On a constructive representation of an orthogonal trigonometric Schauder basis for $C_{2\pi}$

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Dedicated to the memory of Professor Dr. Siegfried Prößdorf

A particular class of orthogonal trigonometric Schauder bases for $C_{2\pi}$ is given by periodic wavelet packet functions. These bases are of minimal growth of the polynomial degree. The focus of attention is their construction and the estimation of the Lebesgue constant. The corresponding approximation error is asymptotically optimal.

1. Introduction

The Fourier series of a continuous function does not converge, in general, in the supremum norm. Faber [Fa] proved for $C_{2\pi}$ that any polynomial set $\{t_\mu \,:\, \mu \in \mathbb{N}\}$ with $\deg t_\mu \le \mu/2$ cannot be a basis. So, a long standing question was which minimal degree does one need in order to preserve pointwise convergence? Privalov gave a conclusive answer by two results [Pri1, Pri2]. First, he showed that, for any basis $\{t_\mu \,:\, \mu \in \mathbb{N}\}$ in $C_{2\pi}$, there exists an $\varepsilon > 0$ such that for sufficiently large μ, one has $\deg t_\mu \ge (1+\varepsilon)\mu/2$. Second, for any such $\varepsilon > 0$ he verified the existence of a Schauder basis satisfying $\deg t_\mu \le (1+\varepsilon)\mu/2$.

Lorentz and Sahakian showed in [LS] that the additional condition on the basis to be orthogonal does not affect the growth of the degree. Their proof was based on Meyer wavelets and corresponding wavelet packets on the real line which then were periodized.

We were studying de la Vallée Poussin means and related polynomial wavelets in [PS1, PS2, Se1, Se2] which led us to a similar basis with optimal growth of the degree. Our construction by means of periodic wavelet and wavelet packet spaces allows an asymptotically optimal estimation of the norm of the corresponding partial sum operator.

For our basis $\{t_k \,:\, k \in \mathbb{N}\}$, we will consider its partial sum operator

$$S_\mu f \quad = \quad \sum_{k=1}^{\mu} \langle f\,,\, t_k\rangle \; t_k \tag{1.1}$$

and verify the Schauder basis property

$$\|f - S_\mu f\|_\infty \quad \to \quad 0 \qquad (\mu \to \infty)$$

for all $f \in C_{2\pi}$. In this paper, we prove it in the form

$$\|f - S_\mu f\|_\infty \;\le\; (1 + \|S_\mu\|_{C\to C})\, E_{\lfloor \frac{\mu}{2}(1-\varepsilon)\rfloor}(f)\,,$$

where the best approximation of $f \in C_{2\pi}$ by trigonometric polynomials of degree at most n,

$$E_n(f) \;:=\; \inf_{p_n \in \mathbb{T}_n} \|f - p_n\|_\infty\,,$$

tends to zero as $n \to \infty$. Here and in the sequel, $\mathbb{T}_n$ denotes the set of trigonometric polynomials of degree at most n, and we set $\lceil r\rceil := \min\{k \in \mathbb{Z} :\; k \ge r\}$ and $\lfloor r\rfloor := \max\{k \in \mathbb{Z} :\; k \le r\}$, for $r \in \mathbb{R}$. In fact, we construct the functions t_k depending on $\varepsilon \in (0, 2/3]$ such that

$$\mathbb{T}_{\lfloor \frac{\mu}{2}(1-\varepsilon)\rfloor} \;\subseteq\; \operatorname{span}\{t_k \;:\; k = 1, \ldots, \mu\}\,, \tag{1.2}$$

and

$$\deg t_\mu \;\le\; \frac{\mu}{2}(1+\varepsilon)\,.$$

Our estimation of the uniform boundedness of the partial sum operator in the sup-norm,

$$\|S_\mu\|_{C\to C} \;=\; \sup_{\|f\|_\infty = 1} \|S_\mu f\|_\infty \;<\; 15 + \frac{4}{\pi^2}\ln\frac{1}{\varepsilon} \tag{1.3}$$

is asymptotically optimal as $\varepsilon \to 0$ (cf. [Pri1]).

In Section 2, we define the orthonormal Schauder basis of optimal degree and state the main theorem. Then, Section 3 shall explain the constructive aspects of our trigonometric wavelet packets. Section 4 is exclusively devoted to the proof of the estimation of the operator norm. We emphasize here that the essential point of our construction to obtain (1.3) is the combination of the ideas of a periodic multiresolution analysis with the explicite determination of frequencies as given in Lemma 3.4.

Let us mention that our polynomial system $\{t_\mu\}$ is also a basis in some separable Hölder spaces. The investigation of approximation in Hölder spaces was initiated by Siegfried Prößdorf in [Prö]. To describe a particular case let, for $0 < \alpha < 1$,

$$C^\alpha = \{f \in C_{2\pi} : \|f\|_{\infty,\alpha} := \|f\|_\infty + \sup_{x\ne y} |x-y|^{-\alpha}|f(x) - f(y)| < \infty\}.$$

With the help of the results in [Pre] and [PP] one can easily prove that our orthonormal system of polynomials $\{t_\mu\}$ is a basis in the subspace

$$\tilde{C}^\alpha = \{f \in C^\alpha : \lim_{y\to x} |x-y|^{-\alpha}|f(x) - f(y)| = 0 \quad \text{for all } x\}.$$

2. Orthogonal bases of optimal degree

We are going to define the Schauder basis polynomials t_μ depending on the number ε to which the degree of the polynomials must be tuned.

Let $\varepsilon > 0$ be given and fix $\lambda \in \mathbb{N}$,

$$\lambda \ := \ \begin{cases} 1 & \text{if } \varepsilon \geq \frac{2}{3}\,, \\ \lceil \log_2 \frac{4}{3\varepsilon} \rceil & \text{if } \varepsilon < \frac{2}{3}\,, \end{cases}$$

i.e., $4 \leq 3\varepsilon \cdot 2^\lambda < 8$ for $\varepsilon \leq 2/3$. For any $\varepsilon > 2/3$, we get by definition the same λ and therewith the same Schauder basis as for $\varepsilon = 2/3$. Hence, we can restrict ourselves to $\varepsilon \leq 2/3$ in the rest of the paper.

We denote *dimension numbers*

$$N_j \ := \ 3 \cdot 2^j \qquad (j \in \mathbb{N}_0)\,,$$

and, for our fixed $\lambda \in \mathbb{N}$,

$$N_j^{\lambda,p} \ := \ N_{j-\lambda}(2^\lambda + p) \qquad (j \geq \lambda; \quad p = 0, \ldots, 2^\lambda)\,.$$

Then, for all $\mu \in \mathbb{N}$, $\mu > 2N_\lambda$ there is a *one-to-one correspondence of* μ *and the triple* (j,p,s), with $j \geq \lambda$, $p \in \{0, \ldots, 2^\lambda - 1\}$ and $s \in \{1, \ldots, 2N_{j-\lambda}\}$, by the equality

$$\mu \ = \ 2N_{j-\lambda}(2^\lambda + p) + s \ = \ 2N_j^{\lambda,p} + s\,.$$

We define the *shift operator* $T_j \,:\, C_{2\pi} \to C_{2\pi}$ *of level* $j \in \mathbb{N}_0$ by

$$T_j\, f \ := \ f\left(\cdot - \frac{\pi}{N_j}\right)\,.$$

We denote the *bit-reversed number of* $p \in \{0, \ldots, 2^\lambda - 1\}$ *with respect to* λ *digits* by

$$b(\lambda,p) \ := \ \sum_{\ell=0}^{\lambda-1} \epsilon_{\lambda-\ell-1}(p)\; 2^\ell\,, \tag{2.1}$$

where $\epsilon_\ell(p)$ are the bits (or binary digits) of p in its binary decomposition

$$p \ = \ \sum_{\ell=0}^{\lambda-1} \epsilon_\ell(p)\; 2^\ell\,, \qquad \text{i.e.,} \quad \epsilon_\ell(p) \ = \ \left\lfloor \frac{p}{2^\ell} \right\rfloor \bmod 2\,.$$

Further, let

$$M_j \ := \ \begin{cases} 1 & \text{if } j < \lambda\,, \\ 2^{j-\lambda} & \text{if } j \geq \lambda\,, \end{cases} \tag{2.2}$$

and

$$j(p) \;:=\; \begin{cases} j & \text{for } p = 0,\ldots,2^\lambda - 2\,, \\ j+1 & \text{for } p = 2^\lambda - 1\,. \end{cases} \tag{2.3}$$

