 0$, $P_0^{(0)}(x) = 1$.

In addition to the inner product and norms defined above on $\mathbb{S}^q$, we will need the following related inner product and norms for $[-1, 1]$, with weight function $w_q(x) := (1 - x^2)^{\frac{q}{2}-1}$:

$$\langle f, g\rangle_{w_q} := \int_{-1}^{1} f(x)\overline{g(x)}w_q(x)dx, \tag{2.9}$$

$$\|f\|_{w_q,p} := \begin{cases} \left\{\int_{-1}^{1} |f(x)|^p w_q(x)dx\right\}^{1/p}, & \text{if } 1 \leq p < \infty, \\ \operatorname*{ess\,sup}_{x\in[-1,1]} |f(x)|, & \text{if } p = \infty. \end{cases} \tag{2.10}$$

2.2. Quadrature

In this subsection, we review certain facts about polynomials which will be utilized often in the sequel.

Let $\mathcal{C}$ be a finite set of distinct points on $\mathbb{S}^q$. The mesh norm of $\mathcal{C}$ is defined to be

$$\delta_{\mathcal{C}} := \sup_{x\in\mathbb{S}^q} \operatorname{dist}(x, \mathcal{C}). \tag{2.11}$$

The following theorem summarizes the quadrature formula ((2.12) below) given in [9]. In the sequel, we adopt the following convention regarding constants. The letters $c, c_1, \cdots$ will denote positive constants depending only on the dimension q, and the different norms involved in the formula. Their value will be different at different occurrences, even within the same formula. The expression $A \sim B$ will mean $cA \leq B \leq c_1A$.

Theorem 2.1. There exist constants α_q and N_q with the following property. Let $\mathcal{C}$ be a finite set of distinct points on $\mathbb{S}^q$, and n be an integer with $N_q \le n \le \alpha_q \delta_{\mathcal{C}}^{-1}$. Then there exist nonnegative weights, $\{a_\xi\}_{\xi \in \mathcal{C}}$, such that for every $P \in \Pi_n^q$,

$$\frac{1}{\omega_q}\int_{\mathbb{S}^q} P(\mathbf{x})d\mu_q(\mathbf{x}) = \sum_{\xi\in\mathcal{C}} a_\xi P(\xi). \tag{2.12}$$

Further,

$$|\{\xi \ : \ a_\xi \neq 0\}| \sim n^q \sim \dim(\Pi_n^q). \tag{2.13}$$

In [9], we have discussed algorithms to compute a_ξ.

2.3. Cesàro means

If $f \in L^1(\mathbb{S}^q)$, $\ell \ge 0$ is an integer, $1 \le k \le d_\ell^q$ is an integer, we write

$$\hat{f}(\ell,k) := \int_{\mathbb{S}^q} f(\mathbf{x})\overline{Y_{\ell,k}(\mathbf{x})}d\mu_q(\mathbf{x}),$$

and

$$\mathsf{P}_\ell(f) := \sum_{k=1}^{d_\ell^q} \hat{f}(\ell,k)Y_{\ell,k}. \tag{2.14}$$

For integers $k \ge 0$ and $n \ge 0$, we define the Cesàro (C,k) means, $\sigma_n^{[k]}(f)$, of the series $\sum \mathsf{P}_\ell(f)$ by

$$s_n^{[k]}(f) := \sum_{\ell=0}^{n} \binom{n-\ell+k}{k}\mathsf{P}_\ell(f), \qquad \sigma_n^{[k]}(f) := \left\{\binom{n+k}{k}\right\}^{-1} s_n^{[k]}(f). \tag{2.15}$$

We observe that $s_n^{[0]}(f) = \mathsf{Proj}_{\Pi_n^q}(f)$, and define $s_n^{[-1]}(f) := \mathsf{P}_n(f)$.

Theorem 2.2. Let $1 \le p \le \infty$, $f \in L^p(\mathbb{S}^q)$, and k_q be the smallest integer greater than $(q-1)/2$. Then

$$\sup_{n\ge 0} \|\sigma_n^{[k_q]}(f)\|_{\mathbb{S}^q,p} \le c\|f\|_{\mathbb{S}^q,p}. \tag{2.16}$$

Proof. The estimate (2.16) is well known in the case $p = \infty$ [23, §9.7(1)]. Using the definitions, it is easy to verify that

$$\sigma_n^{[k]}(f,\mathbf{x}) = \int_{\mathbb{S}^q} f(\mathbf{y})K_n(\mathbf{x}\cdot\mathbf{y})d\mu_q(\mathbf{y})$$

for a suitable kernel function K_n. The general result now follows from [11, Theorem 1]. □

3. Polynomial frames

Let $\{N_j\}$ be an increasing sequence of numbers, with $N_j \to \infty$ as $j \to \infty$. Our scaling spaces are defined by $V_j := \Pi^q_{N_j}$. The wavelet spaces W_j are defined by

$$W_j := \{P \in V_{j+1} \ : \ \int_{\mathbb{S}^q} P(\mathbf{x})R(\mathbf{x})d\mu_q(\mathbf{x}) = 0, \quad R \in V_j\}.$$

Note that $V_{j+1} = V_1 \oplus \left(\bigoplus_{k=1}^{j} W_k\right)$ where the spaces W_k are mutually orthogonal.

