Polynomial Schauder basis of optimal degree with Jacobi orthogonality

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Abstract

In our paper we construct a polynomial Schauder basis $(p_{\alpha,\beta,n})_{n\in\mathbb{N}_0}$ of optimal degree with Jacobi orthogonality. A candidate for such a basis is given by the use of some wavelet theoretical methods, which were already successful in case of Tchebysheff and Legendre orthogonality. To prove that this sequence is in fact a Schauder basis for $C[-1,1]$ and as the main difficulty of the whole proof we show the uniform boundedness of its Lebesgue constants

$$\sup_{x\in[-1,1],n\in\mathbb{N}_0}\Big\|\sum_{j=0}^{n}p_{\alpha,\beta,j}(x)p_{\alpha,\beta,j}\Big\|_{L^1_{\omega_{\alpha,\beta}}[-1,1]}<\infty.$$

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1 Introduction

In this paper our main goal is to construct a Schauder basis for the Banach space $(C[-1,1],\|\cdot\|_\infty)$ whose elements consist of algebraic polynomials, are of minimal degree and orthogonal with respect to the weighted inner product $\langle f,g\rangle_{\omega_{\alpha,\beta}} := \langle f,g\rangle :=$

$\int_{-1}^{1} f(x)g(x)\omega_{\alpha,\beta}(x)\mathrm{d}x$, where $\omega_{\alpha,\beta}$ with $\alpha, \beta \geq -\frac{1}{2}$ and $\max\{\alpha, \beta\} > -\frac{1}{2}$ is the usual Jacobi weight with

$$\omega_{\alpha,\beta}(x) = \begin{cases} (1-x)^{\alpha}(1+x)^{\beta} & \text{for all } -1 < x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

As usual we define a Schauder basis to be a sequence of functions $(p_n)_{n\in\mathbb{N}_0} \subseteq C[-1,1]$ such that there exists for every $f \in C[-1,1]$ a uniquely determined sequence $(c_n)_{n\in\mathbb{N}_0} \subseteq \mathbb{R}$ with $f = \sum_{n=0}^{\infty} c_n p_n$ with respect to the $\|\cdot\|_\infty$-norm on $[-1,1]$. If, in addition, the functions p_n are polynomials, then we call the sequence $(p_n)_{n\in\mathbb{N}_0}$ a polynomial Schauder basis. A polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ is said to be optimal for a given $\varepsilon > 0$ if

$$\deg p_n \leq (1+\varepsilon)n \text{ for all } n \in \mathbb{N}_0,$$

where $\deg p_n$ denotes the degree of the polynomial p_n.

Optimal polynomial bases have been an object of intensive discussions in numerical and applied mathematics for over 100 years. For example one of the first famous but negative results was given in 1914 by Faber [3], who showed that there is no polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ with $\deg p_n = n$ for every $n \in \mathbb{N}_0$. In Ul'yanov's formulation the problem was posed again in [26], namely to find a minimal possible growth of degrees of polynomials forming a Schauder basis or an orthogonal Schauder basis for $C[a,b]$. The breakthrough in this area of research was achieved by Privalov in 1987 and 1990: in [16] Privalov showed that for any polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ (orthogonal or not) there exists $\varepsilon > 0$ such that $\max_{j\leq n} \operatorname{grad} p_j \geq (1+\varepsilon)n$ for all sufficiently large n. In [17] Privalov succeeded in showing that the aforementioned result is optimal in the sense that for any $\varepsilon > 0$ there exists a (not necessarily orthogonal) optimal polynomial Schauder basis $(p_n)_{n\in\mathbb{N}_0}$ for $C[-1,1]$, i.e. a polynomial basis with $\deg p_n \leq (1+\varepsilon)n$ for all sufficiently large $n \in \mathbb{N}$. Since that time the research in this area increased rapidly (compare for example Offin and Oskolkow in [11], Privalov in [18] and Ul'yanov in [27]). With these results in mind there was also progress in a more general problem, namely to construct an optimal Schauder basis with a given orthogonality relation for the basis functions: For the trigonometric case $C_{2\pi}$, Lorentz und Sahakian solved the problem in 1994 (see [8], [15]), for the algebraic case $C[-1,1]$ with the four Tchebysheff weights (i.e. $|\alpha| = |\beta| = \frac{1}{2}$ in (1)), that means in a situation rather similar to the trigonometric case, the problem was solved by us in 1996 and 2000 in [7] and [5] respectively. In 2003 Khabiboulline generalized these results in [6] for the Tchebysheff weights multiplied with a Szegö weight. Lastly, in 1999 Skopina in [22], [23] and [24] and a little bit later and independently Woźniakowski succeeded in [28] in solving the problem for the Legendre weight (i.e. $\alpha = \beta = 0$ in (1)) (and thus solving Ul'yanov's problem at all), which was in some sense the simplest non-trigonometric case for the class of Jacobi weights. Now, the main goal of this paper is to generalize the aforementioned results to the case of Jacobi orthogonality.

The general idea is to use wavelet-like constructions which were already successful in cases of the Tchebysheff and Legendre weights. Therefore, we look for polynomials which could be interpreted as wavelet-like or scaling-like functions, of the form

$$\psi_k^{(\alpha,\beta)} := M^{-\frac{1}{2}} \sum_{s=-2M}^{2M} g(\frac{s}{M}) \cos((3M+s)\theta_k) p_{N-M+s}^{(\alpha,\beta)} \text{ for } k=0,...,2M-1,$$

$$\varphi_k^{(\alpha,\beta)} := \begin{cases} g^*(\frac{k}{N}) p_k^{(\alpha,\beta)} + g^*(\frac{2N-k}{N}) p_{2N-k}^{(\alpha,\beta)} & \text{for } k=0,...,N-1, \\ p_N^{(\alpha,\beta)} & \text{for } k=N \end{cases}$$

with $\theta_k := \frac{2k+1}{4M}$ for $k=0,...,2M-1$ and where the $\cos((3M+j)\theta_k)$ could be interpreted as a generalized translation and the g, g^* are suitably constructed with supp $g, g^* \subseteq [-2,2]$ such that single polynomial scales are introduced. Moreover, the Jacobi polynomials $p_s^{(\alpha,\beta)}$ are normalized by

$$\int_{-1}^{1} p_r^{(\alpha,\beta)}(x) p_s^{(\alpha,\beta)}(x) \omega_{\alpha,\beta}(x) \mathrm{d}x = \delta_{rs}$$

and where we set $L_{\alpha,\beta} := L^1_{\omega_{\alpha,\beta}}[-1,1]$ with $\|\cdot\|_{\alpha,\beta} := \|\cdot\|_{L^1_{\omega_{\alpha,\beta}}[-1,1]}$.
The principle idea of that construction consists of
1. a dyadic splitting into wavelet spaces,
2. decomposition of each wavelet space,
3. re-sorting into a new scaling space and a last wavelet packet,
4. choice of a new basis for the scaling space
and - following this plan -
5. estimation of the norms of the projection operators of the scaling spaces and the wavelet spaces.
Compared with the trigonometric and the Tchebysheff cases we only need two window functions g and $\widetilde{g}$ which fulfill rather strong smoothness and symmetry conditions depending on the Jacobi parameter α, β. In fact, we use only two types of functions $g, \widetilde{g}$ defined by

$$g : \mathbb{R} \to \mathbb{R}_0^+, \quad g \text{ even}, \quad g \in C_c^s(\mathbb{R}), \; s \in \mathbb{N}_0, \tag{2}$$

$$\text{supp } g \subseteq [-2,2], \quad g(0) = 1, \tag{3}$$

$$g^2(1+x) + g^2(1-x) = 1 \quad \text{for all } \; x \in [0,1], \tag{4}$$

and

$$\widetilde{g}(x) := \begin{cases} 0 & \text{for } x \leq -3/2, \\ g(2x+1) & \text{for } -3/2 \leq x \leq -1/2, \\ 1 & \text{for } -1/2 \leq x \leq 0, \\ g(x) & \text{for } x \geq 0, \end{cases}$$

where we define $C_c^s(\mathbb{R})$ to be the space of s-times differentiable functions ($s \in \mathbb{N}_0$) with compact support, such that the existence of these functions for a given degree of smoothness s is clear from elementary calculus.
The first four steps of the construction in the Tchebysheff cases can be applied in a one to one way to our case and give us a sequence $(p_{\alpha,\beta,n})_{n\in\mathbb{N}_0}$ consisting of orthonormalized polynomials which are already optimal in their degree (and which fulfill a certain reproduction condition for the polynomials in $C[-1,1]$, see (78) in Theorem 4.1, such that the completeness of our polynomial system is given by construction). Thus, taking into account a standard lemma from the theory of Banach spaces to show that this sequence is a Schauder basis for $C[-1,1]$ is to prove the uniform boundedness of the Lebesgue

constants of $(p_{\alpha,\beta,n})_{n\in\mathbb{N}_0}$, i.e.

$$\sup_{x\in[-1,1],m\in\mathbb{N}_0}\left\|\sum_{n=0}^{m}p_{\alpha,\beta,n}(x)p_{\alpha,\beta,n}(\cdot)\right\|_{\alpha,\beta}\le c\varepsilon^{-2\max\{\alpha,\beta\}-1}.$$

All in all and again taking over the construction from the Tchebysheff cases, we are left to prove the uniform boundedness of two types of Lebesgue-like constants

$$\sup_{s\in[0,\pi]}\int_{-1}^{1}\Big|\sum_{k=0}^{2M-1}\psi_k^{(\alpha,\beta)}(\cos s)\psi_k^{(\alpha,\beta)}(\cos t)\Big|\omega_{\alpha,\beta}(t)\sin t\mathrm{d}t<A, \tag{5}$$

$$\sup_{s\in[0,\pi]}\int_{-1}^{1}\Big|\sum_{k=0}^{N}\varphi_k^{(\alpha,\beta)}(\cos s)\varphi_k^{(\alpha,\beta)}(\cos t)\Big|\omega_{\alpha,\beta}(t)\sin t\mathrm{d}t<A, \tag{6}$$

where the constant $A>0$ depends after the adequate choice of N,M only on the given ε. To prove (5) we approximate the Jacobi polynomials and wavelets $\psi_k^{(\alpha,\beta)}(\cos t)$ (the strategy in case of (6) is essentially the same) by different types of well-localized trigonometric polynomial kernel functions, making heavy use of the relationship between Bessel functions and hypergeometric series on the one hand and Jacobi polynomials on the other hand as well as of certain asymptotics and integral representations for Jacobi polynomials. Unlike in the case of Tchebysheff polynomials reducing the Jacobi polynomials to trigonometric polynomials is not that easy. To overcome this difficulty we cut off the integral of integration in (5) in dependence of s in several parts and combine the aforementioned approximating functions for the $\psi_k^{(\alpha,\beta)}(\cos t)$.

The paper is organized as follows: In Section 2 we evaluate the Lebesgue-like constants of the wavelet-like functions and prove inequality (5), in Section 3 we evaluate the Lebesgue-like constants of the scaling-like functions and prove the inequality (6) and in Section 4 we prove the main result of our paper, Theorem 4.1.