Definition 2.1. Let $\varepsilon > 0$ be given. The set of polynomials $\{t_\mu \;:\; \mu \in I\!N\}$ is defined as follows. Let

$$\begin{aligned} t_1 &:= 1\,, \\ t_{2k} &:= \sqrt{2}\cos k\,\cdot \qquad \text{for } k = 1,\ldots,N_\lambda\,, \\ t_{2k+1} &:= \sqrt{2}\sin k\,\cdot \qquad \text{for } k = 1,\ldots,N_\lambda - 1\,, \end{aligned}$$

and for all $j \geq \lambda$, $\;p = 0,\ldots,2^\lambda - 1$, $\;s = 1,\ldots,2N_{j-\lambda}$, let

$$t_{2N_j^{\lambda,p}+s} \;:=\; \frac{1}{\sqrt{N_{j-\lambda}}}\, T_{j-\lambda}^{s-1}\, T_{j+1}^{b(\lambda+1,2^\lambda+p)} \left(\sum_{k=-M_j+1}^{M_j-1} \frac{M_j + k}{\sqrt{M_j^2 + k^2}} \cos(N_j^{\lambda,p} + k)\,\cdot \right.$$

$$\left. +\, \sqrt{2} \sum_{k=N_j^{\lambda,p}+M_j}^{N_j^{\lambda,p+1}-M_{j(p)}} \cos k\,\cdot + \sum_{k=-M_{j(p)}+1}^{M_{j(p)}-1} \frac{M_{j(p)} - k}{\sqrt{M_{j(p)}^2 + k^2}} \cos(N_j^{\lambda,p+1} + k)\,\cdot \right).$$

Note that, for $\mu > 2N_\lambda$, the polynomials t_μ are shifted cosine sums which obviously satisfy

$$t_{2N_j^{\lambda,p}+s} \;\in\; I\!\!T_{N_j^{\lambda,p+1}+M_{j(p)}-1} \quad \text{and} \quad t_{2N_j^{\lambda,p}+s} \;\perp\; I\!\!T_{N_j^{\lambda,p}-M_j}\,.$$

Theorem 2.2. *For given $\varepsilon > 0$, the polynomial system $\{t_\mu \;:\; \mu \in I\!N\}$ from Definition 2.1 is an orthonormal Schauder basis in $C_{2\pi}$ of optimal degree, i.e., for all $\mu, \nu \in I\!N$, it holds that*

$$\langle t_\mu\,,\, t_\nu\rangle \;=\; \delta_{\mu,\nu}\,, \tag{2.4}$$

and

$$\deg t_\mu \;\leq\; \frac{\mu}{2}(1+\varepsilon)\,. \tag{2.5}$$

In particular, for all $f \in C_{2\pi}$ we have

$$\left\| f - \sum_{k=1}^{\mu} \langle f\,,\, t_k\rangle\; t_k \right\|_\infty \;<\; \left(16 + \frac{4}{\pi^2}\;\ln\frac{1}{\varepsilon}\right)\; E_{\lfloor \frac{\mu}{2}(1-\varepsilon)\rfloor}(f)\,. \tag{2.6}$$

Proof. First, we prove the degree inequality (2.5) which is trivial for $\mu \leq 2N_\lambda$. For $\mu > 2N_\lambda$, we deduce for $p = 0,\ldots,2^\lambda - 2$

$$\begin{aligned} \frac{\deg t_{2N_j^{\lambda,p}+s}}{2N_j^{\lambda,p} + s} &= \frac{N_j^{\lambda,p+1} + M_j - 1}{2N_j^{\lambda,p} + s} \;<\; \frac{N_j^{\lambda,p} + N_{j-\lambda} + M_j}{2N_j^{\lambda,p}} \\ &= \frac{1}{2}\left(1 + \frac{4 \cdot 2^{j-\lambda}}{3 \cdot 2^{j-\lambda}(2^\lambda + p)}\right) \;\leq\; \frac{1}{2}\left(1 + \frac{4}{3 \cdot 2^\lambda}\right) \;\leq\; \frac{1}{2}(1+\varepsilon)\,, \end{aligned}$$

since $\lambda \ \geq \ \log_2 \frac{4}{3\varepsilon}$. For $p = 2^\lambda - 1$, we have

$$\frac{\deg t_{2N_j^{\lambda,p}+s}}{2N_j^{\lambda,p}+s} \;=\; \frac{N_{j+1}+M_{j+1}-1}{2N_j^{\lambda,2^\lambda-1}+s} \;<\; \frac{N_{j+1}+M_{j+1}}{2(N_{j+1}-N_{j-\lambda})}$$
$$= \; \frac{1}{2}\left(1+\frac{5\cdot 2^{j-\lambda}}{3\cdot 2^{j-\lambda}(2^{\lambda+1}-1)}\right) \;=\; \frac{1}{2}\left(1+\frac{5}{3\cdot(2^{\lambda+1}-1)}\right).$$

Since $\lambda \geq 1$,

$$\frac{2^\lambda}{2^{\lambda+1}-1} \;=\; \frac{2^{\lambda+1}}{2(2^{\lambda+1}-1)} \;=\; \frac{1}{2}\left(1+\frac{1}{2^{\lambda+1}-1}\right) \;\leq\; \frac{1}{2}\left(1+\frac{1}{3}\right) \;=\; \frac{2}{3} \;<\; \frac{4}{5}.$$

Hence, (2.5) follows analogously for $p = 2^\lambda - 1$.

The orthonormality (2.4) can be deduced from the construction of $\{t_\mu \;:\; \mu \in \mathbb{N}\}$ which we will explain in Section 3. It could also be proved straightforward.

The estimation in (2.6) is a consequence of the results in (1.2) and (1.3), which are given in Lemma 3.4 and Theorem 4.4, respectively. □

3. Construction methods

3.1. General theory

In this section, let us focus on Hilbert space theory in $L^2_{2\pi}$. Our construction is strongly based upon multiresolution, wavelet and wavelet packet spaces in $L^2_{2\pi}$ being shift-invariant spaces spanned by one function and its translates. The wavelet spaces and the wavelet packet spaces, respectively, ought to be mutually orthogonal. Finally, the translates of the wavelet packets have to be orthonormalized and will be orthonormal translates of another wavelet packet of the same space.

A periodic multiresolution analysis (PMRA) shall be, as usual, a chain of spaces of the following kind.

Definition 3.1. A *periodic multiresolution analysis (PMRA) in* $L^2_{2\pi}$ is a sequence of subspaces $\{V_j\}_{j\in\mathbb{N}_0}$ of $L^2_{2\pi}$ with the properties:

(MR1) For all $j \in \mathbb{N}_0$ there exists a function $\phi_j \in V_j$, such that $\{T_j^s\,\phi_j \;:\; s = 0,\ldots,2N_j-1\}$ is a basis for V_j .

(MR2) For all $j \in \mathbb{N}_0$ it holds $V_j \subset V_{j+1}$.

(MR3) It holds $\operatorname{clos}_{L^2_{2\pi}}\left(\bigcup_{j\in\mathbb{N}_0} V_j\right) \;=\; L^2_{2\pi}$.

While sticking to shift-invariance, we have to give up dilation invariance for the case of periodic functions, simply due to the fact that $\phi \in L^2_{2\pi} \implies \phi(\frac{\cdot}{2}) \in L^2_{4\pi}$.

So, in each level j, ϕ_j can be any function that belongs to the superior space V_{j+1} and has linearly independent translates with respect to the shift operator T_j. Hence, such a *scaling function* ϕ_j has to satisfy a two-scale relation

$$\phi_j \;=\; \sum_{s=0}^{N_{j+2}-1} \alpha_{j,s}\; T_{j+1}^s \phi_{j+1}\,.$$

In terms of Fourier coefficients

$$c_k(f) := \langle f, e^{ik\cdot}\rangle := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\; e^{-ikx}\; dx$$

it means

$$c_{k+mN_{j+2}}(\phi_j) \;=\; \hat{\alpha}_{j,k}\; c_{k+mN_{j+2}}(\phi_{j+1})\,, \tag{3.1}$$

for all $k = 0,\ldots,N_{j+2}-1$, $m \in \mathbb{Z}$, where $(\alpha_{j,s})_{s=0}^{N_{j+2}-1}$ is a vector of complex numbers, and $(\hat{\alpha}_{j,k})_{k=0}^{N_{j+2}-1}$ is its discrete Fourier transform. The linear independence of the $2N_j$ translates of ϕ_j, i.e. (MR1), is equivalent to

$$\sum_{m\in\mathbb{Z}} |c_{k+mN_{j+1}}(\phi_j)|^2 \;>\; 0 \qquad (k = 0,\ldots,N_{j+1}-1)\,, \tag{3.2}$$

and then (MR2) holds iff there exist $(\hat{\alpha}_{j,k})_{k=0}^{N_{j+2}-1}$ such that (3.1) is satisfied for all $j \in \mathbb{N}_0$.