In this section, we develop frames for the spaces V_j and W_j. As in [12], our frames both for the scaling and the wavelet spaces will be of the form $\sum g_{j,\ell} P_\ell(\mathbf{x}\cdot\mathbf{y}_{j,\ell})$ where the $\{\mathbf{y}_{j,\ell}\}$ are scattered points on the sphere. Following [12], we find it convenient to adopt the following notations and terminology. A matrix G will be called a *scaling matrix* if $g_{j,k} \neq 0$ for $k = 0, \cdots, N_j$, $j = 0, 1, \cdots$, and $g_{j,k} = 0$ otherwise. Similarly, a matrix G will be called a *frame matrix* if $g_{j,k} \neq 0$ for $k = N_{j-1}+1, \cdots, N_j$, $j = 1, 2, \cdots$, and $g_{j,k} = 0$ otherwise. We also assume that $g_{0,k} \neq 0$, $k = 0, \cdots, N_0$ and $g_{0,k} = 0$ otherwise. For an integer k and a matrix G, we define the matrix $G^{[k]}$ by

$$G^{[k]}_{j,\ell} = \begin{cases} (g_{j,\ell})^k, & \text{if } g_{j,\ell} \neq 0, \\ 0, & \text{otherwise.} \end{cases}$$

We assume in the sequel that we have a sequence of scattered point sets $\{\mathcal{C}_j\}$ and the nonnegative weights $\{w_{j,\xi}\}$ such that the following quadrature formula holds (cf. Theorem 2.1):

$$\int_{\mathbb{S}^q} P(\mathbf{x})\overline{R(\mathbf{x})}d\mu_q(\mathbf{x}) = \sum_{\xi\in\mathcal{C}_j} w_{j,\xi}P(\xi)\overline{R(\xi)}, \qquad P, R \in V_j, \ j = 0, 1, 2, \cdots. \tag{3.1}$$

Finally, we define the kernel functions for $\mathbf{x} \in \mathbb{S}^q$, $j = 0, 1, \cdots$, by

$$K_j(G; \mathbf{x}, \mathbf{y}) := \sum_{\ell=0}^{N_j} g_{j,\ell}\frac{d^q_\ell}{\omega_q}\mathcal{P}_\ell(\mathbf{x}\cdot\mathbf{y}) = \sum_{\ell=0}^{N_j} g_{j,\ell}\sum_{k=1}^{d^q_\ell} Y_{\ell,k}(\mathbf{x})\overline{Y_{\ell,k}(\mathbf{y})}, \tag{3.2}$$

and the operators

$$\tau_j(G; f, \mathbf{x}) := \int_{\mathbb{S}^q} K_j(G; \mathbf{x}, \mathbf{y})f(\mathbf{y})d\mu_q(\mathbf{y}), \qquad f \in L^1(\mathbb{S}^q). \tag{3.3}$$

We observe that

$$\tau_j(G; f, \mathbf{x}) = \sum_{\ell=0}^{N_j} g_{j,\ell}\sum_{k=1}^{d^q_\ell} \hat{f}(\ell, k)Y_{\ell,k}(\mathbf{x}). \tag{3.4}$$

Thus, the operators τ_j may be thought of as "windowed Fourier transform" operators. The following theorem gives frame bounds for the operators τ_j associated with either scaling or frame matrices. An analogous theorem for the interval was given in [12].

Theorem 3.1. Let $j \geq 0$ be an integer, and let $\mathcal{C}_j$, and $\{w_{j,\xi}\}$ be as in (3.1).
(a) If G is a scaling matrix and $P \in V_j$, then for $\mathbf{x} \in \mathbb{S}^q$,

$$P(\mathbf{x}) = \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} \tau_j(G; P, \xi) K_j(G^{[-1]}; \mathbf{x}, \xi). \tag{3.5}$$

Moreover, with

$$B_{j,2}(G) := \max_{0 \leq \ell \leq N_j} |g_{j,\ell}|, \tag{3.6}$$

we have the frame bounds:

$$\begin{aligned} & B_{j,2}(G^{[-1]})^{-1} \|P\|_{\mathbb{S}^q,2} \leq \|\tau_j(G; P)\|_{\mathbb{S}^q,2} \\ & := \left\{ \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} |\tau_j(G; P, \xi)|^2 \right\}^{1/2} \leq B_{j,2}(G) \|P\|_{\mathbb{S}^q,2}. \end{aligned} \tag{3.7}$$

The bounds in (3.7) are exact.
(b) If G is a frame matrix, $j \geq 1$, and $P \in W_{j-1}$, then for $\mathbf{x} \in \mathbb{S}^q$,

$$P(\mathbf{x}) = \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} \tau_j(G; P, \xi) K_j(G^{[-1]}; \mathbf{x}, \xi). \tag{3.8}$$

Moreover,

$$\begin{aligned} & B_{j,2}(G^{[-1]})^{-1} \|P\|_{\mathbb{S}^q,2} \leq \|\tau_j(G; P)\|_{\mathbb{S}^q,2} \\ & = \left\{ \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} |\tau_j(G; P, \xi)|^2 \right\}^{1/2} \leq B_{j,2}(G) \|P\|_{\mathbb{S}^q,2}. \end{aligned} \tag{3.9}$$

The bounds in (3.9) are exact.