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2 Estimates for inequality (5)

The main result of this section is given in Theorem 2.5. First of all, we prove a useful asymptotic representation for the normalization factors of the classical orthogonal Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$ and the classical orthonormal Jacobi polynomials respectively with

$$\int_{-1}^{1}P_r^{(\alpha,\beta)}(x)P_s^{(\alpha,\beta)}(x)\omega_{\alpha,\beta}(x)\mathrm{d}(x)=h_r^{(\alpha,\beta)}\delta_{rs},$$

$$\int_{-1}^{1}p_r^{(\alpha,\beta)}(x)p_s^{(\alpha,\beta)}(x)\omega_{\alpha,\beta}(x)\mathrm{d}(x)=\delta_{rs}$$

(see [25, Chap. 4]) and

$$h_n^{(\alpha,\beta)}=\frac{2^{\alpha+\beta+1}\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{n!\Gamma(\alpha+\beta+n+1)(\alpha+\beta+2n+1)} \tag{7}$$

(see [13, Lemma 3.4.]). From now on and for the rest of the paper we use $\lambda:=\frac{\alpha+\beta+1}{2}$.

Lemma 2.1 *Let $\alpha, \beta \geq -\frac{1}{2}$ and $n \in \mathbb{N}$. Then*

$$\left(h_n^{(\alpha,\beta)}\right)^{-\frac{1}{2}} = c_0 n^{\frac{1}{2}} + c_1 n^{-\frac{1}{2}} + \mathcal{O}\left(n^{-\frac{3}{2}}\right) \tag{8}$$

and

$$\left(h_n^{(\alpha,\beta)}\right)^{-\frac{1}{2}} \cdot P_n^{(\alpha,\beta)}(1) = d_0 \cdot n^{\alpha+\frac{1}{2}} + d_1 n^{\alpha-\frac{1}{2}} + \mathcal{O}\left(n^{\alpha-\frac{3}{2}}\right) \tag{9}$$

for $n \to \infty$.

Proof of (8): From (7) we get

$$\left(h_n^{(\alpha,\beta)}\right)^{-\frac{1}{2}} = c_{\alpha,\beta} \cdot \sqrt{n} \cdot \sqrt{(1+\frac{\lambda}{n}) \cdot \frac{n^\alpha \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}}{n^\alpha \frac{\Gamma(n+\beta+1)}{\Gamma(n+2\lambda)}}}. \tag{10}$$

In case $\alpha > 0, \beta \geq -\frac{1}{2}$ we get from the asymptotic [12, Chap. 4, §5]

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} B_n^{(\sigma)}(a) \frac{\Gamma(b-a+n)}{\Gamma(b-a)} \frac{1}{z^{b-a+n}}, \quad z \to \infty$$

($a, b, z \in \mathbb{R}$ with $b - a > 0$, $\sigma := a - b + 1$ and $z + a > 0$; $B_n^{(l)}(\alpha)$ denote the generalized Bernoulli polynomials with the special case $B_0^{(l)}(\alpha) = 1$ for all $l, \alpha \in \mathbb{R}$)

$$n^\alpha \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} = 1 - B_1^{(1-\alpha)}(1) \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{n} + \mathcal{O}\left(n^{-2}\right)$$

and

$$n^\alpha \frac{\Gamma(n+\beta+1)}{\Gamma(n+2\lambda)} = 1 - B_1^{(1-\alpha)}(\beta+1) \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)} \frac{1}{n} + \mathcal{O}\left(n^{-2}\right)$$

for $n \to \infty$. Taking into account a standard lemma from the theory of asymptotics (see [12, Chap. 1, §7]) we conclude that with $n^\alpha \frac{\Gamma(u+\beta+1)}{\Gamma(u+\alpha+\beta+1)}$ also $n^{-\alpha} \frac{\Gamma(u+\alpha+\beta+1)}{\Gamma(u+\beta+1)}$ could be asymptotically expanded and we get for the term under the square-root in (10) an asymptotic representation of the form $1 + a_1 \frac{1}{n} + \mathcal{O}(\frac{1}{n^2})$ such that (8) is proved. The other cases including (9) could be proved analogously. ■

In the next theorem we prove some localization properties of the polynomials $\psi_k^{(\alpha,\beta)}(\cos\theta)$. From now on c denotes a positive constant depending only an α, β and g.

Theorem 2.2 *Let $g \in C_c^r(\mathbb{R})$ with $\operatorname{supp} g \subseteq [-2, 2]$ and $r \geq 2$, $N, M \in \mathbb{N}$ with $N > 4M$ and $k = 0, ..., 2M - 1$. Then*
1. for $\alpha > -\frac{1}{2}$, $\beta > -1$ and $0 < \theta \leq \frac{\pi}{2}$

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \leq c \left(\frac{N}{M}\right)^{\alpha+1/2} \int_0^\theta \frac{M^{\alpha+1}(\cos\phi - \cos\theta)^{\alpha-\frac{1}{2}}}{(1+M|\phi-\theta_k|)^r (1-\cos\theta)^\alpha (1+\cos\phi)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\phi, \tag{11}$$

2. for $-\frac{1}{2} \leq \alpha < 0$, $\beta > -\frac{1}{2}$ and $0 < \theta < \pi$

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \leq c \left(\frac{N}{M}\right)^{\beta+1/2} \int_\theta^\pi \frac{M^{\beta+1}(\cos\theta - \cos\phi)^{\beta-\frac{1}{2}}}{(1+M|\phi-\theta_k|)^r (1+\cos\theta)^\beta (1-\cos\phi)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\phi, \tag{12}$$

3. for $\alpha > 0$, $\beta > -\frac{1}{2}$ *and* $0 < \theta < \pi$

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \le c \left(\frac{N}{M}\right)^{\beta+1/2} \int_\theta^\pi \frac{M^{\beta+1}(\cos\theta - \cos\phi)^{\beta-\frac{1}{2}}}{(1+M|\phi-\theta_k|)^r (1+\cos\theta)^\beta (1-\cos\theta)^\alpha (1-\cos\phi)^{\frac{\beta-\alpha}{2}}} \mathrm{d}\phi, \tag{13}$$

4. for $\alpha = 0$, $\beta > -\frac{1}{2}$ *and* $0 < \theta < \pi$

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \le c \left(\frac{N}{M}\right)^{\beta+1/2} \int_\theta^\pi \frac{M^{\beta+1}(\cos\theta - \cos\phi)^{\beta-\frac{1}{2}}}{(1+M|\phi-\theta_k|)^r (1+\cos\theta)^\beta (1-\cos\theta)^{\alpha+1} (1-\cos\phi)^{\frac{\beta-\alpha-2}{2}}} \mathrm{d}\phi, \tag{14}$$

5. for $\alpha = -\frac{1}{2}$, $\beta \ge \frac{1}{2}$ *and* $0 < \theta \le \frac{\pi}{2}$

$$|\psi_k^{(-\frac{1}{2},\beta)}(\cos\theta)| \le c\frac{N}{M} \int_0^\theta \frac{M^{3/2}}{(1+M|\theta-\theta_k|)^r (1-\cos\theta)^{\frac{1}{2}} (1+\cos\theta)^{\frac{1}{4}+\frac{\beta}{2}}} \mathrm{d}\phi, \tag{15}$$

6. for $\alpha > -\frac{1}{2}$, $\beta = -\frac{1}{2}$ *and* $0 < \theta \le \pi$

$$|\psi_k^{(\alpha,-\frac{1}{2})}(\cos\theta)| \le c \left(\frac{N}{M}\right)^{\alpha+1/2} (1+\cos\frac{\theta}{2})^{-\alpha} \int_{\frac{\theta}{2}}^\pi \frac{M^{\alpha+1}}{(1+M|2\vartheta-\theta_k|)^r} \times \begin{cases} \frac{(\cos\frac{\theta}{2}-\cos\vartheta)^{\alpha-\frac{1}{2}}}{(1-\cos\vartheta)^\alpha}\mathrm{d}\vartheta & \text{for } -\frac{1}{2} < \alpha < 0, \\ (1-\cos\frac{\theta}{2})^{-\alpha}\frac{(\cos\frac{\theta}{2}-\cos\vartheta)^{\alpha-\frac{1}{2}}}{(1-\cos\vartheta)^0}\mathrm{d}\vartheta & \text{for } \alpha > 0, \\ (1-\cos\frac{\theta}{2})^{-\alpha-1}\frac{(\cos\frac{\theta}{2}-\cos\vartheta)^{\alpha-\frac{1}{2}}}{(1-\cos\vartheta)^{-1}}\mathrm{d}\vartheta & \text{for } \alpha = 0. \end{cases} \tag{16}$$

Proof of (11)**:** Using the identity from [4, p. 208]

$$\frac{P_k^{(\alpha,\beta)}(\cos\theta)}{P_k^{(\alpha,\beta)}(1)} = d_{\alpha,\beta}(1-\cos\theta)^{-\alpha} \int_0^\theta \cos((k+\lambda)\phi) \cdot \frac{(\cos\phi-\cos\theta)^{\alpha-\frac{1}{2}}}{(1+\cos\phi)^{\frac{\alpha+\beta}{2}}} \times {}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta}{2}; \alpha+\frac{1}{2}; \frac{\cos\phi-\cos\theta}{1+\cos\phi}\right) \mathrm{d}\phi \tag{17}$$

for $0 < \theta < \pi$ and $\alpha > -\frac{1}{2}$, where ${}_2F_1$ is the hypergeometric function (see [2, Chap. 2]), we get

$$\cos((3M+j)\theta_k) \cdot \cos((N-M+j)\phi + \lambda\phi) = \frac{1}{2}\Re e\Big(\mathrm{e}^{\mathrm{i}(3M\theta_k+(N-M)\phi+\lambda\phi)} \cdot \mathrm{e}^{\mathrm{i}j(\phi+\theta_k)} + \mathrm{e}^{\mathrm{i}(-3M\theta_k+(N-M)\phi+\lambda\phi)} \cdot \mathrm{e}^{\mathrm{i}j(\phi-\theta_k)}\Big)$$

and hence

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \le c \int_0^\theta \frac{(|K_M^{(\alpha,\beta)}(\phi-\theta_k)| + |K_M^{(\alpha,\beta)}(\phi+\theta_k)|)(\cos\phi-\cos\theta)^{\alpha-\frac{1}{2}}}{(1-\cos\theta)^\alpha(1+\cos\phi)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\phi,$$

defining

$$K_M^{(\alpha,\beta)}(\theta) := \sum_{j=-2M}^{2M} g\left(\frac{j}{M}\right) p_{N-M+j}^{(\alpha,\beta)}(1)\mathrm{e}^{\mathrm{i}j\theta}.$$

Taking into account that

$$|K_M^{(\alpha,\beta)}(\theta \pm \phi)| \leq c \left(\frac{N}{M}\right)^{\alpha+1/2} \frac{M^{\alpha+3/2}}{(1+M|\theta-\phi|)^r}, \tag{18}$$

which could be proved in almost the same way as in [14, Lemmas 2.2., 2.3.], and that (see [2, Theorems 2.1.1, 2.1.2])

$$\sup_{|x|\leq 1} |\,{}_2F_1(a,b;c;x)| < \infty \quad \text{for all} \quad a,b,c \in \mathbb{R} \quad \text{with} \quad c>0 \quad \text{and} \quad c > a+b \tag{19}$$

as well as $(\cos\phi - \cos\theta)/(1+\cos\phi) \leq \frac{1}{2}$ for $0 \leq \phi \leq \theta \leq \pi/2$, we get (11).