The wavelet spaces contain what is remaining when cutting V_j out of V_{j+1}.

Definition 3.2. Let $\{V_j\}_{j\in\mathbb{N}_0}$ in $L^2_{2\pi}$ be a given PMRA. For all $j \in \mathbb{N}_0$, the *wavelet space* W_j *of level* j is the orthogonal complement of V_j in V_{j+1}, i.e.,

$$W_j \;=\; V_{j+1} \ominus V_j\,.$$

From this definition it follows that W_j is invariant with respect to the shift T_j and that there exists a function ψ_j such that $W_j = \operatorname{span}\{T_j^s\,\psi_j :\; s = 0,\ldots,2N_j-1\}$. Such a function ψ_j shall be called *wavelet*, and again, no dilation property is required.

Given any PMRA and, in particular, coefficient vectors $(\hat{\alpha}_{j,k})_{k=0}^{N_{j+2}-1}$ for which (3.1) holds, $\psi_j \in L^2_{2\pi}$ is a wavelet iff we have

(W) *The Fourier coefficients of* ψ_j *satisfy*

$$c_k(\psi_j) \;=\; \frac{\sigma_{j,(k \bmod N_{j+2})}\; \overline{\hat{\alpha}}_{j,((k+N_{j+1}) \bmod N_{j+2})}}{\sum_{\ell\in\mathbb{Z}} |c_{k+\ell N_{j+2}}(\phi_{j+1})|^2}\; c_k(\phi_{j+1}) \qquad (k \in \mathbb{Z})$$

with some numbers $\sigma_{j,k} \in \mathbb{C}\setminus\{0\}$, $k = 0,\ldots,N_{j+2}-1$, *for which*

$$\sigma_{j,k} \quad = \quad -\sigma_{j,k+N_{j+1}} \qquad (k=0,\ldots,N_{j+1}-1).$$

So we have exactly $2N_j$ degrees of freedom for the choice of ψ_j.

For the concept of periodic wavelet packets, we do not simply adopt the decomposition algorithms from the scaling spaces as done in $L^2(\mathbb{R})$ (see e.g. [CMW, CW]), but we persist on splitting any T_j-invariant subspace into two (orthogonal) T_{j-1}-invariant subspaces.

Definition 3.3. For a given wavelet space $W_j = W_j^{0,0}$ of level j , $j \in \mathbb{N}$, and for any number $\lambda \in \mathbb{N}$ with $\lambda \le j$, the *wavelet packet spaces of level j and depth λ* are the subspaces $\{W_j^{\lambda,p}\}_{p=0}^{2^\lambda-1}$ of $L^2_{2\pi}$ satisfying the following properties:

(WP1) For all $p=0,\ldots,2^\lambda-1$, there exists a function $\psi_j^{\lambda,p} \in W_j^{\lambda,p}$ such that $\{T^s_{j-\lambda}\,\psi_j^{\lambda,p}:\ s=0,\ldots,2N_{j-\lambda}-1\}$ is a basis for $W_j^{\lambda,p}$.

(WP2) For all $\ell=0,\ldots,\lambda-1$ and $p=0,\ldots,2^\ell-1$, we have

$$W_j^{\ell,p} \quad = \quad W_j^{\ell+1,2p} \oplus W_j^{\ell+1,2p+1}\,.$$

The functions $\psi_j^{\lambda,p}$ are called *wavelet packets.*

The properties (WP1) and (WP2) can be written equivalently in the following form:

(WP) *There exist vectors* $(\hat\beta^{\ell,p}_{j,k})_{k=0}^{N_{j-\ell+2}-1} \in \mathbb{C}^{N_{j-\ell+2}}$, *for all* $\ell=1,\ldots,\lambda$ *and* $p=0,\ldots,2^\ell-1$, *such that*

$$c_k(\psi_j^{\ell,p}) \quad = \quad \hat\beta^{\ell,p}_{j,(k \bmod N_{j-\ell+2})}\; c_k(\psi_j^{\ell-1,\lfloor \frac{p}{2}\rfloor}) \qquad (k \in \mathbb{Z}),$$

with

$$\begin{aligned} &\hat\beta^{\ell,2\lfloor\frac{p}{2}\rfloor}_{j,k}\;\overline{\hat\beta^{\ell,2\lfloor\frac{p}{2}\rfloor+1}_{j,k}}\sum_{m\in\mathbb{Z}}|c_{k+mN_{j-\ell+2}}(\psi_j^{\ell-1,\lfloor\frac{p}{2}\rfloor})|^2 \\ &= \; -\,\hat\beta^{\ell,2\lfloor\frac{p}{2}\rfloor}_{j,k+2N_{j-\ell}}\;\overline{\hat\beta^{\ell,2\lfloor\frac{p}{2}\rfloor+1}_{j,k+2N_{j-\ell}}}\sum_{m\in\mathbb{Z}}|c_{k+2N_{j-\ell}+mN_{j-\ell+2}}(\psi_j^{\ell-1,\lfloor\frac{p}{2}\rfloor})|^2 \end{aligned}$$

and

$$\left|\hat\beta^{\ell,p}_{j,k}\right|^2 + \left|\hat\beta^{\ell,p}_{j,k+2N_{j-\ell}}\right|^2 \quad > \quad 0\,,$$

for all $k=0,\ldots,2N_{j-\ell}-1$.

Last but not least we need to know how to orthonormalize the translates of a function without loosing the translation invariance property of the basis. Given $f \in L^2_{2\pi}$ for which the translates $\{T_j^s f : s = 0,\ldots,2N_j - 1\}$ are linearly independent, i.e., (3.2) is satisfied by $f = \phi_j$, we obtain orthonormal translates $\{T_j^s g : s = 0,\ldots,2N_j - 1\}$ by choosing

$$c_{k+2N_j\ell}(g) \;=\; c_{k+2N_j\ell}(f)\left(\sum_{m\in\mathbb{Z}} |c_{k+2N_jm}(f)|^2\right)^{-\frac{1}{2}}, \tag{3.3}$$

for all $k = 0,\ldots,2N_j - 1$ and $\ell \in \mathbb{Z}$.

3.2. Particular polynomial functions

Now, let us define special scaling functions that will lead us to our polynomials in Definition 2.1. A *de la Vallée Poussin mean* is given, for any $N, M \in \mathbb{N}$ with $N \geq M$, by

$$\varphi_N^M(x) \;=\; 1 + 2\sum_{k=1}^{N-M} \cos kx + \sum_{k=-M+1}^{M-1} \frac{M-k}{M} \cos(N+k)x\,. \tag{3.4}$$

Its interpolating property $\varphi_N^M(\frac{s\pi}{N}) = 2N\,\delta_{s,0}$, for $s = 0,\ldots,2N-1$, is clear from

$$\varphi_N^M(x) \;=\; \begin{cases} \dfrac{\sin Nx \sin Mx}{2M\sin^2\frac{x}{2}} & \text{if } x \notin 2\pi\mathbb{Z}\,,\\ 2N & \text{if } x \in 2\pi\mathbb{Z} \end{cases} \tag{3.5}$$

and yields also the linear independence of its translates $\{\varphi_N^M(\cdot - \frac{s\pi}{N}) : s = 0,\ldots,2N-1\}$.

With the numbers $N_j = 3\cdot 2^j$, the $\phi_j = \varphi_{N_j}^{M_j}$ are scaling functions of a PMRA iff $M_j + M_{j+1} \leq N_j + 1$ for all $j \in \mathbb{N}_0$ (see [Se3]) which is satisfied by M_j as given in (2.2). Indeed, since

$$c_k(\varphi_{N_j}^{M_j}) \;=\; \begin{cases} 1 & \text{if} \quad 0 \leq |k| \leq N_j - M_j\,,\\ \frac{M_j-(|k|-N_j)}{2M_j} & \text{if} \quad N_j - M_j < |k| < N_j + M_j\,,\\ 0 & \text{if} \quad N_j + M_j \leq |k|\,, \end{cases}$$

condition (3.2) holds, and with

$$\hat{\alpha}_{j,k} \;=\; \begin{cases} 1 & \text{if} \quad 0 \leq k \leq N_j - M_j\,,\\ \frac{M_j-(k-N_j)}{2M_j} & \text{if} \quad N_j - M_j < k < N_j + M_j\,,\\ 0 & \text{if} \quad N_j + M_j \leq k \leq 3N_j - M_j\,,\\ \frac{M_j+(k-3N_j)}{2M_j} & \text{if} \quad 3N_j - M_j < k < 3N_j + M_j\,,\\ 1 & \text{if} \quad 3N_j + M_j \leq k < 4N_j\,, \end{cases}$$

(3.1) is true, for all $j \in \mathbb{N}_0$. Moreover, one easily verifies

$$\mathbb{T}_{N_j-M_j} \subset V_j \subset \mathbb{T}_{N_j+M_j-1} .$$

Due to $N_j - M_j \geq 2N_j/3$ and the Weierstrass theorem, (MR3) is valid, too.