Proof. First, we observe that if $P \in V_j$, $P = \sum_{\ell=0}^{N_j} \sum_{k=1}^{d_\ell^q} a_{\ell,k} Y_{\ell,k}$, then

$$\tau_j(G; P, \mathbf{x}) = \sum_{\ell=0}^{N_j} \sum_{k=1}^{d_\ell^q} g_{j,\ell} a_{\ell,k} Y_{\ell,k}(\mathbf{x}). \tag{3.10}$$

Since $\tau_j(G; P) \in V_j$, the quadrature formula (3.1) implies that

$$\begin{aligned} & \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} \tau_j(G; P, \xi) K_j(G^{[-1]}; \mathbf{x}, \xi) \\ & = \int_{\mathbb{S}^q} \tau_j(G; P, \mathbf{y}) K_j(G^{[-1]}; \mathbf{x}, \mathbf{y}) d\mu_q(\mathbf{y}) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\ell=0}^{N_j} \sum_{k=1}^{d_\ell^q} g_{j,\ell} a_{\ell,k} \int_{\mathbb{S}^q} Y_{\ell,k}(\mathbf{y}) K_j(G^{[-1]}; \mathbf{x}, \mathbf{y}) d\mu_q(\mathbf{y}) \\
&= \sum_{\ell=0}^{N_j} \sum_{k=1}^{d_\ell^q} g_{j,\ell} a_{\ell,k} g_{j,\ell}^{-1} Y_{\ell,k}(\mathbf{x}) = P(\mathbf{x}). \qquad (3.11)
\end{aligned}
$$

This proves (3.5). In view of the quadrature formula (3.1), we see that

$$\sum_{\xi \in \mathcal{C}_j} w_{j,\xi} |\tau_j(G; P, \xi)|^2 = \int_{\mathbb{S}^q} |\tau_j(G; P, \mathbf{x})|^2 d\mu_q(\mathbf{x}).$$

The upper bound in (3.7) now follows from (3.10) and the Parseval identity. The lower bound follows from the upper bound and the fact that $P = \tau_j(G^{[-1]}; \tau_j(G; P))$. The exactness of the bounds is clear from the proof. The part (b) is proved in exactly the same way. □

Remark 3.2. If G is a frame matrix, any function $f \in L^2(\mathbb{S}^q)$ has the frame expansion

$$\sum_{j=0}^{\infty} \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} \tau_j(G; f, \xi) K_j(G^{[-1]}; \mathbf{x}, \xi). \qquad (3.12)$$

Thus, the operator τ_j may be thought of as a continuous frame operator. We observe that the frame bounds are independent of q. The reconstruction and decomposition algorithms in this case can be obtained easily using the quadrature formulas. We state these algorithms for the sake of completeness, but omit the proofs. For matrices G and H, we define the matrix $G \circ H$ by $(G \circ H)_{j,\ell} = g_{j,\ell} h_{j+1,\ell}$.

Theorem 3.3. Let $j \geq 0$ be an integer, G be a scaling matrix, and H be a frame matrix. We have for $P \in V_{j+1}$,

$$
\begin{aligned}
\tau_j(G; P, \mathbf{x}) &= \sum_{\xi \in \mathcal{C}_{j+1}} w_{j+1,\xi} \tau_{j+1}(G; P, \xi) K_j(G \circ G^{[-1]}; \mathbf{x}, \xi), \\
\tau_j(H; P, \mathbf{x}) &= \sum_{\xi \in \mathcal{C}_{j+1}} w_{j+1,\xi} \tau_{j+1}(G; P, \xi) K_j(H \circ G^{[-1]}; \mathbf{x}, \xi), \\
\tau_{j+1}(G; P, \mathbf{x}) &= \sum_{\xi \in \mathcal{C}_j} w_{j,\xi} \tau_j(G; P, \xi) K_{j+1}(G^{[-1]} \circ G; \mathbf{x}, \xi) \\
&\quad + \sum_{\xi \in \mathcal{C}_{j+1}} w_{j+1,\xi} \tau_j(H; P, \xi) K_{j+1}(H^{[-1]} \circ G; \mathbf{x}, \xi). \qquad (3.13)
\end{aligned}
$$

In practice, we may have only samples of f at scattered data. In [9,10], we have described certain quasi-interpolatory polynomial-valued operators defined on the scattered samples of f. We may use these to get a near-best approximation to f from V_J for a sufficiently large J that the data permits, and use this approximation in place of f in (3.12). We note that the points at which the function is sampled need not be the same as those appearing in (3.12). In view of these remarks, and the singularity detection applications as in [12], it is worthwhile to discuss the stability of the frame operators τ_j with respect to norms other than the L^2-norm. Our discussion below is based on the ideas in [12] and [13], but the results are sharper compared to those in [12] because we use Theorem 2.2 as in [13], rather than the weaker results on strong $(C,1)$-summability of Jacobi expansions as in [12].