Proof of (12): In view of $P_n^{(\alpha,\beta)}(-x) = (-1)^n P_n^{(\beta,\alpha)}(x)$ we conclude from (17) after a transformation of variable $\phi \to \pi - \phi$

$$\begin{aligned} \frac{P_k^{(\alpha,\beta)}(\cos\theta)}{P_k^{(\beta,\alpha)}(1)} = c_{\beta,\alpha}(1+\cos\theta)^{-\beta} \int_\theta^\pi \cos\left(k\phi - \lambda(\pi-\phi)\right) \frac{(\cos\theta-\cos\phi)^{\beta-\frac{1}{2}}}{(1-\cos\phi)^{\frac{\alpha+\beta}{2}}} \\ \times\ {}_2F_1\left(\frac{\alpha+\beta+1}{2}, \frac{\alpha+\beta}{2}; \beta+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1-\cos\phi}\right) \mathrm{d}\phi \end{aligned} \tag{20}$$

for $0<\theta<\pi$, $\alpha > -1$ and $\beta > -\frac{1}{2}$. With respect to (19) the proof of (12) runs along the same lines.

Proof of (13): Applying (see [2, Theorem 2.2.5])

$${}_2F_1\left(a,b;c;x\right) = (1-x)^{c-a-b}\ {}_2F_1\left(c-a, c-b; c; x\right)$$

with

$$a := \frac{\alpha+\beta}{2}, \quad b := \frac{\alpha+\beta+1}{2} \quad c := \alpha + \frac{1}{2} \quad \text{and} \quad x := \frac{\cos\phi - \cos\theta}{1+\cos\phi}$$

to (17), we get after a transformation of variable $\phi \to \pi - \phi$

$$\begin{aligned} \frac{P_k^{(\alpha,\beta)}(\cos\theta)}{P_k^{(\beta,\alpha)}(1)} = c_{\beta,\alpha} \int_\theta^\pi \frac{\cos\left(k\phi - \lambda(\pi-\phi)\right)(\cos\theta-\cos\phi)^{\beta-\frac{1}{2}}}{(1+\cos\theta)^\beta (1-\cos\theta)^\alpha (1-\cos\phi)^{\frac{\beta-\alpha}{2}}} \\ \times\ {}_2F_1\left(\frac{\beta-\alpha+1}{2}, \frac{\beta-\alpha}{2}; \beta+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1-\cos\phi}\right) \mathrm{d}\phi \end{aligned} \tag{21}$$

and with the arguments from (11) the results follow.

Proof of (14): Due to the fact (see [2, Theorem 2.1.3]) that

$$\lim_{x\to 1-} \frac{{}_2F_1(a,b;a+b;x)}{\log(\frac{1}{1-x})} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)},$$

for all $a,b \in \mathbb{R}$ with $a+b>0$, there exists especially for $\alpha = 0$, $\beta > -\frac{1}{2}$ and $0 < \theta \leq \phi \leq \pi$ a constant $c>0$ with

$$\left|{}_2F_1\left(\frac{\beta-\alpha+1}{2}, \frac{\beta-\alpha}{2}; \beta+\frac{1}{2}; \frac{\cos\theta-\cos\phi}{1-\cos\phi}\right)\right| \leq c\frac{1-\cos\phi}{1-\cos\theta}. \tag{22}$$

With the same arguments from the proof of (11) we obtain (14).

Proof of (15): From trigonometric addition theorems and the identities [1, 22.7.18]

$$P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta}{2n+\alpha+\beta} P_n^{(\alpha+1,\beta)}(x) - \frac{n+\beta}{2n+\alpha+\beta} P_{n-1}^{(\alpha+1,\beta)}(x)$$

and

$$P^{(\alpha+1,\beta)}_{N-M+j}(1) = \frac{N-M+j+\alpha}{\alpha+1} P^{(\alpha,\beta)}_{N-M+j}(1), \quad P^{(\alpha+1,\beta)}_{N-M+j-1}(1) = \frac{N-M+j}{\alpha+1} P^{(\alpha,\beta)}_{N-M+j}(1) \tag{23}$$

(see [13, Corollary 3.2]) we get at first

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \leq c_1(1-\cos\theta)^{-\alpha-1}\sqrt{M}\int_0^\theta |K_{1,M}^{(\alpha,\beta)}(\phi\pm\theta_k)| \frac{(\cos\phi-\cos\theta)^{\alpha+\frac{1}{2}}}{(1+\cos\theta)^{\frac{\alpha+1+\beta}{2}}}\mathrm{d}\phi$$
$$+ c_2(1-\cos\theta)^{-\alpha-1}\sqrt{M}\int_0^\theta |K_{2,M}^{(\alpha,\beta)}(\phi\pm\theta_k)| \frac{(\cos\phi-\cos\theta)^{\alpha+\frac{1}{2}}}{(1+\cos\theta)^{\frac{\alpha+1+\beta}{2}}}\mathrm{d}\phi,$$

where we set

$$K_{1,M}^{(\alpha,\beta)}(\theta) := \sum_{j=-2M}^{2M} g(\frac{j}{M}) p^{(\alpha,\beta)}_{N-M+j}(1)\left(\frac{N+\alpha}{M} - 1 + \frac{j}{M}\right)\frac{\frac{N+\alpha+\beta}{M}+\frac{j}{M}}{\frac{N+\frac{\alpha+\beta}{2}}{M}+\frac{j}{M}}\mathrm{e}^{\mathrm{i}j\theta},$$
$$K_{2,M}^{(\alpha,\beta)}(\theta) := \sum_{j=-2M}^{2M} g(\frac{j}{M}) p^{(\alpha,\beta)}_{N-M+j}(1)\left(\frac{N}{M} - 1 + \frac{j}{M}\right)\frac{\frac{N+\beta}{M}+\frac{j}{M}}{\frac{N+\frac{\alpha+\beta}{2}}{M}+\frac{j}{M}}\mathrm{e}^{\mathrm{i}j\theta}.$$

Taking into account the fact that (see (18))

$$|K_{i,M}^{(\alpha,\beta)}(\phi\pm\theta)| \leq c\frac{N}{(1+M|\phi-\theta|)^r}, \quad i=1,2$$

and arguing along the lines of (11), we obtain (15).
Proof of (16): At first we apply the identity (see [1, 22.5.22])

$$\frac{P_{2n}^{(\alpha,\alpha)}(x)}{P_{2n}^{(\alpha,\alpha)}(1)} = \frac{P_n^{(\alpha,-\frac{1}{2})}(2x^2-1)}{P_n^{(\alpha,-\frac{1}{2})}(1)} \tag{24}$$

to $p^{(\alpha,-1/2)}_{N-M+j}(\cos\theta)$. For $\alpha > 0$ we use (21) and (19) and argue analogously to (11), for $\alpha = 0$ we use (21) and (22) and lastly for $0 > \alpha > -\frac{1}{2}$ we use (20) and (19). ■
In the next theorem we prove some localization properties of the polynomials $\psi_k^{(\alpha,\beta)}(\cos\theta)$ by combining some asymptotics for Jacobi polynomials in terms of Bessel functions $J_\alpha(x)$ (for a precise definition see [2, Chap. 4.5]) or of Bessel functions in terms of trigonometric functions.

Theorem 2.3 *Let $\alpha,\beta \geq -\frac{1}{2}$, $g \in C_c^r(\mathbb{R})$ with* supp $g \subseteq [-2,2]$*, $N, M \in \mathbb{N}$ with $N > 4M$ and $N > \alpha+\beta+1$ and $k = 0,...,2M-1$. Then for all $n \in \mathbb{N}$*

$$|\psi_k^{(\alpha,\beta)}(\cos\theta)| \leq c\theta^{-\alpha}\left(\frac{1}{M(M\theta)^{\frac{1}{2}}} + \frac{M}{(M\theta)^{n+\frac{1}{2}}} + \frac{M}{(M\theta)^{\frac{1}{2}}(1+M|\theta-\theta_k|)^r}\right) \tag{25}$$

for $\frac{1}{M} \leq \theta \leq \frac{5\pi}{6}$. Especially for $\alpha = -\frac{1}{2}$, $\beta > -\frac{1}{2}$ and $0 \leq \theta \leq \frac{5}{6}\pi$ we have

$$|\psi_k^{(-\frac{1}{2},\beta)}(\cos\theta)| \leq c_1\frac{M^{\frac{1}{2}}}{(1+M|\theta-\theta_k|)^r} + c_2 M^{-\frac{3}{2}}. \tag{26}$$

Proof of (25)**:** From [29, Theorem 1] it follows that for all $\alpha, \beta \in \mathbb{R}$ with $\alpha > -1$, $m \in \mathbb{N}$, $\varepsilon > 0$ and $\theta \in (0, \pi - \varepsilon]$

$$P_n^{(\alpha,\beta)}(\cos\theta) = \left(\frac{\theta}{2}\right)^{-\alpha} \left\{ J_\alpha(\tilde{N}\theta) \sum_{k=0}^{m-1} \frac{a_{k,0}(\theta^2)}{\tilde{N}^k} + \theta J'_\alpha(\tilde{N}\theta) \sum_{k=1}^{m-1} \frac{b_{k,0}(\theta^2)}{\tilde{N}^k} + \varepsilon_m(\theta) \right\} \tag{27}$$

and

$$|\varepsilon_m(\theta)| \leq C_m \tilde{N}^{-m} \left\{ |J_\alpha(\tilde{N}\theta)| + \theta |J'_\alpha(\tilde{N}\theta)| \right\}, \tag{28}$$

where $\tilde{N} := n + \frac{\alpha+\beta+1}{2}$ and the functions $a_{k,0}(\theta^2)$, $b_{k,0}(\theta^2)$ are analytic in θ^2 for $|\theta| < \pi$. We also take into account [2, Chap. 4.8] that for all $n \in \mathbb{N}$, $x \geq 1$ we have

$$J_\alpha(x) = \sqrt{\frac{2}{\pi x}} \sum_{\nu=0}^{n-1} (-1)^\nu A_\nu(\alpha) \frac{\cos\left(x - (\frac{1}{2} + \alpha + \nu)\frac{\pi}{2}\right)}{x^\nu} + \widetilde{\varepsilon}_n(x), \quad n \in \mathbb{N},\ x \geq 1, \tag{29}$$

where

$$|\widetilde{\varepsilon}_n(x)| \leq c_{\alpha,n} x^{-(n+\frac{1}{2})} \qquad \text{for all } x \geq 1. \tag{30}$$

Combining (27) for $m = 2$ and $\varepsilon = \frac{\pi}{6}$ with (29), the identities (see [2, Chap. 4.6])

$$\alpha J_\alpha(z) + z J'_\alpha(z) = z J_{\alpha-1}(z), \quad \alpha \in \mathbb{C}, z \in \mathbb{C} \backslash \mathbb{R}^-,$$