Based on the PMRA generated by de la Vallée Poussin means, we further define functions

$$\psi_j = T_{j+1}\left(\varphi_{N_{j+1}}^{M_{j+1}} - \varphi_{N_j}^{M_j}\right) \tag{3.6}$$

for all $j \in \mathbb{N}_0$ and

$$\psi_j^{\lambda,p} = T_{j+1}^{b(\lambda+1,2^\lambda+p)}\left(\varphi_{N_j^{\lambda,p+1}}^{M_{j(p)}} - \varphi_{N_j^{\lambda,p}}^{M_j}\right) \tag{3.7}$$

for all $j \geq \lambda$ and $p = 0, \ldots, 2^\lambda - 1$, where $b(\lambda, p)$ is the bit-reversed number (2.1). Using the above theory, we now show that the functions ψ_j and $\psi_j^{\lambda,p}$ are wavelets and wavelet packets, respectively.

The functions ψ_j in (3.6) are wavelets since they satisfy property (W) if we choose

$$\sigma_{j,(k \bmod N_{j+2})} = e^{-\frac{ik\pi}{N_{j+1}}} \times \begin{cases} \sum_{\ell\in\mathbb{Z}} |c_{k+\ell N_{j+2}}(\varphi_{N_{j+1}}^{M_{j+1}})|^2 & \text{if } \ N_j \leq k < 3N_j, \\ \sum_{\ell\in\mathbb{Z}} |c_{k+N_{j+1}+\ell N_{j+2}}(\varphi_{N_{j+1}}^{M_{j+1}})|^2 & \text{if } -N_j \leq k < N_j. \end{cases}$$

These functions ψ_j have already been used, e.g. in [Pri3]. For the functions $\psi_j^{\lambda,p}$ in (3.7), we have property (WP) in form of

$$c_k(\psi_j^{\lambda,p}) = e^{-\frac{i(p \bmod 2)k\pi}{N_{j-\lambda+1}}} \hat{\beta}^{\lambda,2\lfloor\frac{p}{2}\rfloor}_{j,((k+(p \bmod 2)N_{j-\lambda+1}) \bmod N_{j-\lambda+2})} c_k\left(\psi_j^{\lambda-1,\lfloor\frac{p}{2}\rfloor}\right)$$

with the factors

$$\hat{\beta}^{\lambda,2\lfloor\frac{p}{2}\rfloor}_{j,k} = \begin{cases} 1 & \text{if } \quad 0 \leq k \leq N_{j-\lambda} - M_j , \\ \frac{M_j-(k-N_{j-\lambda})}{2M_j} & \text{if } \quad N_{j-\lambda} - M_j < k < N_{j-\lambda} + M_j , \\ 0 & \text{if } \quad N_{j-\lambda} + M_j \leq k \leq 3N_{j-\lambda} - M_j , \\ \frac{M_j+(k-3N_{j-\lambda})}{2M_j} & \text{if } \quad 3N_{j-\lambda} - M_j < k < 3N_{j-\lambda} + M_j , \\ 1 & \text{if } \quad 3N_{j-\lambda} - M_j \leq k < 4N_{j-\lambda} . \end{cases}$$

If we compute the Fourier coefficients of each $\psi_j^{\lambda,p}$ explicitely, then we see that their absolute value is equal to the Fourier coefficients of $\varphi_{N_j^{\lambda,p+1}}^{M_{j(p)}} - \varphi_{N_j^{\lambda,p}}^{M_j}$, and the modulation that is multiplied in the iteration step from $\psi_j^{\lambda-1,\lfloor\frac{p}{2}\rfloor}$ to $\psi_j^{\lambda,p}$ is exactly $e^{-\frac{i(p \bmod 2)k\pi}{N_{j-\lambda+1}}}$. Since

$$c_k(T_j f) = e^{-\frac{ik\pi}{N_j}} c_k(f) ,$$

it follows that the total shift of $\varphi^{M_{j(p)}}_{N_j^{\lambda,p+1}} - \varphi^{M_j}_{N_j^{\lambda,p}}$ to gain $\psi_j^{\lambda,p}$ is indeed

$$T_{j-\lambda+1}^{p \bmod 2} \; \dots \; T_j^{\lfloor \frac{p}{2^{\lambda-1}} \rfloor \bmod 2} \, T_{j+1} \;=\; T_j^{b(\lambda,p)} \, T_{j+1} \;=\; T_{j+1}^{b(\lambda+1,2^\lambda+p)} \,.$$

So, our functions in (3.7) are wavelet packet functions.

Now, we want to orthonormalize $\{T_{j-\lambda}^s \psi_j^{\lambda,p} \;:\; s = 0,\dots,2N_{j-\lambda}-1\}$. Their Fourier coefficients are changed following the rule in (3.3), i.e., they are divided by the square root of

$$\sum_{m\in\mathbb{Z}} |c_{k+2N_{j-\lambda}m}(\psi_j^{\lambda,p})|^2 = \begin{cases} \frac{M_{j(p)}^2+k^2}{2M_{j(p)}^2} & 0 \le |k| < M_{j(p)}\,, \\ 1 & M_{j(p)} \le |k| \le N_{j-\lambda} - M_j\,, \\ \frac{M_j^2+(k-N_{j-\lambda})^2}{2M_j^2} & N_{j-\lambda} - M_j < \; k \; < N_{j-\lambda} + M_j\,, \end{cases}$$

for $k = -N_{j-\lambda}+M_j,\dots,N_{j-\lambda}+M_j-1 \bmod 2N_{j-\lambda}$. Thus we obtain the functions in Definition 2.1 for $\mu > 2N_\lambda$ via their Fourier coefficients

$$c_k(t_{2N_j^{\lambda,p}+1}) \;=\; c_k(\psi_j^{\lambda,p}) \left(\sum_{m\in\mathbb{Z}} |c_{k+2N_{j-\lambda}m}(\psi_j^{\lambda,p})|^2 \right)^{-\frac{1}{2}} \qquad (k\in\mathbb{Z})\,.$$

So, for any $j \ge \lambda$, we have by wavelet and wavelet packet decomposition

$$\begin{aligned} V_j \;&=\; V_\lambda \oplus \bigoplus_{h=\lambda}^{j-1} W_h \\ &=\; V_\lambda \oplus \bigoplus_{h=\lambda}^{j-1} \bigoplus_{q=0}^{2^\lambda-1} W_h^{\lambda,q} \qquad (3.8) \\ &=\; \operatorname{span}\{t_k \;:\; k = 1,\dots,2N_j\}\,, \end{aligned}$$

and, for any $p \in \{1,\dots,2^\lambda-1\}$,

$$\bigoplus_{q=0}^{p-1} W_j^{\lambda,q} \;=\; \operatorname{span}\{t_k \;:\; k = 2N_j+1,\dots,2N_j^{\lambda,p}\}\,. \qquad (3.9)$$

Finally, we determine the trigonometric polynomials which are included in our polynomial spaces. As pointed out in the introduction, the change of basis from wavelet packets to polynomials consisting of one or two frequencies is the main ingredient to obtain optimal Lebesgue constants.