We introduce the forward difference operator Δ^q for a sequence $\{a_j\}_{j=0}^\infty$ by

$$\Delta a_j := \Delta^1 a_j := a_{j+1} - a_j,\ \Delta^q a_j = \Delta(\Delta^{q-1} a_j),\ q = 2, 3, \cdots,\ j = 0, 1, 2, \cdots.$$

When we apply the forward difference operator to a multiply indexed sequence, we will write in the subscript the index to which it is applied. Thus, for example, we write $\Delta_\ell g_{j,\ell} = g_{j,\ell+1} - g_{j,\ell}$, etc. For a matrix G, we define

$$B_{q;j,\infty}(G) := \sum_\ell \ell^{k_q} |\Delta_\ell^{k_q+1} g_{j,\ell}|, \tag{3.14}$$

where k_q is the smallest integer greater than $(q-1)/2$ (cf. Theorem 2.2). We further define

$$B_{q;j,p}(G) = \begin{cases} B_{j,2}^{2/p} B_{q;j,\infty}(G)^{1-2/p}, & \text{if } 2 \le p \le \infty, \\ B_{j,2}^{2-2/p} B_{q;j,\infty}(G)^{2/p-1}, & \text{if } 1 \le p \le 2. \end{cases} \tag{3.15}$$

Theorem 3.4. Let $j \ge 0$ be an integer, and $1 \le p \le \infty$. For a scaling (respectively frame) matrix G and $P \in V_j$ (respectively $P \in W_{j-1}$), there are constants $c_1 > 0$ and $c_2 > 0$ such that

$$c_1 B_{q;j,p}(G^{[-1]})^{-1} \|P\|_{\mathbb{S}^q,p} \le \|\tau_j(G;P)\|_{\mathbb{S}^q,p} \le c_2 B_{q;j,p} \|P\|_{\mathbb{S}^q,p}. \tag{3.16}$$

Proof. We consider the case when G is a scaling matrix. The proof for the case when G is a frame matrix is the same. We observe that $g_{j,\ell} = 0$ if $\ell \ge N_j + 1$, and hence, for $f \in L^1(\mathbb{S}^q)$,

$$\tau_j(G;f) = \sum_{\ell=0}^{N_j} g_{j,\ell} \mathsf{P}_\ell(f) = \sum_{\ell=0}^{\infty} g_{j,\ell} s_\ell^{[-1]}(f). \tag{3.17}$$

Further, it is well known [25, Formula III(1.12)] that $s_\ell^{[k]}(f) - s_{\ell-1}^{[k]}(f) = s_\ell^{[k-1]}(f)$, $\ell \geq 1$ and $k \geq -1$. We define $s_\ell^{[k]} = 0$ for $\ell < 0$ and $k \geq -1$. Then a repeated summation by parts in (3.17) yields

$$\tau_j(G; f) = (-1)^{k_q+1} \sum_{\ell=0}^{\infty} \left(\Delta_\ell^{k_q+1} g_{j,\ell}\right) s_\ell^{[k_q]}(f). \tag{3.18}$$

In view of Theorem 2.2, this implies that for $f \in L^\infty(\mathbb{S}^q)$,

$$\|\tau_j(G; f)\|_{\mathbb{S}^q,\infty} \leq c B_{q;j,\infty} \|f\|_{\mathbb{S}^q,\infty}. \tag{3.19}$$

In view of (3.7), and the fact that $\tau_j(G; f) = \tau_j(G; \mathsf{Proj}_{V_j}(f))$, we conclude that

$$\|\tau_j(G; f)\|_{\mathbb{S}^q,2} \leq B_{j,2} \|f\|_{\mathbb{S}^q,2}, \qquad f \in L^2(\mathbb{S}^q). \tag{3.20}$$

An application of Riesz-Thorin interpolation theorem now gives

$$\|\tau_j(G; f)\|_{\mathbb{S}^q,p} \leq c B_{q;j,p} \|f\|_{\mathbb{S}^q,p}, \qquad f \in L^p(\mathbb{S}^q),$$

in the case $2 \leq p \leq \infty$. The same estimate holds for $1 \leq p \leq 2$ by duality (cf. [11]). This proves the upper bound in (3.16). The lower bound follows from the observation that $P = \tau_j(G^{[-1]}; \tau_j(G; P))$ for all $P \in V_j$. □

4. Localization

In this section, we examine the question of space-frequency localization of the kernels $K_j(G; \mathbf{x}, \mathbf{y})$, where G is either a scaling or frame matrix. The question of space-frequency localization is important for any nonstationary MRA as is the case on the sphere.

Before discussing the question theoretically, we first provide a few numerical examples to illustrate the space localization. First, we observe that $K_j(G; \mathbf{x}, \mathbf{y})$ is a function of $\mathbf{x} \cdot \mathbf{y}$ alone; i.e.,

$$K_j(G; \mathbf{x}, \mathbf{y}) = \mathcal{K}_j(G; \mathbf{x} \cdot \mathbf{y}) := \sum_{\ell=0}^{N_j} g_{j,\ell} \frac{d_\ell^q}{\omega_q} \mathcal{P}_\ell(\mathbf{x} \cdot \mathbf{y}). \tag{4.1}$$

Therefore, $K_j(G; \mathbf{x}, \mathbf{y})$ is localized near $\mathbf{y}$ if and only if $\mathcal{K}_j(G; t)$ is localized near 1. Let $\kappa_j(G) := \max_{x \in [-1,1]} |\mathcal{K}_j(G; x)|$. In Figure 1, we illustrate the graph of $\mathcal{K}_j(G; t)/\kappa_j(G)$ near $t = 1$ (left) and away from $t = 1$ (right), in the case when $q = 2$. The values $g_{j,\ell}$ are the sampled values on the interval $[0, 1]$ of a cardinal B-spline of order s, supported on $[1/2, 1]$. We take $N_j = 64$. The solid line represents the case $s = 1$, which corresponds to the constructions in the papers of Potts, Steidl, and Tasche [17,18], and the dashed line corresponds to $s = 3$.