(8) and trigonometric addition theorems, we get at first

$$\begin{aligned} |\psi_k^{(\alpha,\beta)}(\cos\theta)| \leq c_2 \theta^{-\alpha} \Bigg(& \frac{1}{M(M\theta)^{\frac{1}{2}}} + \frac{M}{(M\theta)^{n+\frac{1}{2}}} + \sum_{l=0}^{n-1} \frac{|K^{(\alpha,\beta)}_{M,\frac{1}{2},-l-\frac{1}{2}}(\theta+\theta_k)|}{(M\theta)^{l+\frac{1}{2}}} \\ & + \sum_{l=0}^{n-1} \frac{|K^{(\alpha,\beta)}_{M,\frac{1}{2},-l-\frac{3}{2}}(\theta+\theta_k)| + |K^{(\alpha,\beta)}_{M,-\frac{1}{2},-l-\frac{1}{2}}(\theta+\theta_k)|}{(M\theta)^{l+\frac{1}{2}} M} \Bigg) \\ + c_2 \theta^{-\alpha} \Bigg(& \frac{1}{M(M\theta)^{\frac{1}{2}}} + \frac{M}{(M\theta)^{n+\frac{1}{2}}} + \sum_{l=0}^{n-1} \frac{|K^{(\alpha,\beta)}_{M,\frac{1}{2},-l-\frac{1}{2}}(\theta-\theta_k)|}{(M\theta)^{l+\frac{1}{2}}} \\ & + \sum_{l=0}^{n-1} \frac{|K^{(\alpha,\beta)}_{M,\frac{1}{2},-l-\frac{3}{2}}(\theta-\theta_k)| + |K^{(\alpha,\beta)}_{M,-\frac{1}{2},-l-\frac{1}{2}}(\theta-\theta_k)|}{(M\theta)^{l+\frac{1}{2}} M} \Bigg), \end{aligned}$$

where we set

$$K^{(\alpha,\beta)}_{M,\gamma_1,\gamma_2}(\theta) := \sum_{j=-2M}^{2M} g\left(\frac{j}{M}\right) \left(\frac{N}{M} - 1 + \frac{j}{M}\right)^{\gamma_1} \left(\frac{N}{M} - 1 + \frac{\lambda}{M} + \frac{j}{M}\right)^{\gamma_2} \mathrm{e}^{\mathrm{i}j\theta}.$$

Again with nearly the same arguments as in [14, Lemmas 2.2., 2.3.] we show for $\phi, \theta \in [0, \pi]$ with $\phi \neq \theta$

$$|K^{(\alpha,\beta)}_{M,\gamma_1,\gamma_2}(\theta \pm \varphi)| \leq c \left(\frac{N}{M}\right)^{\gamma_1+\gamma_2} \frac{M}{(1 + M|\theta - \varphi|)^r},$$

where $c = c_{\alpha,\beta,r} \cdot \max_{0 \leq \nu \leq r} \|g^{(\nu)}\|_{L^1(\mathbb{R})}$ and (25) is proved.
To show (26) we use the arguments from the proof of (25), taking into account the simplification $J_{-\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}} \cos x$ for all $x \in \mathbb{R} \setminus \{0\}$ and thus getting $\widetilde{\varepsilon}_1(x) \equiv 0$ in view

of (30). ■

From now on let

$$\Psi(s,t,\alpha,\beta,M) := \sum_{k=0}^{2M-1} |\psi_k^{(\alpha,\beta)}(\cos s)| \cdot |\psi_k^{(\alpha,\beta)}(\cos t)|\omega_{\alpha,\beta}(\cos t)\sin t.$$

With the identity $P_n^{(\alpha,\beta)}(x) = (-1)^n P_n^{(\beta,\alpha)}(-x)$ we get after some computations

$$\int_0^\pi \Psi(s,t,\alpha,\beta,M)\mathrm{d}t = \int_0^\pi \Psi(s^*,t,\beta,\alpha,M)\mathrm{d}t \tag{31}$$

for all $s \in [\frac{\pi}{2},\pi]$, $\alpha,\beta \geq -\frac{1}{2}$, setting $s^* := \pi - s$.

Theorem 2.4 *Let $r > \max\{2\alpha+2, \alpha+\beta+3\}$, $g \in C_c^r(\mathbb{R})$ and $\operatorname{supp} g \subseteq [-2,2]$, $N,M \in \mathbb{N}$ with $N > 4M$ and $k = 0,...,2M-1$. Then*

1. for $\alpha > -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$

$$\sup_{s\in[0,\frac{\pi}{2}]} \int_0^{\frac{3}{2M}} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}, \tag{32}$$

2. for $\alpha,\beta > -\frac{1}{2}$

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2}s}^{\pi} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}, \tag{33}$$

3. for $\alpha,\beta > -\frac{1}{2}$

$$\sup_{s\in[0,\frac{1}{M}]} \int_{\frac{3}{2M}}^{\pi} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}, \tag{34}$$

4. for $\alpha > -\frac{1}{2}$, $\beta \geq -\frac{1}{2}$

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2M}}^{\frac{3}{2}s} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{\max\{\alpha,\beta\}+\frac{1}{2}}, \tag{35}$$

5. for $\alpha = -\frac{1}{2}$, $\beta > -\frac{1}{2}$

$$\sup_{s\in[0,\frac{\pi}{2}]} \int_0^{\frac{5}{6}\pi} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c, \tag{36}$$

6. for $\alpha = -\frac{1}{2}$, $\beta > -\frac{1}{2}$

$$\sup_{s\in[0,\frac{\pi}{2}]} \int_{\frac{5}{6}\pi}^{\pi} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\beta+1}, \tag{37}$$

7. for $\alpha > -\frac{1}{2}$, $\beta = -\frac{1}{2}$

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2}s}^{\pi} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}, \tag{38}$$

8. for $\alpha > -\frac{1}{2}$, $\beta = -\frac{1}{2}$

$$\sup_{s\in[0,\frac{1}{M}]} \int_{\frac{3}{2M}}^{\pi} \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}. \tag{39}$$

Proof of (32)**:** From

$$\left|P_\nu^{(\alpha,\beta)}(x)\right| \leq c \cdot \begin{cases} \min\left\{(\nu+1)^\alpha, (1-x)^{-\frac{\alpha}{2}-\frac{1}{4}}(\nu+1)^{-\frac{1}{2}}\right\} & \text{for } 0 \leq x \leq 1, \\ \min\left\{(\nu+1)^\beta, (1+x)^{-\frac{\beta}{2}-\frac{1}{4}}(\nu+1)^{-\frac{1}{2}}\right\} & \text{for } -1 \leq x \leq 0, \end{cases} \tag{40}$$

$(\alpha, \beta \geq -\frac{1}{2},\ \nu \in \mathbb{N}_0)$ (see [25, Theorem 7.32.2]) and $h_n^{(\alpha,\beta)} = \mathcal{O}\left(\frac{1}{n}\right)$ for $n \to \infty$ (see [13, Lemma 3.5.]) we conclude $\left|\psi_k^{(\alpha,\beta)}(\cos t)\right| \leq cM^{\frac{1}{2}}N^{\alpha+\frac{1}{2}}$ for all $0 \leq t \leq \frac{3}{2M}$. From (11) we get in addition

$$|\psi_k^{(\alpha,\beta)}(\cos s)| \leq c\left(\frac{N}{M}\right)^{\alpha+\frac{1}{2}} \int_0^s \frac{M^{\alpha+1}(\cos\phi - \cos s)^{\alpha-\frac{1}{2}}}{(1-\cos s)^\alpha (1+M|\phi-\theta_k|)^r (1+\cos\phi)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\phi.$$

Integrating and taking into account the elementary inequality

$$\sum_{k=0}^{2M-1} \frac{1}{(1+M|\phi-\theta_k|)^r} \leq \sum_{k:\theta_k\leq\phi} \frac{1}{(1+M|\phi-\theta_k|)^r} + \sum_{k:\theta_k>\phi} \frac{1}{(1+M|\phi-\theta_k|)^r} \leq c\sum_{j=1}^{\infty} \frac{1}{j^r} < \infty$$

for $r \geq 2$ and $0 \leq \phi \leq \pi$ and the fact that $\omega_{\alpha,\beta}(\cos t)\sin t \sim t^{2\alpha+1}$, we get (32).
Proof of (33): Let $\alpha > 0$, the proof in case of $-\frac{1}{2} < \alpha < 0$ and $\alpha = 0$ runs along the same lines. At first we apply (11) to $\psi_k^{(\alpha,\beta)}(\cos s)$ and (13) to $\psi_k^{(\alpha,\beta)}(\cos t)$. In view of

$$\sum_{k=0}^{2M-1} \frac{1}{(1+M|\phi-\theta_k|)^r} \frac{1}{(1+M|\vartheta-\theta_k|)^r} \leq c\frac{1}{(1+M|\phi-\vartheta|)^r} \tag{41}$$

(see [21, Lemma 2]),

$$\vartheta - \phi \geq \frac{\vartheta}{3} \tag{42}$$

and

$$\sup_{s\in(0,\frac{\pi}{2}]} (1-\cos s)^{-\alpha} \int_0^s \frac{(\cos\phi-\cos s)^{\alpha-\frac{1}{2}}}{(1+\cos\phi)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\phi < \infty, \qquad \text{for } \alpha > -\frac{1}{2}, \beta \geq -\frac{1}{2}, \tag{43}$$

we conclude

$$\sum_{k=0}^{2M-1} \left|\psi_k^{(\alpha,\beta)}(\cos s)\psi_k^{(\alpha,\beta)}(\cos t)\right| \leq \int_t^\pi \frac{c\left(\frac{N}{M}\right)^{\alpha+\beta+1} M^{\alpha+\beta+2}(\cos t - \cos\vartheta)^{\beta-\frac{1}{2}}}{(1+M|\vartheta|)^r(1+\cos t)^\beta(1-\cos t)^\alpha(1-\cos\vartheta)^{\frac{\beta-\alpha}{2}}} \mathrm{d}\vartheta. \tag{44}$$

Applying the corresponding parts of the proof of [14, Theorem 2.1.], we obtain (33).
Proof of (34): In case of $s \in (0, \frac{1}{M}]$ we argue similarly to the proofs of (33) and especially of (32).
Proof of (35): Applying (25) to $\psi_k^{(\alpha,\beta)}(\cos t)$ for an $n > \alpha + 3/2$ and showing