Lemma 3.4. *For given $\varepsilon > 0$, the span of the first μ polynomials t_k from Definition 2.1 contains all trigonometric polynomials of degree less or equal to $(1-\varepsilon)\mu/2$, i.e.,*

$$\mathbb{T}_{\lfloor \frac{\mu}{2}(1-\varepsilon) \rfloor} \;\subseteq\; \operatorname{span}\{t_k \;:\; k = 1,\dots,\mu\}\,. \qquad (3.10)$$

Moreover, for $j \geq \lambda$ *,*

$$
\begin{aligned}
&\operatorname{span}\{t_k \;:\; k = 1,\dots,2N_j\} \\
&\quad = \mathbb{T}_{N_j - M_j} \oplus \operatorname{span}\{\theta^{\lambda,2^\lambda-1}_{j-1,N_{j-\lambda-1}+k} \;:\; k = -M_j+1,\dots,M_j-1\}\,,
\end{aligned}
\tag{3.11}
$$

and for $j \geq \lambda$ *,* $p \in \{1,\dots,2^\lambda - 1\}$*,*

$$
\begin{aligned}
&\operatorname{span}\{t_k \;:\; k = 1,\dots,2N_j^{\lambda,p}\} \\
&\quad = \mathbb{T}_{N_j^{\lambda,p} - M_j} \oplus \operatorname{span}\{\theta^{\lambda,p-1}_{j,N_{j-\lambda}+k} \;:\; k = -M_j+1,\dots,M_j-1\}\,,
\end{aligned}
\tag{3.12}
$$

with

$$\theta^{\lambda,p}_{j,N_{j-\lambda}} = \sqrt{2}\; T^{b(\lambda+1,2^\lambda+p)}_{j+1}\;\cos N_j^{\lambda,p+1}\cdot\,,$$

$$
\begin{aligned}
\theta^{\lambda,p}_{j,N_{j-\lambda}-k} \quad &= \\
T^{b(\lambda+1,2^\lambda+p)}_{j+1} &\left(\tfrac{M_{j(p)}+k}{\sqrt{M_{j(p)}^2+k^2}}\cos(N_j^{\lambda,p+1}-k)\cdot + \tfrac{M_{j(p)}-k}{\sqrt{M_{j(p)}^2+k^2}}\cos(N_j^{\lambda,p+1}+k)\cdot\right), \\
\theta^{\lambda,p}_{j,N_{j-\lambda}+k} \quad &= \\
T^{b(\lambda+1,2^\lambda+p)}_{j+1} &\left(\tfrac{M_{j(p)}+k}{\sqrt{M_{j(p)}^2+k^2}}\sin(N_j^{\lambda,p+1}-k)\cdot - \tfrac{M_{j(p)}-k}{\sqrt{M_{j(p)}^2+k^2}}\sin(N_j^{\lambda,p+1}+k)\cdot\right)
\end{aligned}
$$

for $k = 1,\dots,M_{j(p)}-1$*, where the shift operator in* $\theta^{\lambda,2^\lambda-1}_{j,N_{j-\lambda}+k}$ *can be omitted.*

Proof. Here and in the proof of Lemma 4.3 we need coefficient functions given in the interval $[-\frac{\pi}{3}, 2\pi - \frac{\pi}{3}]$ by

$$
g_p(x) \;=\; \begin{cases}
\frac{1+\frac{3x}{\pi}}{\sqrt{1+(\frac{3x}{\pi})^2}} & \text{if} \quad -\frac{\pi}{3} \leq x < \frac{\pi}{3}\,, \\
\sqrt{2} & \text{if} \quad \frac{\pi}{3} \leq x < \pi - \frac{\pi}{3}\,, \\
\frac{1-3(\frac{x}{\pi}-1)}{\sqrt{1+(3(\frac{x}{\pi}-1))^2}} & \text{if} \quad \pi - \frac{\pi}{3} \leq x < \pi + \frac{\pi}{3}\,, \\
0 & \text{if} \quad \pi + \frac{\pi}{3} \leq x \leq 2\pi - \frac{\pi}{3}\,,
\end{cases}
\tag{3.13}
$$

for $p = 0,\dots,2^\lambda - 2$, and

$$
g_{2^\lambda-1}(x) \;=\; \begin{cases}
\frac{1+\frac{3x}{\pi}}{\sqrt{1+(\frac{3x}{\pi})^2}} & \text{if} \quad -\frac{\pi}{3} \leq x < \frac{\pi}{3}\,, \\
\sqrt{2} & \text{if} \quad \frac{\pi}{3} \leq x < \pi - \frac{2\pi}{3}\,, \\
\frac{2-3(\frac{x}{\pi}-1)}{\sqrt{4+(3(\frac{x}{\pi}-1))^2}} & \text{if} \quad \pi - \frac{2\pi}{3} \leq x < \pi + \frac{2\pi}{3}\,, \\
0 & \text{if} \quad \pi + \frac{2\pi}{3} \leq x \leq 2\pi - \frac{\pi}{3}\,.
\end{cases}
\tag{3.14}
$$

Hence, we can rewrite our basis functions

$$t_{2N_j^{\lambda,p}+s} \;=\; \frac{1}{\sqrt{N_{j-\lambda}}}\; T^{s-1}_{j-\lambda}\; T^{b(\lambda+1,2^\lambda+p)}_{j+1} \sum_{k=-M_j+1}^{N_{j-\lambda}+M_{j(p)}-1} g_p\left(\tfrac{k\pi}{N}\right)\cos(N_j^{\lambda,p}+k)\cdot\;.$$

Rewriting the shift by $T_{j-\lambda}^{s-1}$ and regrouping the frequencies, we obtain, now for $s = 0, \ldots, 2N_{j-\lambda} - 1$,

$$\begin{aligned}
&\sqrt{N_{j-\lambda}}\, T_{j+1}^{-b(\lambda+1,2^\lambda+p)}\, t_{2N_j^{\lambda,p}+s+1} \\
&= \sum_{k=-M_j+1}^{N_{j-\lambda}+M_{j(p)}-1} g_p(\tfrac{k\pi}{N}) \cos(N_{j-\lambda}(2^\lambda+p)+k)\left(\cdot - \tfrac{s\pi}{N_{j-\lambda}}\right) \\
&=(-1)^{ps} \sum_{k=-M_j+1}^{N_{j-\lambda}+M_{j(p)}-1} g_p(\tfrac{k\pi}{N})\Big(\cos\tfrac{ks\pi}{N_{j-\lambda}} \cos(N_j^{\lambda,p}+k)\cdot + \sin\tfrac{ks\pi}{N_{j-\lambda}} \sin(N_j^{\lambda,p}+k)\cdot\Big) \\
&= (-1)^{ps}\Bigg\{ \sum_{k=1}^{M_j-1} \cos\tfrac{ks\pi}{N_{j-\lambda}} \Big(g_p(\tfrac{k\pi}{N})\cos(N_j^{\lambda,p}+k)\cdot + g_p(-\tfrac{k\pi}{N})\cos(N_j^{\lambda,p}-k)\cdot\Big) \\
&\qquad\qquad + \sum_{k=1}^{M_j-1} \sin\tfrac{ks\pi}{N_{j-\lambda}} \Big(g_p(\tfrac{k\pi}{N})\sin(N_j^{\lambda,p}+k)\cdot - g_p(-\tfrac{k\pi}{N})\sin(N_j^{\lambda,p}-k)\cdot\Big) \\
&+ \sqrt{2} \sum_{k=M_j}^{N_{j-\lambda}-M_{j(p)}} \Big(\cos\tfrac{ks\pi}{N_{j-\lambda}}\cos(N_j^{\lambda,p}+k)\cdot + \sin\tfrac{ks\pi}{N_{j-\lambda}}\sin(N_j^{\lambda,p}+k)\cdot\Big) \\
&+ \sqrt{2}\cos N_j^{\lambda,p}\cdot \; + \sqrt{2}(-1)^s \cos N_j^{\lambda,p+1}\cdot \\
&+\sum_{k=1}^{M_{j(p)}-1}\cos\tfrac{(N_{j-\lambda}-k)s\pi}{N_{j-\lambda}} \Big(g_p(\pi-\tfrac{k\pi}{N})\cos(N_j^{\lambda,p+1}-k)\cdot + g_p(\pi+\tfrac{k\pi}{N})\cos(N_j^{\lambda,p+1}+k)\cdot\Big) \\
&+\sum_{k=1}^{M_{j(p)}-1}\sin\tfrac{(N_{j-\lambda}-k)s\pi}{N_{j-\lambda}} \Big(g_p(\pi-\tfrac{k\pi}{N})\sin(N_j^{\lambda,p+1}-k)\cdot \\
&\qquad\qquad\qquad - g_p(\pi+\tfrac{k\pi}{N})\sin(N_j^{\lambda,p+1}+k)\cdot\Big)\Bigg\}.
\end{aligned}$$

Here, we observe a change of basis in $W_j^{\lambda,p}$ from $\{t_{2N_j^{\lambda,p}+s} \,:\, s = 1, \ldots, 2N_{j-\lambda}\}$ to another orthonormal, more frequency-localized basis, $\{\theta_{j,k}^{\lambda,p} \,:\, k = 0, \ldots, 2N_{j-\lambda} - 1\}$, by means of the cosine and sine transform in form of

$$t_{2N_j^{\lambda,p}+s+1} = \frac{(-1)^{ps}}{\sqrt{N_{j-\lambda}}}\left(\sum_{k=0}^{N_{j-\lambda}} \cos\tfrac{ks\pi}{N_{j-\lambda}}\,\theta_{j,k}^{\lambda,p} + \sum_{k=1}^{N_{j-\lambda}-1}\sin\tfrac{ks\pi}{N_{j-\lambda}}\,\theta_{j,2N_{j-\lambda}-k}^{\lambda,p}\right).$$