In Figure 2, we illustrate the curious phenomenon that the localization increases with the dimension of the sphere. We use for $g_{j,\ell}$'s the sampled values of a cardinal B-spline of order 5, supported on $[1/2, 1]$, $N_j = 64$, and plot the

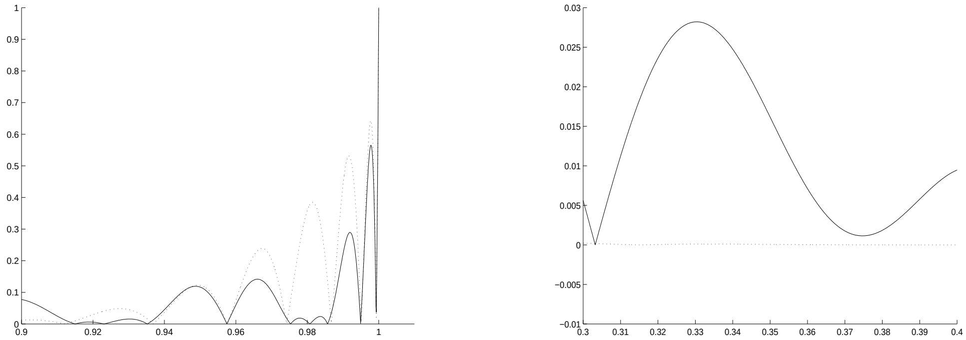

Figure 1. The graph of $\mathcal{K}_j(G;t)/\kappa_j(G)$ for different G: (left) Near $t=1$, (right) away from $t=1$. Solid line is for $s=1$, dashed line for $s=3$.

graph of $\mathcal{K}_j(G;t)/\kappa_j(G)$ near $t=1$ (left) and away from $t=1$ (right), for various values of q.

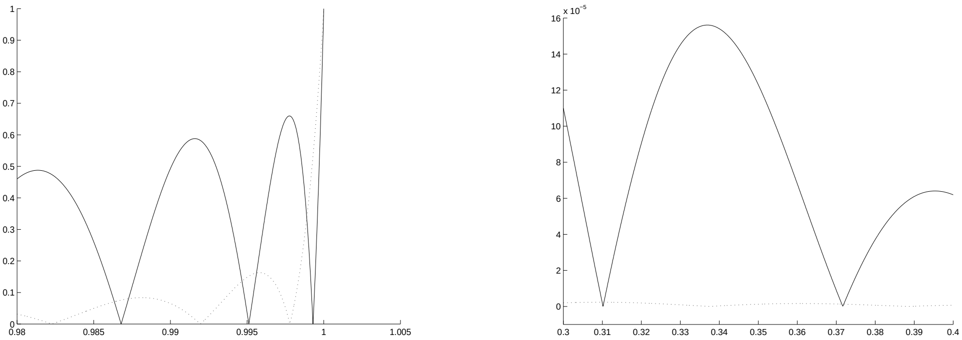

Figure 2. The graph of $\mathcal{K}_j(G;t)$ for different dimensions : (left) Near $t=1$, (right) away from $t=1$. The solid line is for $q=2$, the dashed line for $q=6$.

Next, we discuss the question of localization from a theoretical point of view. Without loss of generality, we may assume that $\mathbf{y} = \mathbf{e}_{q+1} := (0,\cdots,0,1)$. Following [15], we may measure the frequency localization of K_j by

$$\mathsf{var}_F(G;j) := \left\{ \sum_{\ell=0}^{\infty} |g_{\ell,k}|^2 d_\ell^q \right\}^{-1} \sum_{\ell=0}^{\infty} \ell(\ell+q-1)|g_{\ell,k}|^2 d_\ell^q. \tag{4.2}$$

Obviously, we have the estimate

$$\mathsf{var}_F(G;j) \le cN_j^2. \tag{4.3}$$

A measurement for space localization is defined by the expression $\mathsf{var}_S(G;j)$ as in the following formula (4.5). Let

$$T(G;j) := \|K_j(G;\cdot,\mathbf{e}_{q+1})\|_{\mathbb{S}^q,2}^{-2}\int_{\mathbb{S}^q}\mathbf{x}|K_j(G;\mathbf{x},\mathbf{e}_{q+1})|^2 d\mu_q(\mathbf{x}). \tag{4.4}$$

Then we define (cf. [15])

$$\mathsf{var}_S(G;j) := \frac{1-\|T(G;j)\|^2}{\|T(G;j)\|^2}, \tag{4.5}$$

where $\|T(G;j)\|$ is the Euclidean norm of the $q+1$-dimensional vector $T(G;j)$. In the case when $q=2$, it is proved in [15] that $\mathsf{var}_S(G;j)\mathsf{var}_F(G;j)\geq 1$ for all scaling or frame matrices G and integer $j\geq 0$. Rösler and Voit [20] have recently shown that the lower bound of 1 is sharp. In the following theorem, we give some sufficient conditions on a frame matrix G to ensure that $\mathsf{var}_S(G;j)\mathsf{var}_F(G;j)\leq c$.