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2M}}^{\frac{3}{2}s} \sum_{k=0}^{2M-1} |\psi_k^{(\alpha,\beta)}(\cos s)| \left(\frac{M(Mt)^{-\frac{1}{2}}t^{\alpha+1}}{(1+M|t-\theta_k|)^r} + (Mt)^{-\frac{1}{2}}t^{\alpha+1}\right) \mathrm{d}t < \infty$$

and

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2M}}^{\frac{3}{2}s} \sum_{k=0}^{2M-1} |\psi_k^{(\alpha,\beta)}(\cos s)| M(M\theta)^{-n-\frac{1}{2}}t^{\alpha+1}\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{\alpha+1/2},$$

we get (35).
Proof of (36): Here, we apply (26) to $\psi_k^{(-\frac{1}{2},\beta)}(\cos s)$ and $\psi_k^{(-\frac{1}{2},\beta)}(\cos t)$ and notice (41).
Proof of (37): In case $\alpha = -\frac{1}{2}$ and $\frac{1}{2} \geq \beta > -\frac{1}{2}$ we apply (26) to $|\psi_k^{(-\frac{1}{2},\beta)}(\cos s)|$ and (12) to $|\psi_k^{(-\frac{1}{2},\beta)}(\cos t)|$, notice (41), the analog to (42) as well as the inequality

$$\sup_{t\in[\frac{5\pi}{6},\pi]} \int_t^\pi (1+\cos t)^{-\beta} \frac{(\cos t - \cos\theta)^{\beta-\frac{1}{2}}}{(1-\cos\theta)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\theta < \infty.$$

In case $\alpha = -\frac{1}{2}$ and $\beta > \frac{1}{2}$ we apply (15) to $\psi_k^{(\alpha,\beta)}(\cos s)$ and (12) to $\psi_k^{(\alpha,\beta)}(\cos t)$. As with (44) we show first

$$\sum_{k=0}^{2M-1} |\psi_k^{(\alpha,\beta)}(\cos s)|\, |\psi_k^{(\alpha,\beta)}(\cos t)| \leq c \left(\frac{N}{M}\right)^{\beta+3/2} \int_t^\pi \frac{M^{\beta+5/2}(\cos t - \cos\vartheta)^{\beta-\frac{1}{2}}}{(1+M|\vartheta|)^r (1+\cos t)^\beta (1-\cos\vartheta)^{\frac{\alpha+\beta}{2}}} \mathrm{d}\vartheta$$

and take over the corresponding parts of the proof of [14, Theorem 2.1.].
Proof of (38): Here we apply (16) to $\psi_k^{(\alpha,\beta)}(\cos t)$ and (11) to $\psi_k^{(\alpha,\beta)}(\cos s)$. Arguing analogously to the proof of (44), we get first

$$\sum_{k=0}^{2M-1} |\psi_k^{(\alpha,-\frac{1}{2})}(\cos s)\psi_k^{(\alpha,-\frac{1}{2})}(\cos t)| \leq c(\frac{N}{M})^{2\alpha+1} \int_{\frac{t}{2}}^\pi \frac{M^{2\alpha+2}}{(1+2M|\vartheta|)^r} (1+\cos\frac{t}{2})^{-\alpha}$$
$$\times \begin{cases} \frac{(\cos\frac{t}{2}-\cos\vartheta)^{\alpha-\frac{1}{2}}}{(1-\cos\vartheta)^\alpha} \mathrm{d}\vartheta & \text{for } -\frac{1}{2} < \alpha < 0, \\ (1-\cos\frac{t}{2})^{-\alpha} \frac{(\cos\frac{t}{2}-\cos\vartheta)^{\alpha-\frac{1}{2}}}{(1-\cos\vartheta)^0} \mathrm{d}\vartheta & \text{for } \alpha > 0, \\ (1-\cos\frac{t}{2})^{-\alpha-1} \frac{(\cos\frac{t}{2}-\cos\vartheta)^{\alpha-\frac{1}{2}}}{(1-\cos\vartheta)^{-1}} \mathrm{d}\vartheta & \text{for } \alpha = 0, \end{cases}$$

taking over the corresponding arguments from the proof of [14, Theorem 2.1.], we obtain (36). Finally, for the proof of (39) we argue as in the proof of (34). ■

Now we formulate the main result of this section.

Theorem 2.5 *Let* $\alpha, \beta \geq -\frac{1}{2}$ *with* $\max\{\alpha,\beta\} > -\frac{1}{2}$, $N, M \in \mathbb{N}$ *with* $N > 4M$, $g \in C_c^r(\mathbb{R})$ *with* $r > \max\{\alpha+\beta+3, 2\alpha+2, 2\beta+2, 2\}$ *and* $\operatorname{supp} g \subseteq [-2,2]$. *Then*

$$\sup_{s\in[0,\pi]} \int_0^\pi \Psi(s,t,\alpha,\beta,M)\mathrm{d}t \leq c \left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}. \tag{45}$$

Proof: Let $s \in [0, \frac{\pi}{2}]$ first. In dependence of α, β and s we divide the integral in (45) as follows:

$$\int_0^\pi = \begin{cases} \int_0^{\frac{3}{2M}} + \int_{\frac{3}{2M}}^\pi & \text{for } \alpha,\beta > -\frac{1}{2}, s \in [0, \frac{1}{M}], \\ \int_0^{\frac{3}{2M}} + \int_{\frac{3}{2M}}^{\frac{3}{2}s} + \int_{\frac{3}{2}s}^\pi & \text{for } \alpha,\beta > -\frac{1}{2}, s \in [\frac{1}{M}, \frac{\pi}{2}], \\ \int_0^{\frac{5}{6}\pi} + \int_{\frac{5}{6}\pi}^\pi & \text{for } \alpha = -\frac{1}{2}, \beta > -\frac{1}{2}, s \in [0, \frac{\pi}{2}], \\ \int_0^{\frac{3}{2M}} + \int_{\frac{3}{2M}}^\pi & \text{for } \alpha > -\frac{1}{2}, \beta = -\frac{1}{2}, s \in [0, \frac{1}{M}], \\ \int_0^{\frac{3}{2M}} + \int_{\frac{3}{2M}}^{\frac{3}{2}s} + \int_{\frac{3}{2}s}^\pi & \text{for } \alpha > -\frac{1}{2}, \beta = -\frac{1}{2}, s \in [\frac{1}{M}, \frac{\pi}{2}]. \end{cases}$$

First case: The summand on the left can be estimated by (32), the summand on the right by (34). Second case: the summand on the left can be estimated by (32), the summand in the middle by (35) and the summand on the right by (33). Third case: The summand on the left can be estimated by (36), the summand on the right by (37). Fourth case: The summand on the left can be estimated by (32), the summand on the right by (39). Fifth case: The summand on the left can be estimated by (32), the summand in the middle by (35) and the summand on the right by (38). In case $s \in [\frac{\pi}{2}, \pi]$ we reduce the integral (45) with the help of (31) to one of the foregoing cases. ■

3 Estimates for inequality (6)

With the following lemma we divide the left-hand side of (6) into two summands which we will evaluate separately.

Lemma 3.1 *Let* $\alpha, \beta \geq -\frac{1}{2}$, $N, M \in \mathbb{N}$ *with* $N > 3M$ *and* g *be a function fulfilling* (2), (3),(4). *Then*

$$\left|\sum_{k=0}^{N} \varphi_k^{(\alpha,\beta)}(\cos s)\varphi_k^{(\alpha,\beta)}(\cos t)\right| \leq |\Phi(\alpha,\beta,N,M,s,t)| + \left|\widetilde{\Phi}(\alpha,\beta,N,M,s,t)\right| + \left(p_0^{(\alpha,\beta)}\right)^2, \quad (46)$$

where we defined

$$\begin{aligned}
\Phi(\alpha,\beta,N,M,s,t) &:= \sum_{k=-M}^{M} g\left(\frac{M-k}{M}\right) g\left(\frac{M+k}{M}\right) p_{N+k}^{(\alpha,\beta)}(\cos s) p_{N-k}^{(\alpha,\beta)}(\cos t), \\
\widetilde{\Phi}(\alpha,\beta,N,M,s,t) &:= \sum_{k=1}^{2N} \left(g^*\left(\frac{k}{N}\right)\right)^2 p_k^{(\alpha,\beta)}(\cos s) p_k^{(\alpha,\beta)}(\cos t) \text{ and} \\
g^*(x) &:= \begin{cases} g(0) & \text{for } 0 \leq x \leq 1 - \frac{M}{N}, \\ g(\frac{N}{M}(x-1)+1) & \text{for } x > 1 - \frac{M}{N}, \\ g^*(-x) & \text{for } x \leq 0. \end{cases}
\end{aligned}$$

Proof: The proof is obtained from an easy calculation, taking into account that with g the function g^* also fulfills the properties (2), (3), (4). ■

Now, we prove

Theorem 3.2 *Let* $g \in C_c^r(\mathbb{R})$ *with* $r > 2$, $\operatorname{supp} g \subseteq [-2,2]$ *and* $N, M \in \mathbb{N}$ *with* $N > 4M$. *Then*
1. for $\alpha, \beta \geq -\frac{1}{2}$

$$\sup_{s \in [0,\frac{\pi}{2}]} \int_0^{\frac{3}{2M}} \left|\Phi(\alpha,\beta,N,M,s,t)\right| \omega_{\alpha,\beta}(\cos t) \sin t dt \leq c \left(\frac{N}{M}\right)^{2\alpha+1}, \quad (47)$$

2. *for* $\alpha, \beta > -\frac{1}{2}$

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2}s}^{\pi} \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c\left(\frac{N}{M}\right)^{2\alpha+1}, \tag{48}$$

3. *for* $\alpha, \beta > -\frac{1}{2}$

$$\sup_{s\in[0,\frac{1}{M}]} \int_{\frac{3}{2M}}^{\pi} \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c\left(\frac{N}{M}\right)^{2\alpha+1}, \tag{49}$$

4. *for* $\alpha, \beta \ge -\frac{1}{2}$

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2M}}^{\frac{3}{2}s} \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c, \tag{50}$$

5. *for* $\alpha = -\frac{1}{2}, \beta > -\frac{1}{2}$

$$\sup_{s\in[0,\frac{\pi}{2}]} \int_{0}^{\frac{5}{6}\pi} \left|\Phi(\alpha,\beta,N,M,s,t)\right|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c, \tag{51}$$

6. *for* $\alpha = -\frac{1}{2}, \beta > -\frac{1}{2}$

$$\sup_{s\in[0,\frac{\pi}{2}]} \int_{\frac{5}{6}\pi}^{\pi} \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c\left(\frac{N}{M}\right)^{2\beta+1}, \tag{52}$$

7. *for* $\alpha > -\frac{1}{2}, \beta = -\frac{1}{2}$

$$\sup_{s\in[\frac{1}{M},\frac{\pi}{2}]} \int_{\frac{3}{2}s}^{\pi} \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c\left(\frac{N}{M}\right)^{2\alpha+1}, \tag{53}$$

8. *for* $\alpha > -\frac{1}{2}$, $\beta = -\frac{1}{2}$

$$\sup_{s\in[0,\frac{1}{M}]} \int_{\frac{3}{2M}}^{\pi} \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \le c\left(\frac{N}{M}\right)^{2\alpha+1}. \tag{54}$$