Hence, the union of the smallest PMRA space V_λ and the wavelet packet spaces, as given in (3.8) and (3.9), provides us with a basis of the cosine and sine functions $\{t_k \,:\, k = 1, \ldots, 2N_\lambda\}$ and

$$\{\cos(N_h^{\lambda,q}+k)\cdot,\ \sin(N_h^{\lambda,q}+k)\cdot,:\ k = M_h, \ldots, N_{h-\lambda} - M_{h(q)}\} \subset W_h^{\lambda,q}$$

for all indices h, q which are included. Here, the meaning of $h(q)$ is the same as of $j(p)$ in (2.3). Moreover, in the overlapping ranges of frequencies, we have bases

$$\begin{aligned}
S_h^{\lambda,q} &:= \{\theta_{h,N_{h-\lambda}+k}^{\lambda,q} \,:\, k = -M_{h(q)}+1, \ldots, M_{h(q)}-1\}, \\
T_h^{\lambda,q} &:= \{\theta_{h,k}^{\lambda,q} \,:\, k = 0, \ldots, M_h - 1\} \cup \{\theta_{h,2N_{h-\lambda}-k}^{\lambda,q} \,:\, k = 1, \ldots, M_h - 1\}
\end{aligned}$$

for which $S_h^{\lambda,q-1}, T_h^{\lambda,q} \subset \mathbb{T}_{N_h^{\lambda,q}+M_h-1} \ominus \mathbb{T}_{N_h^{\lambda,q}-M_h+1}$ and $S_h^{\lambda,q-1} \perp T_h^{\lambda,q}$ hold, where for $q=0$, we have to write $S_{h-1}^{\lambda,2^\lambda-1}$. From the dimensions, we see that, for $h \le \lambda$ and $q = 0, \dots, 2^\lambda - 1$,

$$\operatorname{span}(S_h^{\lambda,q-1} \cup T_h^{\lambda,q}) \;=\; \mathbb{T}_{N_h^{\lambda,q}+M_h-1} \ominus \mathbb{T}_{N_h^{\lambda,q}-M_h+1}\,.$$

The shift operator in the definition of $\theta_{j-1,N_{j-\lambda}+k}^{\lambda,2^\lambda-1}$, for $k = -M_j+1, \dots, M_j-1$, can be omitted because $T_j^{b(\lambda+1,2^\lambda+2^\lambda-1)} = T_j^{2^{\lambda+1}-1}$, and a shift of $\theta_{j-1,N_{j-\lambda}+k}^{\lambda,2^\lambda-1}$ by a multiple of T_j yields a linear combination of $\{\theta_{j-1,N_{j-\lambda}+\ell}^{\lambda,2^\lambda-1} :\ |\ell| = k\}$.

Inclusion (3.10) is trivial for $\mu \le 2N_\lambda$. For $\mu > 2N_\lambda$, it follows from (3.11) and (3.12) due to

$$N_j^{\lambda,p} - M_j \;\ge\; \frac{\mu}{2}(1-\varepsilon) \;\ge\; \left\lfloor \frac{\mu}{2}(1-\varepsilon) \right\rfloor,$$

which can be shown similar to the proof of the inequality (2.5) for the degree. □

4. Norm of the Fourier sum operator

First, we prove some auxiliary results.

Lemma 4.1. *For any $N, M \in \mathbb{N}$ $N \ge M$, the functions φ_N^M in (3.4) satisfy*

$$\left\|\varphi_{2N}^{2M}\right\|_1 \;=\; \left\|\varphi_N^M\right\|_1$$

and

$$\left\|\varphi_N^1\right\|_1 \;<\; 1.6 + \frac{4}{\pi^2}\ln N\,. \tag{4.1}$$

P r o o f. From (3.5) it follows

$$\varphi_{2N}^{2M}(x) \;=\; \frac{\sin 2Nx \sin 2Mx}{4M\sin^2\frac{x}{2}} \;=\; \frac{\sin N(2x)\sin M(2x)}{M\sin^2\frac{2x}{2}}\;\cos^2\frac{x}{2}$$
$$=\; \varphi_N^M(2x)\;(1+\cos x)\,.$$

Since φ_N^M and φ_{2N}^{2M} are even and 2π-periodic, and $\cos\frac{y}{2} = -\cos\frac{2\pi-y}{2}$, we have

$$\left\|\varphi_{2N}^{2M}\right\|_1 \;=\; \frac{1}{\pi}\int_0^\pi |\varphi_N^M(2x)|\;(1+\cos x)\,dx \;=\; \frac{1}{\pi}\int_0^\pi |\varphi_N^M(y)|\;(1+\cos\tfrac{y}{2})\,dy$$
$$=\; \left\|\varphi_N^M\right\|_1 + \frac{1}{4\pi}\int_0^{2\pi} |\varphi_N^M(y)|\left(\cos\frac{y}{2} - \cos\frac{2\pi-y}{2}\right)\,dy \;=\; \left\|\varphi_N^M\right\|_1\,.$$

Concerning (4.1), we follow [Zy], Chap. II.12, and obtain

$$\begin{aligned}
\|\varphi_N^1\|_1 &= \frac{1}{\pi}\int_0^\pi \left|\frac{\sin Nx \sin x}{2\sin^2 x/2}\right| dx \\
&= \frac{2}{\pi}\int_0^\pi \frac{|\sin Nx|}{x}\,dx - \frac{2}{\pi}\int_0^\pi |\sin Nx| \left(\frac{1}{x} - \frac{1}{2}\cot\frac{x}{2}\right) dx \\
&< \frac{2}{\pi}\sum_{k=0}^{N-1}\int_0^{\pi/N} \frac{|\sin Nx|}{x + k\pi/N}\,dx \\
&< \frac{2}{\pi}\int_0^{\pi/N} \frac{\sin Nx}{x}\,dx + \frac{2N}{\pi^2}\sum_{k=1}^{N-1}\frac{1}{k}\int_0^{\pi/N} \sin Nx\,dx \\
&= \frac{2}{\pi}\,\mathrm{Si}(\pi) + \frac{4}{\pi^2}\sum_{k=1}^{N-1}\frac{1}{k} \;<\; 1.6 + \frac{4}{\pi^2}\ln N\,.
\end{aligned}$$

□

Lemma 4.2. *For any $M \in \mathbb{N}$ it holds that*

$$\frac{1}{2M}\left\|\sum_{s=0}^{2M-1}\left|1 + 2\sum_{k=1}^{M-1}\frac{M^2-k^2}{M^2+k^2}\cos k\left(\cdot - \frac{s\pi}{M}\right)\right|\right\|_\infty \;<\; 2.1\,.$$

Proof. The function to be estimated in the supremum norm is obviously $\frac{\pi}{M}$-periodic, so it suffices to consider its supremum over the interval $[0, \frac{\pi}{M})$. Defining a 2π-periodic absolutely continuous (coefficient) function by

$$g(x) \;=\; \frac{1-(\frac{x}{\pi})^2}{1+(\frac{x}{\pi})^2} \qquad (x \in [-\pi,\pi])\,,$$

we can write

$$\begin{aligned}
&\left\|\sum_{s=0}^{2M-1}\left|1 + 2\sum_{k=1}^{M-1}\tfrac{M^2-k^2}{M^2+k^2}\cos k\left(\cdot - \tfrac{s\pi}{M}\right)\right|\right\|_\infty \\
&= \sup_{x\in[0,\frac{\pi}{M})}\sum_{s=0}^{2M-1}\left|\sum_{k=-M+1}^{M} g\left(\tfrac{k\pi}{M}\right)\cos k(x - \tfrac{s\pi}{M})\right| \\
&= \sup_{\xi\in[0,1)}\sum_{s=0}^{2M-1}\left|\sum_{k=-M+1}^{M}\left(f^\xi\left(\tfrac{k\pi}{M}\right)\cos\tfrac{ks\pi}{M} + h^\xi\left(\tfrac{k\pi}{M}\right)\sin\tfrac{ks\pi}{M}\right)\right|\,,
\end{aligned}$$

where the functions

$$f^\xi(x) \;=\; g(x)\;\cos\xi x\,, \qquad h^\xi(x) \;=\; g(x)\;\sin\xi x \qquad (x\in(-\pi,\pi])$$

are even and odd, respectively, and are extended 2π-periodically. Expanding them in Fourier series,

$$f^\xi(x) \;=\; \tfrac{1}{2}a_0(f^\xi) + \sum_{n=1}^{\infty} a_n(f^\xi)\cos nx\,, \qquad h^\xi(x) \;=\; \sum_{n=1}^{\infty} b_n(h^\xi)\sin nx$$