Theorem 4.1. Let G be a frame matrix, and

$$f_{j,\ell} := g_{j,\ell}(d_\ell^q/\omega_q)^{1/2}\left\{\sum_{\ell=0}^{\infty}|g_{j,\ell}|^2(d_\ell^q/\omega_q)\right\}^{-1/2}, \qquad j=0,1,\cdots,\ \ell=0,1,\cdots. \tag{4.6}$$

If

$$\frac{f_{j,\ell+1}+f_{j,\ell-1}}{2} = f_{j,\ell}\left(1+\mathcal{O}(N_j^{-2})\right), \tag{4.7}$$

then there exists $c>0$ such that

$$\mathsf{var}_S(G;j)\leq cN_{j-1}^{-2}, \qquad j=1,2,\cdots. \tag{4.8}$$

Remark 4.2. Before giving a proof of Theorem 4.1, we give an example of a frame matrix G that satisfies the condition (4.7). Let $N_{j-1}<(1-c)N_j$ for some constant $c\in(0,1)$, $j=1,2,\cdots$. We write

$$g_{j,\ell} := \begin{cases} (d_\ell^q)^{-1/2}\sin\left(\pi\dfrac{\ell-N_{j-1}}{N_j-N_{j-1}+1}\right), & \text{if } N_{j-1}\leq\ell\leq N_j+1,\ j=1,2,\cdots,\\ 0, & \text{otherwise.}\end{cases}$$

It is then easy to verify that G is a frame matrix, and the condition (4.7) is satisfied. In general, it is much easier to define a matrix G that satisfies the weaker condition:

$$\frac{f_{j,\ell+1}+f_{j,\ell-1}}{2} = f_{j,\ell}+\mathcal{O}(N_j^{-2}). \tag{4.9}$$

For example, one may take any two times continuously differentiable function g such that $g(0) = g(1) = 0$, and define

$$g_{j,\ell} := \begin{cases} (d_\ell^q)^{-1/2} g\left(\dfrac{\ell - N_{j-1}}{N_j - N_{j-1} + 1}\right), & \text{if } N_{j-1} \le \ell \le N_j + 1,\ j = 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

The following proof will show that if $N_{j-1} < (1-c)N_j$, then

$$\mathsf{var}_S(G; j) \le c N_{j-1}^{-1}, \qquad j = 1, 2, \cdots. \tag{4.10}$$

This is analogous to the estimate obtained in [16] for periodic wavelets and scaling functions.

Proof of Theorem 4.1. In this proof, we write $p_\ell(t) := (d_\ell^q \omega_{q-1}/\omega_q)^{1/2} \mathcal{P}_\ell(q + 1; t)$, so that

$$\int_{-1}^{1} p_\ell(t) p_\nu(t) w_q(t) dt = \delta_{\ell,\nu}, \qquad \ell, \nu = 0, 1, \cdots.$$

It is well known [23] that

$$t p_{\ell-1}(t) = \rho_\ell p_\ell(t) + \rho_{\ell-1} p_{\ell-2}(t), \tag{4.11}$$

where

$$\rho_\ell := \sqrt{\frac{\ell(\ell + q - 2)}{(2\ell + q - 1)(2\ell + q - 3)}} = \frac{1}{2} + \mathcal{O}((\ell+1)^{-2}). \tag{4.12}$$

We also need the fact that for any $f \in L^1_{w_q}[-1, 1]$,

$$\int_{\mathbb{S}^q} f(\mathbf{x} \cdot \mathbf{e}_{q+1}) d\mu_q(\mathbf{x}) = \omega_{q-1} \int_{-1}^{1} f(t) w_q(t) dt. \tag{4.13}$$

First, we observe that

$$\|K_j(G; (\cdot), \mathbf{e}_{q+1})\|^2_{\mathbb{S}^q, 2} = \sum_{\ell=0}^{\infty} |g_{j,\ell}|^2 (d_\ell^q / \omega_q).$$

Hence, defining the matrix F by $F_{j,\ell} = f_{j,\ell}(d_\ell^q/\omega_q)^{-1/2}$, we see that $T(G; j) = T(F; j)$ and also that $\|K_j(F; (\cdot), \mathbf{e}_{q+1})\|_{\mathbb{S}^q,2} = 1$. In the remainder of this proof, we write $T(F; j) := (t_1, \cdots, t_{q+1})$. It is easy to verify that $t_\nu = 0$, $\nu = 1, \cdots, q$. Using (4.13), we obtain that

$$\begin{aligned} t_{q+1} &= \omega_{q-1}^{-1} \int_{\mathbb{S}^q} \mathbf{x} \cdot \mathbf{e}_{q+1} \left| \sum_{\ell=0}^{\infty} f_{j,\ell} p_\ell(\mathbf{x} \cdot \mathbf{e}_{q+1}) \right|^2 d\mu_q(\mathbf{x}) \\ &= \int_{-1}^{1} t \left| \sum_{\ell=0}^{\infty} f_{j,\ell} p_\ell(t) \right|^2 w_q(t) dt. \end{aligned}$$