Proof: From (40) we immediately conclude (47).
Proof of (48): Here, we apply the same strategies as in the proofs of (11) and (12) or (13) or (14) to get at first

$$|\Phi(\alpha,\beta,N,M,s,t)| \le c(1-\cos s)^{-\alpha}(1+\cos t)^{-\beta}\int_t^{\pi}\int_0^{s}\left|\widetilde{K}_{1,M}^{(\alpha,\beta)}(\theta\pm\phi)\right|$$

$$\times\frac{(\cos\phi-\cos s)^{\alpha-\frac{1}{2}}}{(1+\cos\phi)^{\frac{\alpha+\beta}{2}}}\begin{cases}\frac{(\cos t-\cos\theta)^{\beta-\frac{1}{2}}}{(1-\cos\theta)^{\frac{\alpha+\beta}{2}}}\mathrm{d}\phi\mathrm{d}\theta & \text{for } -\frac{1}{2}<\alpha<0,\\ (1-\cos t)^{-\alpha}\frac{(\cos t-\cos\theta)^{\beta-\frac{1}{2}}}{(1-\cos\theta)^{\frac{\alpha-\beta}{2}}}\mathrm{d}\phi\mathrm{d}\theta & \text{for } \alpha>0,\\ \frac{(\cos t-\cos\theta)^{\beta-\frac{1}{2}}}{(1-\cos\theta)^{\frac{\alpha+\beta}{2}}}\frac{1-\cos\theta}{1-\cos t}\mathrm{d}\phi\mathrm{d}\theta & \text{for } \alpha=0,\end{cases}$$

where we define

$$\widetilde{K}_{1,M}^{(\alpha,\beta)}(\theta) := \sum_{j=-M}^{M} g(1-\frac{j}{M})g(1+\frac{j}{M})p_{N+j}^{(\alpha,\beta)}(1)p_{N-j}^{(\beta,\alpha)}(1)\mathrm{e}^{\mathrm{i}j\theta}.$$

Similar to the proof of [14, Lemmas 2.2., 2.3.] we show again

$$|\widetilde{K}_{1,M}^{(\alpha,\beta)}(\phi\pm\theta)| \le c\left(\frac{N}{M}\right)^{\alpha+\beta+1}\frac{M^{\alpha+\beta+2}}{(1+M|\phi-\theta|)^r}$$

for all $0\le\phi,\theta\le\pi$. From this, taking into account $|\vartheta-\phi|=\vartheta-\phi\ge\frac{\vartheta}{4}$ and (43) and arguing along the same lines as in the corresponding parts of the proof of [14, Theorem 2.1.] we get (48).
Proof of (49): Inequality (42) is fulfilled by the choice of the limits of the integration so that we can take over the proof of (48).
Proof of (50): With $m=1$, $n=1$ and $\varepsilon_0=\frac{\pi}{6}$ we conclude from (27), (28), (29) and (30)

$$P_{N-j}^{(\alpha,\beta)}(\cos t) = \frac{ca_{0,0}(t^2)\cos\left((N-j+\lambda)\,t-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}\right)t^{-\alpha}}{\sqrt{(N-j+\lambda)\,t}}+\epsilon_1(t)t^{-\alpha}$$

and

$$P_{N+j}^{(\alpha,\beta)}(\cos s) = \frac{ca_{0,0}(s^2)\cos\left((N+j+\lambda)\,s-\left(\alpha+\frac{1}{2}\right)\frac{\pi}{2}\right)s^{-\alpha}}{\sqrt{(N+j+\lambda)\,s}}+\epsilon_2(s)s^{-\alpha},$$

where

$$|\epsilon_1(t)|\le c_{\varepsilon_0}\frac{1}{Nt\sqrt{Nt}} \quad\text{and}\quad |\epsilon_2(s)|\le c_{\varepsilon_0}\frac{1}{Ns\sqrt{Ns}} \tag{55}$$

for all $s\in[\frac{1}{M},\frac{\pi}{2}]$, $t\in[\frac{3}{2M},\frac{5}{6}\pi]$. From this, taking into account trigonometric addition theorems, the asymptotic representation

$$(h_{N-j}^{(\alpha,\beta)})^{-1/2}(h_{N+j}^{(\alpha,\beta)})^{-1/2} = \sqrt{N-j}\sqrt{N+j}+\mathcal{O}(1)$$

(which is an immediate consequence of (8)) and

$$\left|\sum_{j=-M}^{M} g(1-\frac{j}{M})g(1+\frac{j}{M})\frac{\left(\frac{N}{M}-\frac{j}{M}\right)^{\frac{1}{2}}}{\left(\frac{N+\lambda}{M}-\frac{j}{M}\right)^{\frac{1}{2}}}\frac{\left(\frac{N}{M}+\frac{j}{M}\right)^{\frac{1}{2}}}{\left(\frac{N+\lambda}{M}+\frac{j}{M}\right)^{\frac{1}{2}}}\mathrm{e}^{\mathrm{i}j(\theta\pm\phi)}\right| \le c\frac{M}{(1+M|\theta-\phi|)^r}$$

for all $|\theta|,|\phi|\le\pi$ (which is again proved as in [14, Lemmas 2.2., 2.3.]) we finally conclude

$$|\Phi(\alpha,\beta,N,M,s,t)| \le \frac{1}{2}\frac{M}{(1+M|s-t|)^r}\frac{t^{-\alpha}s^{-\alpha}}{\sqrt{s}\sqrt{t}}+c\left(\frac{1}{s\sqrt{st}}+\frac{1}{t\sqrt{st}}\right)s^{-\alpha}t^{-\alpha} \tag{56}$$

and after integration we obtain (50).
Proof of (51): We argue similarly to the proof of (50). Since $\alpha=-\frac{1}{2}$ we have $\widetilde{\varepsilon_1}\equiv 0$ in (29). Thus we can improve (55) to

$$|\epsilon_1(t)|\le c\frac{1}{N\sqrt{Nt}} \quad\text{and}\quad |\epsilon_2(s)|\le c\frac{1}{N\sqrt{Ns}}$$

such that we get in contrast to (56)

$$|\Phi(\alpha,\beta,N,M,s,t)| \le \frac{M}{2(1+M|s-t|)^r} + c$$

and by integration we obtain (51).
Proof of (52): In case $\beta \ge \frac{1}{2}$ we apply (23) to $p_{N+j}^{(\alpha,\beta)}(\cos s)$ and (20) to $p_{N-j}^{(\alpha,\beta)}(\cos t)$. From trigonometric addition theorems and the identities in (23) we conclude

$$\Big|\Phi(\alpha,\beta,N,M,s,t)\Big| \le cM \int_t^\pi \int_0^s (|\widetilde{K}_{3,M}^{(\alpha,\beta)}(\theta\pm\phi)| + |\widetilde{K}_{4,M}^{(\alpha,\beta)}(\theta\pm\phi)|)$$
$$\times(1-\cos s)^{-\alpha-1}(1+\cos t)^{-\beta}\frac{(\cos\phi-\cos s)^{\alpha+\frac{1}{2}}}{(1+\cos s)^{\frac{\alpha+1+\beta}{2}}}\frac{(\cos t-\cos\theta)^{\beta-\frac{1}{2}}}{(1-\cos\theta)^{\frac{\alpha+\beta}{2}}}\mathrm{d}\phi\mathrm{d}\theta,$$

where we set

$$\widetilde{K}_{3,M}^{(\alpha,\beta)}(\theta) := \sum_{j=-M}^{M} g(1-\frac{j}{M})g(1+\frac{j}{M})\left(\frac{N+\alpha}{M}+\frac{j}{M}\right)\frac{\frac{N+\alpha+\beta}{M}+\frac{j}{M}}{\frac{N+\frac{\alpha+\beta}{2}}{M}+\frac{j}{M}}p_{N+j}^{(\alpha,\beta)}(1)p_{N-j}^{(\beta,\alpha)}(1)\mathrm{e}^{\mathrm{i}j\theta}$$

or

$$\widetilde{K}_{4,M}^{(\alpha,\beta)}(\theta) := \sum_{j=-M}^{M} g(1-\frac{j}{M})g(1+\frac{j}{M})\left(\frac{N}{M}+\frac{j}{M}\right)\frac{\frac{N+\beta}{M}+\frac{j}{M}}{\frac{N+\frac{\alpha+\beta}{2}}{M}+\frac{j}{M}}p_{N+j}^{(\alpha,\beta)}(1)p_{N-j}^{(\beta,\alpha)}(1)\mathrm{e}^{\mathrm{i}j\theta}.$$

Again as in the proof of [14, Lemmas 2.2., 2.3.], we show for $i=3,4$

$$|\widetilde{K}_{i,M}^{(\alpha,\beta)}(\theta\pm\phi)| \le c\left(\frac{N}{M}\right)^{\alpha+\beta+2}\frac{M^{\alpha+\beta+2}}{(1+M|\theta-\phi|)^r} \quad \text{for all } |\theta|,|\phi|\le\pi.$$

From this, taking into account (43), $\theta-\phi\ge\frac{\pi}{3}$ and

$$\sup_{t\in[\frac{\pi}{2},\pi]}\int_t^\pi (1+\cos t)^{-\beta}\frac{(\cos t-\cos\theta)^{\beta-\frac{1}{2}}}{(1-\cos\theta)^{\frac{\beta-\alpha}{2}}}\mathrm{d}\theta < \infty,$$

we get the inequality (52) in case $\beta\ge\frac{1}{2}$.
Now, let $-\frac{1}{2}<\beta<\frac{1}{2}$. From (27) for $m=2$ and $\varepsilon=\frac{\pi}{6}$, the identities $J_{\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\sin z$, $J_{-\frac{1}{2}}(z)=\sqrt{\frac{2}{\pi z}}\cos z$ (see [2, Chap. 4.6]) and (27) we get first

$$P_{N+j}^{(-\frac{1}{2},\beta)}(\cos s) = \left(\frac{s}{2}\right)^{\frac{1}{2}}\left(\frac{\sqrt{2}\cos(N^*s)}{\sqrt{\pi N^* s}}\left(a_{0,0}(s^2)+\frac{a_{1,0}(s^2)}{N^*}\right)+\frac{1}{N^*}\frac{1}{\sqrt{\pi N^* s}}\cos(N^*s)\frac{b_{1,0}(s^2)}{N^*}\right.$$
$$\left.-\sqrt{\frac{2s}{\pi N^*}}\cos(N^*s-\frac{\pi}{2})\frac{b_{1,0}(s^2)}{N^*}+\varepsilon_2\right), \tag{57}$$

where we set $N^* := N+j+\frac{\alpha+\beta+1}{2}$ and notice that

$$|\varepsilon_2(s)| \le c\frac{1}{N^2}\frac{1}{\sqrt{Ns}} \tag{58}$$

because of $\frac{1}{2}N \le N^* \le 2N$ and the choice of N^*.