and using the well-known identities

$$\sum_{k=-M+1}^{M} \cos m \tfrac{k\pi}{M} = 2M\ \delta_{0,m \bmod 2M}\,, \qquad \sum_{k=-M+1}^{M} \sin m \tfrac{k\pi}{M} = 0\,, \quad (m \in \mathbb{Z})\,,$$

we obtain, for arbitrary $\xi \in [0,1)$,

$$\begin{aligned} &\tfrac{1}{2M} \sum_{s=0}^{2M-1} \left| \sum_{k=-M+1}^{M} \left(f^\xi\left(\tfrac{k\pi}{M}\right) \cos \tfrac{ks\pi}{M} + h^\xi\left(\tfrac{k\pi}{M}\right) \sin \tfrac{ks\pi}{M} \right) \right| \\ &= \sum_{s=0}^{2M-1} \left| \tfrac{1}{2} \sum_{m=0}^{\infty} \left(a_{2Mm+s}(f^\xi) + a_{2M(m+1)-s}(f^\xi) + b_{2Mm+s}(h^\xi) - b_{2M(m+1)-s}(h^\xi) \right) \right| \\ &= \sum_{s=0}^{2M-1} \left| \sum_{m=0}^{\infty} \tfrac{1}{2\pi} \int_{-\pi}^{\pi} g(x) \left(\cos(2Mm+s-\xi)x + \cos(2M(m+1)-s+\xi)x \right) dx \right|, \end{aligned}$$

with $b_0(h^\xi) := 0$. These integrals can be estimated by means of the total variation. Since g is a 2π-periodic function having a first derivative in $[-\pi,\pi]$ of finite total variation $V[g']$, it holds that

$$\left| \tfrac{1}{\pi} \int_{-\pi}^{\pi} g(x) \cos rx \; dx \right| \;\le\; \frac{V[g']}{\pi r^2}\,, \qquad (r \in \mathbb{R} \setminus \{0\})\,. \tag{4.2}$$

Note that (4.2) is well-known for $r \in \mathbb{N}$, see e.g. [Fi], Chap. 19. As $g(-\pi) = g(\pi) = 0$, the result can be extended to all $r \in \mathbb{R} \setminus \{0\}$. From the first and second derivative,

$$g'(x) \;=\; -\frac{4x}{\pi^2 \left(1 + (\frac{x}{\pi})^2\right)^2}\,, \qquad g''(x) \;=\; -\frac{4\left(1 - 3(\frac{x}{\pi})^2\right)}{\pi^2 \left(1 + (\frac{x}{\pi})^2\right)^3}\,,$$

we see that g' is monotonously increasing in $[-\pi, -\frac{\pi}{\sqrt{3}}]$ and $[\frac{\pi}{\sqrt{3}}, \pi]$ and monotonously decreasing in $[-\frac{\pi}{\sqrt{3}}, \frac{\pi}{\sqrt{3}}]$. Further, the extrema are at $-\frac{\pi}{\sqrt{3}}$ and $\frac{\pi}{\sqrt{3}}$ where

$$\sup_x |g'(x)| \;=\; g'\left(-\frac{\pi}{\sqrt{3}}\right) \;=\; -\; g'\left(\frac{\pi}{\sqrt{3}}\right) \;=\; \frac{3\sqrt{3}}{4\pi}\,.$$

Hence, by $g'(-\pi) > 0$ and $g'(\pi) < 0$, we obtain

$$V[g'] \;=\; 4 \sup_x |g'(x)| \;=\; \frac{3\sqrt{3}}{\pi}\,.$$

For $-1 \le r \le 1$, instead of (4.2), we estimate the integral by the zeroth Fourier coefficient,

$$a_0(g) \;=\; \frac{2}{\pi} \int_0^{\pi} g(x)\; dx \;=\; 2 \int_0^1 \frac{1-y^2}{1+y^2}\; dy \;=\; \pi - 2\,.$$

So we can summarize

$$\frac{1}{2M}\left\|\sum_{s=0}^{2M-1}\left|1+2\sum_{k=1}^{M-1}\tfrac{M^2-k^2}{M^2+k^2}\cos k\left(\cdot-\tfrac{s\pi}{M}\right)\right|\right\|_\infty$$
$$\leq \quad \pi-2+\frac{V[g']}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2} \;=\; \pi-2+\frac{\sqrt{3}}{2} \;<\; 2.1\,.$$

□

By similar techniques we prove the following estimate.

Lemma 4.3. *For the functions t_k given in Definition* 2.1, *it holds that*

$$\left\|\sum_{s=1}^{2N_{j-\lambda}}\left|t_{2N_j^{\lambda,p}+s}\right|\right\|_\infty \;<\; 2.8\sqrt{2N_{j-\lambda}}\,.$$

Proof. By definition of the functions t_k we have

$$\left\|\sum_{s=1}^{2N_{j-\lambda}}\left|t_{2N_j^{\lambda,p}+s}\right|\right\|_\infty \;=\; \left\|\sum_{s=1}^{2N_{j-\lambda}}\left|T_{j+1}^{-b(\lambda+1,2^\lambda+p)}\;t_{2N_j^{\lambda,p}+s}\right|\right\|_\infty$$
$$= \sup_{x\in[0,2\pi)}\frac{1}{\sqrt{N_{j-\lambda}}}\sum_{s=0}^{2N_{j-\lambda}-1}\left|\sum_{k=-M_j+1}^{M_j-1}\tfrac{M_j+k}{\sqrt{M_j^2+k^2}}\cos(N_j^{\lambda,p}+k)(x-\tfrac{s\pi}{N_{j-\lambda}})\right.$$
$$+\sqrt{2}\sum_{k=N_j^{\lambda,p}+M_j}^{N_j^{\lambda,p+1}-M_{j(p)}}\cos k(x-\tfrac{s\pi}{N_{j-\lambda}})$$
$$\left.+\sum_{k=-M_{j(p)}+1}^{M_{j(p)}-1}\tfrac{M_{j(p)}-k}{\sqrt{M_{j(p)}^2+k^2}}\cos(N_j^{\lambda,p+1}+k)(x-\tfrac{s\pi}{N_{j-\lambda}})\right|.$$

Hence, we have to estimate the supremum of a $\frac{\pi}{N_{j-\lambda}}$-periodic function which can be done on an interval of length $\frac{\pi}{N_{j-\lambda}}$. Because of $2M_j \leq M_j+M_{j+1}\leq N_{j-\lambda}$, the degree of the cosine terms is less than $N_j^{\lambda,p}+N_{j-\lambda}+M_{j(p)}\leq N_{j-\lambda}(2^\lambda+p)+2N_{j-\lambda}-M_j$. For simplicity we set $N=N_{j-\lambda}$ and $M=M_j$. So, using the 2π-periodic absolutely continuous coefficient functions g_p defined in (3.13) and (3.14), for all $p=0,\ldots,2^\lambda-1$, we can write analogously to the proof of the previous lemma,

$$\left\|\sum_{s=0}^{2N_{j-\lambda}-1}\left|t_{2N_j^{\lambda,p}+s}\right|\right\|_\infty$$
$$= \sup_{x\in[0,\frac{\pi}{N})}\frac{1}{\sqrt{N}}\sum_{s=0}^{2N-1}\left|\sum_{k=-M+1}^{2N-M}g_p\left(\tfrac{k\pi}{N}\right)\cos(N(2^\lambda+p)+k)(x-\tfrac{s\pi}{N})\right|$$

$$\begin{aligned}
&= \sup_{\xi\in[0,1)} \frac{1}{\sqrt{N}} \sum_{s=0}^{2N-1} \left| \sum_{k=-M+1}^{2N-M} g_p\left(\tfrac{k\pi}{N}\right) \cos\left((2^\lambda+p)\pi + \tfrac{k\pi}{N}\right)(\xi - s) \right| \\
&= \sup_{\xi\in[0,1)} \frac{1}{\sqrt{N}} \sum_{s=0}^{2N-1} \left| \sum_{k=-M+1}^{2N-M} \left(f_p^\xi\left(\tfrac{k\pi}{N}\right) \cos s\tfrac{k\pi}{N} + h_p^\xi\left(\tfrac{k\pi}{N}\right) \sin s\tfrac{k\pi}{N}\right) \right| ,
\end{aligned}$$

with

$$\begin{aligned}
f_p^\xi(x) &= g_p(x)\ \cos((2^\lambda+p)\xi\pi + \xi x)\,, \\
h_p^\xi(x) &= g_p(x)\ \sin((2^\lambda+p)\xi\pi + \xi x)\,.
\end{aligned}$$