Using (4.11), we obtain that

$$\begin{aligned} t&\left|\sum_{\ell=0}^{\infty} f_{j,\ell}p_\ell(t)\right|^2 \\ &= t\sum_{\ell=0}^{\infty} f_{j,\ell}p_\ell(t)\sum_{\nu=0}^{\infty} f_{j,\nu}p_\nu(t) \\ &= \left(\sum_{\nu=0}^{\infty} f_{j,\nu}p_\nu(t)\right)\left(\sum_{\ell=0}^{\infty} f_{j,\ell}(\rho_{\ell+1}p_{\ell+1}(t)+\rho_\ell p_{\ell-1}(t))\right). \end{aligned}$$

Therefore, the orthogonality relations for $\{p_\ell\}$ imply that

$$t_{q+1} = \sum_{\ell=0}^{\infty} \left(\rho_{\ell+1}f_{j,\ell}f_{j,\ell+1} + \rho_\ell f_{j,\ell}f_{j,\ell-1}\right).$$

Using (4.12), (4.7), and the fact that $f_{j,\ell}=0$ if $\ell<N_{j-1}$ or $\ell>N_j$ we obtain

$$t_{q+1} = \sum_{\ell=0}^{\infty}(1+\mathcal{O}(\ell+1)^{-2})f_{j,\ell}\frac{f_{j,\ell+1}+f_{j,\ell-1}}{2} = \sum_{\ell=N_{j-1}+1}^{N_j}(1+\mathcal{O}(\ell+1)^{-2})f_{j,\ell}^2.$$

Since $\sum_{\ell=0}^{\infty} f_{j,\ell}^2 = 1$, we see that

$$t_{q+1} = 1+\mathcal{O}(N_{j-1}^{-2}). \tag{4.14}$$

Consequently,

$$\|T(G;j)\| = \|T(F;j)\| = |t_{q+1}| = 1+\mathcal{O}(N_{j-1}^{-2}).$$

This leads to (4.8). □

In the case of the scaling matrices, we have the following theorem.

Theorem 4.3. Let $j\ge 0$ be an integer, G be a scaling matrix, X_j be the largest zero of $\mathcal{P}_{N_j+1}(q+1;\cdot)$. Then

$$\|T(G;j)\| \le X_j, \tag{4.15}$$

with equality if and only if for $\ell=0,\cdots,N_j$, $g_{j,\ell}=\alpha_j\mathcal{P}_\ell(q+1;X_j)$ for some scalar α_j. In particular, with this choice of G, there is a $c>0$ such that $\mathsf{var}_S(G;j)\le cN_j^{-2}$.

Proof. Let $T(G;j)=(t_1(G;j),\cdots,t_{q+1}(G;j))$. As in the proof of Theorem 4.1, $t_\nu(G;j)=0$ for $\nu=1,\cdots,q$. Let

$$P(t) := \sum_{\ell=0}^{N_j} g_{j,\ell}(d_\ell^q/\omega_q)\mathcal{P}_\ell(q+1;t).$$

Then, using (4.13), we get as before that

$$t_{q+1}(G;j) = \|P\|_{w_q,2}^{-2} \int_{-1}^{1} t|P(t)|^2 w_q(t)dt.$$

Consequently,

$$\sup_G t_{q+1}(G;j) = \sup_{P\in\Pi^1_{N_j}} \|P\|_{w_q,2}^{-2} \int_{-1}^{1} t|P(t)|^2 w_q(t)dt.$$

This last expression is equal to X_j (cf. [8, Theorem 1.3.3]), with the supremum attained if $P(t) = \beta\mathcal{P}_{N_j+1}(q+1;t)/(t-X_j)$ for some scalar β. In turn, the Gauss quadrature formula for w_q implies that

$$\begin{aligned} P(t) &= \gamma_j \sum_{\ell=0}^{N_j} \mathcal{P}_\ell(q+1;X_j)\mathcal{P}_\ell(q+1;t)/\|\mathcal{P}_\ell(q+1;\cdot)\|_{w_q,2}^2 \\ &= \alpha_j \sum_{\ell=0}^{N_j} \mathcal{P}_\ell(q+1;X_j)d_\ell^q\mathcal{P}_\ell(q+1;t). \end{aligned}$$

□

5. An example – singularity detection

To illustrate the ideas of this paper, we show how the frame operators τ_j (see (3.3)) can be used to detect the singularities of the (inverted) "tornado function"

$$f_{\mathbf{y},\alpha,r}(\mathbf{x}) := \frac{(\mathbf{x}\cdot\mathbf{y}-\alpha)_+^r}{r!}.$$

The graph of this function clearly has a singularity at $\mathbf{y}$ (in the sense that the gradient is not defined). In addition, the function is infinitely differentiable except at the vectors $D_\alpha := \{\mathbf{x}\colon\ \mathbf{x}\cdot\mathbf{y} = \alpha\}$, where the r-th order derivatives have a jump discontinuity.