From this together with (57), trigonometric addition theorems and applying (20) to $P_{N-j}^{(-\frac{1}{2},\beta)}(\cos t)$ we get

$$\Big|\Phi(\alpha,\beta,N,M,s,t)\Big| \le c_1 \int_t^{\pi} \sum_{i=1}^{3} M^{\frac{1}{2}-i} |\widetilde{K}_{i,M}^{(\beta,-\frac{1}{2})}(s\pm\vartheta)|(1+\cos t)^{-\beta} \frac{(\cos t-\cos\vartheta)^{\beta-\frac{1}{2}}}{(1-\cos\vartheta)^{\frac{\beta+\alpha}{2}}} \mathrm{d}\vartheta$$
$$+ c_2 M N^{\beta-3/2},$$

where we set for $i=5,6,7$

$$\widetilde{K}_{i,M}^{(-\frac{1}{2},\beta)}(\theta) := \sum_{j=-M}^{M} g(1-\frac{j}{M}) g(1+\frac{j}{M}) \left(\frac{N+\lambda}{M}-\frac{j}{M}\right)^{\frac{1}{2}+4-i} \left(h_{N-j}^{(-\frac{1}{2},\beta)} h_{N+j}^{(-\frac{1}{2},\beta)}\right)^{-\frac{1}{2}} P_{N+j}^{(\beta,-\frac{1}{2})}(1)\mathrm{e}^{\mathrm{i}j\theta}.$$

Again, as in the proof of [14, Lemmas 2.2., 2.3.], we prove

$$\Big|\widetilde{K}_{i,M}^{(-\frac{1}{2},\beta)}(s\pm\vartheta)\Big|\,\theta \le c\left(\frac{N}{M}\right)^{\beta+5.5-i} \frac{M^{\beta+2}}{(1+M|s-\vartheta|)^r} \quad \text{for all } |s|,|\vartheta|\le\pi.$$

From this together with (57) and (58), and taking into account $\vartheta - s \ge \frac{\pi}{3}$ as in (33), we get the inequality (52) in case $-\frac{1}{2}<\beta<\frac{1}{2}$.
Proof of (53): From (24), (12) or (13) or (14) respectively and (11), applying trigonometric addition theorems, we conclude

$$\Big|\Phi(\alpha,\beta,N,M,s,t)\Big| \le c(1+\cos\frac{t}{2})^{-\alpha}(1-\cos s)^{-\alpha} \int_0^s \int_{\frac{t}{2}}^{\pi} \frac{(\cos\phi-\cos s)^{\alpha-\frac{1}{2}}}{(1+\cos\phi)^{\frac{\alpha+\beta}{2}}}$$
$$\times|\widetilde{K}_{8,M}(\phi\pm 2\theta)| \begin{cases} \frac{(\cos\frac{t}{2}-\cos\theta)^{\alpha-\frac{1}{2}}}{(1-\cos\theta)^{\alpha}}\mathrm{d}\theta\mathrm{d}\phi & \text{for } -\frac{1}{2}<\alpha<0, \\ (1-\cos\frac{t}{2})^{-\alpha}\frac{(\cos\frac{t}{2}-\cos\theta)^{\alpha-\frac{1}{2}}}{(1-\cos\theta)^{0}}\mathrm{d}\theta\mathrm{d}\phi & \text{for } \alpha>0, \\ (1-\cos\frac{t}{2})^{-\alpha-1}\frac{(\cos\frac{t}{2}-\cos\theta)^{\alpha-\frac{1}{2}}}{(1-\cos\theta)^{-1}}\mathrm{d}\theta\mathrm{d}\phi & \text{for } \alpha=0, \end{cases}$$

where we set

$$\widetilde{K}_{8,M}^{(\alpha,-\frac{1}{2})}(\theta) := \sum_{j=-M}^{M} g(1-\frac{j}{M}) g(1+\frac{j}{M}) p_{N+j}^{(\alpha,-\frac{1}{2})}(1) p_{N-j}^{(\alpha,-\frac{1}{2})}(1)\mathrm{e}^{\mathrm{i}j\theta}.$$

Again, as in the proof of [14, Lemmas 2.2., 2.3.], we show

$$\Big|\widetilde{K}_{8,M}^{(\alpha,-\frac{1}{2})}(\theta\pm\phi)\Big| \le c\left(\frac{N}{M}\right)^{2\alpha+1} \frac{M^{2\alpha+2}}{(1+M|\theta-\phi|)^r}$$

for all $|\theta|,|\phi|\le\pi$. Taking into account that $|2\theta-\phi|\ge\frac{1}{10}\theta$, we argue analogously to (33) in case $\alpha=\beta$ and (53) is proved. ■

With exactly the same arguments from the proof of Theorem 2.5 and taking into account equations analogous to (31), we get as the first main result of this section:

Theorem 3.3 *Let $\alpha,\beta \geq -\frac{1}{2}$ with $\max\{\alpha,\beta\} > -\frac{1}{2}$, $N,M \in \mathbb{N}$ with $N > 4M$ and $N \geq \alpha+\beta+1$ as well as $g \in C_c^r(\mathbb{R})$ with $r \geq 2\max\{\alpha,\beta\}+3$ and $\operatorname{supp} g \subseteq [-2,2]$. Then*

$$\sup_{s\in[0,\pi]} \int_0^\pi \Big|\Phi(\alpha,\beta,N,M,s,t)\Big|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}. \tag{59}$$

Now, we evaluate the second summand of (46). As usually, we define the difference operator Δ for a given sequence $(a_\nu)_{\nu\in\mathbb{N}_0}$ or function $a:\mathbb{R}\to\mathbb{R}$ by

$$\begin{gathered}\Delta a_\nu = \Delta^1 a_\nu = a_{\nu+1} - a_\nu \text{ and } \quad \Delta^k a_\nu = \Delta(\Delta^{k-1}a_\nu) \text{ or}\\ \Delta a(u) = \Delta^1 a(u) = a(u+1) - a(u) \text{ and } \Delta^k a(u) = \Delta(\Delta^{k-1}a(u))\end{gathered} \tag{60}$$

for $k\in\mathbb{N}$ with $k\geq 2$. Obviously, we have for all $m\in\mathbb{N}$ and $l\in\mathbb{N}_0$ with $m+l\leq r$

$$\operatorname{supp} \Delta^m\, g^*_{\frac{M}{N}}(\frac{u}{N}) \subseteq [N-M-m, N+M], \tag{61}$$

$$\|\frac{\mathrm{d}^l}{\mathrm{d}u^l}\Delta^m\, g^*_{\frac{N}{M}}(\frac{u}{N})\|_\infty \leq \frac{1}{N^{m+l}}\left(\frac{N}{M}\right)^{m+l}\|g^{(m+l)}\|_\infty. \tag{62}$$

We formulate the second main result of this section.

Theorem 3.4 *Let $\alpha,\beta\geq -\frac{1}{2}$ with $\max\{\alpha,\beta\} > -\frac{1}{2}$ and $N,M\in\mathbb{N}$ given with $N>4M$ and $N > 2\max\{\alpha,\beta\}+3$. Then*

$$\sup_{s\in[0,\pi]} \int_0^\pi |\tilde{\Phi}(\alpha,\beta,N,M,s,t)|\omega_{\alpha,\beta}(\cos t)\sin t\mathrm{d}t \leq c\left(\frac{N}{M}\right)^{2\max\{\alpha,\beta\}+1}. \tag{63}$$

Proof: We use [10, Theorem 5.1]

$$\begin{aligned}&|\sum_{k=0}^\infty h_k p_k^{(\alpha,\beta)}(\cos\theta)p_k^{(\alpha,\beta)}(\cos\varphi)|\\ &\leq c\sum_{k=0}^\infty \min\big\{(k+1),\frac{1}{|\theta-\varphi|}\big\}^{\max\{\alpha,\beta\}+S+\frac{1}{2}}\sum_{m=1}^S (k+1)^{\max\{\alpha,\beta\}+\frac{1}{2}-S+m}|\Delta^m h_k|,\end{aligned} \tag{64}$$

where Δ is defined by (60), $\alpha,\beta\geq -\frac{1}{2}$, $S\in\mathbb{N}$ and $(h_k)_{k\in\mathbb{N}_0}$ is a sequence of real numbers with $h_k = 0$ for all sufficiently large k.
In case $\alpha = \max\{\alpha,\beta\}\geq 0$ we set $s=0$, $S := \lfloor\alpha+\frac{3}{2}\rfloor + 1$. Taking into account that $\min\{(j+1),\frac{1}{t}\}^{\alpha+S+\frac{1}{2}} \leq c\frac{N^{\alpha+S+\frac{1}{2}}}{(1+Nt)^{\alpha+S+\frac{1}{2}}}$, we get from (64), (61) and (62) first

$$|\tilde{\Phi}(\alpha,\beta,N,M,s,t)| \leq c\left(\frac{N}{M}\right)^{S-1}\frac{N^{2\alpha+2}}{(1+Nt)^{\alpha+S+\frac{1}{2}}},$$

and after integration we obtain (63).
Now let $-\frac{1}{2} < \max\{\alpha,\beta\} = \alpha < 0$. For $0 < t \leq \frac{1}{M}$ we choose $S=1$ and from (64) with $s=0$ we conclude

$$|\tilde{\Phi}(\alpha,\beta,N,M,s,t)| \leq c\frac{N^{\alpha+\frac{3}{2}}}{(1+Nt)^{2\alpha+2}} \leq cN^{2\alpha+1}\frac{1}{t}. \tag{65}$$

For $\frac{1}{M} \le t \le \frac{\pi}{2}$ we choose $S = 2$ and from (64) again with $s = 0$ we conclude

$$|\tilde{\Phi}(\alpha, \beta, N, M, s, t)| \le c \left(\frac{N}{M}\right)^{2\alpha+1} \frac{M^{2\alpha} N^2}{(1 + Nt)^{\alpha+\frac{5}{2}}} \le c \left(\frac{N}{M}\right)^{2\alpha+1} \frac{M^{2\alpha+2}}{(1 + Mt)^{\alpha+\frac{5}{2}}}. \tag{66}$$

Dividing the integral in (63) by $\int_0^\pi = \int_0^{\frac{1}{M}} + \int_{\frac{1}{M}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\pi}$ and applying (65) and (66), the inequality (63) is proved in case $-\frac{1}{2} < \max\{\alpha, \beta\} = \alpha < 0$. The case $\beta > \alpha$ is immediately clear from $P_n^{\alpha,\beta}(-x) = (-1)^n P_n^{\beta,\alpha}(x)$.
The needed assertion now follows from Lemma 4.6. in [9], i.e.,

$$\sup_{x\in[-1,1]} \| \sum_{k=0}^{\infty} h_k p_k^{(\alpha,\beta)}(x) p_k^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta} \le c\| \sum_{k=0}^{\infty} h_k p_k^{(\alpha,\beta)}(1) p_k^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta},$$

where again $\alpha, \beta \ge -\frac{1}{2}$ and $(h_k)_{k\in\mathbb{N}_0}$ is a sequence of real numbers such that $h_k = 0$ for all sufficiently large k. ■

4 Proof of the main result

Here, we give the main result of our paper.

Theorem 4.1 *Let* $\alpha, \beta \ge -\frac{1}{2}$ *with* $\max\{\alpha, \beta\} > -\frac{1}{2}$, $\varepsilon > 0$ *and* $\eta \in \mathbb{N}$ *with* $\frac{3}{2^\eta} \le \varepsilon$, $\eta \ge 3$ *and* $2^\eta > \alpha + \beta + 1$. *Let* g, $\widetilde{g}$ *be functions with*

$$g : \mathbb{R} \to \mathbb{R}_0^+, \quad g \text{ even}, \quad g \in C_c^s(\mathbb{R}) \text{ with } s > 2\max\{\alpha, \beta\} + 3, \tag{67}$$

$$\operatorname{supp} g \subseteq [-2, 2], \quad g(0) = 1, \tag{68}$$

$$g^2(1 + x) + g^2(1 - x) = 1 \text{ for all } x \in [0, 1], \tag{69}$$

and

$$\widetilde{g}(x) := \begin{cases} 0 & \text{for } x \le -3/2, \\ g(2x + 1) & \text{for } -3/2 \le x \le -1/2, \\ 1 & \text{for } -1/2 \le x \le 0, \\ g(x) & \text{for } x \ge 0. \end{cases}$$

For $n \in \mathbb{N}$ *with* $n > 2^\eta$ *choose the uniquely given numbers*

$$l \in \mathbb{N}_0, \quad r \in \{1, ..., 2^{\eta-1}\} \quad \text{and } k \in \{0, ..., 2^{l+1} - 1\} \tag{70}$$

such that

$$n = (2^\eta + 2r - 2)2^l + 1 + k, \tag{71}$$

set

$$M := 2^l, \quad N := (2^\eta + 2r)2^l \quad \text{and} \quad \theta_k := \frac{2k + 1}{4M}\pi \tag{72}$$

and define the sequence of polynomials $(p_{\alpha,\beta,n})_{n\in\mathbb{N}_0}$ *by*

$$p_{\alpha,\beta,n} := p_n^{(\alpha,\beta)} \text{ for all } n \in \mathbb{N} \text{ with } n \le 2^\eta, \tag{73}$$

$$p_{\alpha,\beta,n} := \frac{1}{\sqrt{M}} \sum_{j=-2M}^{2M} \widetilde{g}\left(\frac{j}{M}\right) \cos((3M + j)\theta_k) p_{N-M+j}^{(\alpha,\beta)} \text{ for } r = 1, \tag{74}$$

$$p_{\alpha,\beta,n} := \frac{1}{\sqrt{M}} \sum_{j=-2M}^{2M} g\left(\frac{j}{M}\right) \cos((3M + j)\theta_k) p_{N-M+j}^{(\alpha,\beta)} \text{ for } r > 1. \tag{75}$$

Then

$$\int_{-1}^{1} p_{\alpha,\beta,j}(x)p_{\alpha,\beta,i}(x)(1-x)^{\alpha}(1+x)^{\beta}\mathrm{d}x = \delta_{ij}, \textit{ for all } i,j \in \mathbb{N}_0, \tag{76}$$

$$\deg p_{\alpha,\beta,n} \le n(1+\varepsilon) \quad \textit{for all } n \in \mathbb{N}_0, \tag{77}$$

$$\textit{for each polynomial } q \textit{ there exists an } n_q \in \mathbb{N}_0 \textit{ with } q = \sum_{n=0}^{n_q} \langle q, p_{\alpha,\beta,n}\rangle p_{\alpha,\beta,n}, \tag{78}$$

$$\lim_{n\to\infty} \|f - S_n f\|_\infty = 0 \quad \textit{for all } f \in C[-1,1], \tag{79}$$

$$\|S_n\|_{C\to C} \le c\varepsilon^{-2\max\{\alpha,\beta\}-1}, \tag{80}$$

where we set $S_n f(x) := \sum_{j=0}^{n} \langle f, p_{\alpha,\beta,j}\rangle p_{\alpha,\beta,j}(x)$.

Proof: Without loss of generality we assume $0 < \varepsilon \le \min\{\frac{3}{2(\alpha+\beta+1)}, \frac{3}{8}\}$ and take $\eta \in \mathbb{N}$ to be a uniquely determined number with $2^\eta \ge \max\{2(\alpha+\beta+1), 8\}$ and

$$\frac{3}{2^\eta} \le \varepsilon \le \frac{6}{2^\eta}. \tag{81}$$

For the proof of (76), (77) and (78) we can adopt nearly the same arguments which served to solve the case of the Tchebysheff weights (see [5], [20, Chap. 3] or [7]). Taking into account the reproduction condition from (78), i.e. the completeness of the polynomial sequence and in view of a standard argument from approximation theory the convergence result (79), consequently the whole theorem is proved, if we show the uniform boundedness of the Lebesgue constants of $(p_{\alpha,\beta,n})_{n\in\mathbb{N}_0}$, i.e.

$$\sup_{x\in[-1,1],m\in\mathbb{N}_0} \Big\| \sum_{n=0}^{m} p_{\alpha,\beta,n}(x)p_{\alpha,\beta,n}(\cdot)\Big\|_{\alpha,\beta} < c\varepsilon^{-2\max\{\alpha,\beta\}-1}. \tag{82}$$

Proof of (82): For a given $j \in \mathbb{N}$ there exist uniquely determined numbers $l \in \mathbb{N}_0$ and $r \in \mathbb{N}$ with $1 \le r \le 2^{\eta-1}$ such that $j = l \cdot 2^{\eta-1} + r$. For $j = 0$ let $n_0 := 2^\eta$ and $m_0 = 1$ and for $j \in \mathbb{N}$ let

$$n_j := (2^\eta + 2r)2^l \quad \text{and} \quad m_j := 2^l. \tag{83}$$

Now, we look for certain polynomial subspaces $V^{(j)}, W^{(j)} \subseteq C[-1,1]$ ($j \in \mathbb{N}$) and define for $j \in \mathbb{N}$

$$\begin{aligned} V^{(0)} &:= \operatorname{span}\{p_k^{(\alpha,\beta)} : k = 0,...,n_0\}, \\ V^{(j)} &:= V^{(n_j,m_j)} := \operatorname{span}\Big\{\varphi_k^{(\alpha,\beta)} = \varphi_k^{(\alpha,\beta),(j)} : \quad k = 0,...,N\Big\} \text{ and} \\ W^{(j)} &:= W^{(n_j,m_j)} := \operatorname{span}\Big\{\psi_k^{(\alpha,\beta)} = \psi_k^{(\alpha,\beta),(j)} : \quad k = 0,...,2M-1\Big\}, \end{aligned}$$

where we set $N = n_j$, $M = m_j$ and

$$\begin{aligned} \psi_k^{(\alpha,\beta)} &:= M^{-\frac{1}{2}} \sum_{s=-2M}^{2M} g_r\Big(\frac{s}{M}\Big)\cos((3M+s)\theta_k)p_{N-M+s}^{(\alpha,\beta)} \text{ for } k = 0,...,2M-1, \\ \varphi_k^{(\alpha,\beta)} &:= \begin{cases} g^*(\frac{k}{N})p_k^{(\alpha,\beta)} + g_r^*(\frac{2N-k}{N})p_{2N-k}^{(\alpha,\beta)} & \text{for } k = 0,...,N-1, \\ p_N^{(\alpha,\beta)} & \text{for } k = N \end{cases} \end{aligned}$$

where r is the unique number from (70), g^* is the function defined in Lemma (3.1) and

$$g_r := \begin{cases} g & \text{for } r > 1, \\ \tilde{g} & \text{for } r = 1. \end{cases}$$

Exactly in the same way as in [20, Lemma 3.12] we get

$$V^{(j-1)} \oplus W^{(j)} = V^{(j)} \qquad \text{for all } j \in \mathbb{N}.$$

First let $n \in \mathbb{N}$ with $n > n_0$. Then there exists a $j \in \mathbb{N}$ with $n_{j-1} < n \leq n_j$, that means with $n = n_{j-1} + h + 1$ for an $h = 0, ..., 2m_j - 1$. We divide the orthogonal projection $\mathcal{O}_n$ into the orthogonal projection onto the spaces $\operatorname{span}\{p_0, \ldots, p_{n_{j-1}}\} = V^{(j-1)}$ and $\operatorname{span}\{p_{n_{j-1}+1}, \ldots, p_n\} \subseteq W^{(j)}$. Considering the fact that the operator norm of an orthogonal projection from $V^{(j-1)}$ onto itself depends not on the special choice of the orthonormal basis we get for the Lebesgue constant

$$\begin{aligned}
&\sup_{x\in[-1,1]} \|\sum_{i=0}^{n} p_i^{(\alpha,\beta)}(x) p_i^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta} \\
&\leq \sup_{x\in[-1,1]} \|\sum_{i=0}^{n_{j-1}} p_i^{(\alpha,\beta)}(x) p_i^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta} + \sup_{x\in[-1,1]} \|\sum_{i=n_{j-1}+1}^{n} p_i^{(\alpha,\beta)}(x) p_i^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta} \\
&\leq \sup_{x\in[-1,1]} \|\sum_{k=0}^{n_{j-1}} \varphi_k^{(\alpha,\beta),(j-1)}(x) \varphi_k^{(\alpha,\beta),(j-1)}(\cdot)\|_{\alpha,\beta} + \sup_{x\in[-1,1]} \sum_{k=0}^{2m_j-1} |\psi_k^{(\alpha,\beta),(j)}(x)| \cdot \|\psi_k^{(\alpha,\beta),(j)}(\cdot)\|_{\alpha,\beta}.
\end{aligned}$$

Since g fulfills (67), (68) and (69), we get from an easy calculation that $\widetilde{g}, g^*, \widetilde{g^*} \in C^m[-1,1]$ fulfills (67), (68) and also (69). Applying Lemma 3.1 and Theorems 3.3 and 3.4 to the left and Theorem 2.5 to the right summand and taking into account that by (81) and (83) we have $\frac{n_j}{m_j} \leq 2^{\eta+1} \leq c\varepsilon$ for all $j \in \mathbb{N}$, we get (82).
Now let $n \leq n_0 = 2^\eta$: We use the fact (see [9, Lemma 4.6]) that

$$\sup_{x\in[-1,1]} \|\sum_{k=0}^{\infty} h_k p_k^{(\alpha,\beta)}(x) p_k^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta} \leq c \|\sum_{k=0}^{\infty} h_k p_k^{(\alpha,\beta)}(1) p_k^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta},$$

for $\alpha, \beta \geq -\frac{1}{2}$ and sequences of real numbers $(h_k)_{k\in\mathbb{N}_0}$ with $h_k = 0$ for all sufficiently large k. Combining this with (73), (81) and the fact (see [19]) that

$$\|\sum_{j=0}^{n} p_j^{(\alpha,\beta)}(1) p_j^{(\alpha,\beta)}(\cdot)\|_{\alpha,\beta} \leq c \begin{cases} n^{\max\{\alpha,\beta\}+\frac{1}{2}} & \text{for } \max\{\alpha,\beta\} > -\frac{1}{2}, \\ \log n & \text{for } \alpha = \beta = -\frac{1}{2} \end{cases}$$

for all $n \in \mathbb{N}_0$ and $\alpha, \beta \geq -\frac{1}{2}$, we get

$$\sup_{x\in[-1,1]} \|\sum_{i=0}^{n} p_{\alpha,\beta,i}(x) p_{\alpha,\beta,i}(\cdot)\|_{\alpha,\beta} \leq c\, 2^{\eta(\max\{\alpha,\beta\}+\frac{1}{2})} \leq c\varepsilon^{-\max\{\alpha,\beta\}-\frac{1}{2}}$$

for all $n \leq n_0 = 2^\eta$ and thereby (82). ∎

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