The "Fourier coefficients" $\hat{f}_{\mathbf{y},\alpha,r}$ can be computed easily, enabling us to use the formula (3.4) to compute $\tau_j(G; f_{\mathbf{y},\alpha,r}, \mathbf{x})$. Indeed, let

$$\Gamma_{r,\alpha}(t) := \frac{(t-\alpha)_+^r}{r!}.$$

Using Rodrigues' formula [23, (4.3.1)], one can explicitly compute the coefficients (cf. (2.8) and [12]):

$$C_{m,r}(q;\alpha) := \int_\alpha^1 \Gamma_{r,\alpha}(t)\mathcal{P}_m(q+1;t)w_q(t)dt. \tag{5.1}$$

This gives us the formal expansion

$$\Gamma_{r,\alpha}(t) = \sum_{m=0}^{\infty} \frac{\omega_{q-1} d_m^q}{\omega_q} C_{m,r}(q;\alpha)\mathcal{P}_m(q+1;t). \tag{5.2}$$

Since $f_{\mathbf{y},\alpha,r}(\mathbf{x}) = \Gamma_{r,\alpha}(\mathbf{x}\cdot\mathbf{y})$, a comparison of (5.2) and (2.5) leads us to

$$\hat{f}_{\mathbf{y},\alpha,r}(m,k) = \omega_{q-1} C_{m,r}(q;\alpha)\overline{Y_{m,k}(\alpha)}. \tag{5.3}$$

In the important case when $q = 2$, one obtains (cf. (2.8))

$$C_{m,r} := C_{m,r}(2,\alpha) = \frac{(1-\alpha^2)^{r+1}(m-r-1)!}{2^{r+1}m!} P_m^{(r+1,r+1)}(\alpha).$$

Also, using the fact that $\Gamma_{r,\alpha}(t) = (t-\alpha)\Gamma_{r-1,\alpha}(t)$, and the recurrence formulas for ultraspherical polynomials [23, (4.5.1)], one can deduce the recurrence relations

$$\begin{aligned} C_{m,r} &= \frac{1}{2m+1}(C_{m-1,r-1} - C_{m+1,r-1}), \qquad && r \geq 1, m \geq 1 \\ \alpha C_{m,0} &= \frac{m+3}{2m+3} C_{m+1,0} + \frac{m}{2m+3} C_{m-1,0}, && m \geq 2, \\ C_{1,0} &= \frac{(1-\alpha^2)}{2}. \end{aligned}$$

In our example below, we use $q = 2$, $N_j = 64$, take $g_{j,\ell}$ to be the sampled values on the interval $[0,1]$ of a fifth order cardinal B-spline supported on $[1/2,1]$, and compute $\tau_j(G; f_{\mathbf{e}_3,0.7,4})$. In Figure 3, we show the graph of $y(t)/y_{max}$, where

$$y(t) := \left| \sum_{\ell=0}^{64} g_{64,\ell} C_{\ell,4} P_\ell(t) \right|^4, \qquad t \in [-1,1],$$

and $y_{max} = \max_{t\in[-1,1]} y(t)$. The singularity at 0.7 is detected by the maximum of this graph.

Finally, we remark that in cases where the Fourier coefficients of f cannot be computed or if f exists only as scattered data, quadrature rules such as given in (3.1) can be employed.

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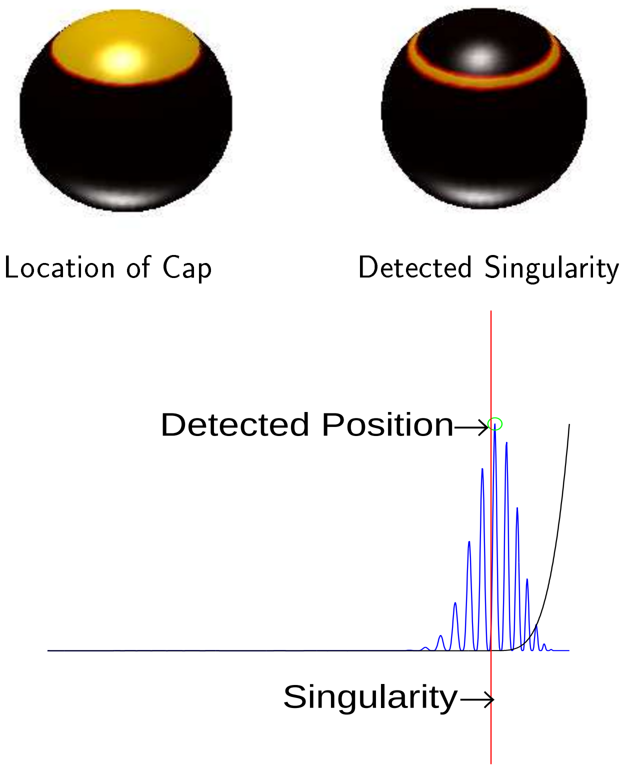

Figure 3. Detection of the singularity of $f_{\mathbf{e}_3,0.7,4}$ at 0.7.

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