Note that the g_p as defined in (3.13) and (3.14) are given in $[-\frac{\pi}{3}, 2\pi - \frac{\pi}{3}]$ with $g_p(-\frac{\pi}{3}) = g_p(2\pi - \frac{\pi}{3}) = 0$. Extending g_p 2π-periodically, the functions f_p^ξ and h_p^ξ are absolutely continuous functions with first derivative of bounded variation on an interval of length 2π. We expand these functions in Fourier series,

$$\begin{aligned}
f_p^\xi(x) &= \tfrac{1}{2}a_0(f_p^\xi) + \sum_{n=1}^{\infty} a_n(f_p^\xi)\cos nx + \sum_{n=1}^{\infty} b_n(f_p^\xi)\sin nx\,, \\
h_p^\xi(x) &= \tfrac{1}{2}a_0(h_p^\xi) + \sum_{n=1}^{\infty} a_n(h_p^\xi)\cos nx + \sum_{n=1}^{\infty} b_n(h_p^\xi)\sin nx\,,
\end{aligned}$$

and use

$$\sum_{k=-M+1}^{2N-M} \cos m\tfrac{k\pi}{N} = 2N\,\delta_{0,m \bmod 2N}\,, \qquad \sum_{k=-M+1}^{2N-M} \sin m\tfrac{k\pi}{N} = 0 \quad (m \in \mathbb{Z}),$$

to rewrite, for any $\xi \in [0,1)$

$$\begin{aligned}
&\sum_{s=0}^{2N-1} \left| \sum_{k=-M+1}^{2N-M} \left(f_p^\xi\left(\tfrac{k\pi}{N}\right)\cos s\tfrac{k\pi}{N} + h_p^\xi\left(\tfrac{k\pi}{N}\right)\sin s\tfrac{k\pi}{N}\right) \right| \\
&= N \sum_{s=0}^{2N-1} \left| \sum_{m=0}^{\infty} \left(a_{2Nm+s}(f_p^\xi) + a_{2N(m+1)-s}(f_p^\xi) + b_{2Nm+s}(h_p^\xi) - b_{2N(m+1)-s}(h_p^\xi)\right) \right| \\
&= N \sum_{s=0}^{2N-1} \left| \sum_{m=0}^{\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} g_p(x)\left(\cos(2Nm+s-\xi)x + \cos(2N(m+1)-s+\xi)x\right) dx \right| .
\end{aligned}$$

Again, we estimate the Fourier coefficients by means of the total variation of the first derivative of g_p. Here we consider the closed interval $[-\frac{\pi}{3}, 2\pi - \frac{\pi}{3}]$. For $p = 0, \ldots, 2^\lambda - 2$, we observe that g_p' is monotonously increasing in $[-\frac{\pi}{3}, \frac{3-\sqrt{17}}{4}\frac{\pi}{3}]$ and $[\pi - \frac{3-\sqrt{17}}{4}\frac{\pi}{3}, \pi + \frac{\pi}{3}]$ and monotonously decreasing in $[\frac{3-\sqrt{17}}{4}\frac{\pi}{3}, \frac{\pi}{3}]$ and in $[\pi - \frac{\pi}{3}, \pi - \frac{3-\sqrt{17}}{4}\frac{\pi}{3}]$. Further, the extrema are at $\frac{3-\sqrt{17}}{4}\frac{\pi}{3}$ and $\pi - \frac{3-\sqrt{17}}{4}\frac{\pi}{3}$ where

$$\sup_x |g_p'(x)| = g_p'\left(\frac{3-\sqrt{17}}{4}\frac{\pi}{3}\right) = -\ g_p'\left(\pi - \frac{3-\sqrt{17}}{4}\frac{\pi}{3}\right) = \frac{1+\sqrt{17}}{(42-6\sqrt{17})^{3/2}}\frac{48}{\pi}\,.$$

Since $g_p'(-\frac{\pi}{3}) > 0$ and $g_p'(\pi + \frac{\pi}{3}) < 0$, the total variation of g_p' is

$$V[g_p'] = 4g_p'\left(\frac{3-\sqrt{17}}{4}\frac{\pi}{3}\right) = \frac{1+\sqrt{17}}{(42-6\sqrt{17})^{3/2}}\frac{192}{\pi}\,.$$

For $p = 2^\lambda - 1$, we have analogously the local minimum

$$g'_{2^\lambda-1}\left(\pi - \tfrac{3-\sqrt{17}}{2}\tfrac{\pi}{3}\right) \;=\; -\tfrac{1+\sqrt{17}}{(42-6\sqrt{17})^{3/2}}\tfrac{24}{\pi}\,.$$

Hence,

$$\begin{aligned} V[g'_{2^\lambda-1}] &= 2g'_{2^\lambda-1}\left(\tfrac{3-\sqrt{17}}{4}\tfrac{\pi}{3}\right) - 2g'_{2^\lambda-1}\left(\pi - \tfrac{3-\sqrt{17}}{2}\tfrac{\pi}{3}\right) \\ &= \tfrac{1+\sqrt{17}}{(42-6\sqrt{17})^{3/2}}\tfrac{144}{\pi}\,. \end{aligned}$$

The zeroth coefficient can be computed explicitely. For $p = 0, \ldots, 2^\lambda - 2$,

$$\begin{aligned} |a_0(g_p)| \;=\; \frac{1}{\pi}\int_{-\frac{\pi}{3}}^{2\pi-\frac{\pi}{3}} g_p(x)\,dx \;&=\; \sqrt{2}(1-\frac{2}{3}) + \frac{2}{3}\int_{-1}^{1}\frac{1+y}{\sqrt{1+y^2}}\,dy \\ &=\; \frac{\sqrt{2}}{3} + \frac{2}{3}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\,. \end{aligned}$$

Analogously,

$$|a_0(g_{2^\lambda-1})| \;=\; \frac{1}{\pi}\int_{-\frac{\pi}{3}}^{2\pi-\frac{\pi}{3}} g_{2^\lambda-1}(x)\,dx \frac{1+y}{\sqrt{1+y^2}}\,dy \;=\; \ln\frac{\sqrt{2}+1}{\sqrt{2}-1}\,.$$

Finally, we get the estimates

$$\left\|\sum_{s=0}^{2N_{j-\lambda}-1}\left|t_{2N_j^{\lambda,p}+s}\right|\right\|_\infty \;\le\; \sqrt{N_{j-\lambda}}\left(|a_0(g_p)| + \frac{V[g'_p]}{\pi}\sum_{k=1}^{\infty}\frac{1}{k^2}\right)$$

$$\le \begin{cases} \sqrt{2N_{j-\lambda}}\left(\frac{1}{3} + \frac{\sqrt{2}}{3}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{16\sqrt{2}(1+\sqrt{17})}{(42-6\sqrt{17})^{3/2}}\right) & \text{for } p = 0, \ldots, 2^\lambda - 2\,, \\ \sqrt{2N_{j-\lambda}}\left(\frac{\sqrt{2}}{2}\ln\frac{\sqrt{2}+1}{\sqrt{2}-1} + \frac{12\sqrt{2}(1+\sqrt{17})}{(42-6\sqrt{17})^{3/2}}\right) & \text{for } p = 2^\lambda - 1\,. \end{cases}$$

A simple calculation proves the Lemma. □

Now, let us estimate the norm of the partial sum operator with respect to our basis of Definition 2.1.

Theorem 4.4. *Let $\varepsilon > 0$ be given. The orthogonal projection operator S_μ in* (1.1) *for the functions $\{t_\mu \,:\, \mu \in \mathbb{N}\}$ from Definition* 2.1, *acting as an operator from $C_{2\pi}$ to $C_{2\pi}$, is uniformly bounded for all $\mu \in \mathbb{N}$. For $\varepsilon \le \frac{2}{3}$, it holds that*

$$\|S_\mu\|_{C\to C} \;<\; 15 + \frac{4}{\pi^2}\,\ln\frac{1}{\varepsilon}\,.$$

P r o o f. With the kernel function

$$K_\mu(x,\xi) := \sum_{k=1}^{\mu} t_k(\xi)\,t_k(x)$$

we have the well-known representation

$$\begin{aligned}\|S_\mu\|_{C\to C} &= \sup_{\|f\|_\infty=1}\ \sup_{x\in[0,2\pi)} \left|\sum_{k=1}^{\mu}\langle f\,,\,t_k\rangle\ t_k(x)\right| \\ &= \sup_{\|f\|_\infty=1}\ \sup_{x\in[0,2\pi)} \left|\frac{1}{2\pi}\int_0^{2\pi} f(\xi)K_\mu(x,\xi)d\xi\right| \\ &= \sup_{x\in[0,2\pi)} \tfrac{1}{2\pi}\int_0^{2\pi}|K_\mu(x,\xi)|\,d\xi \\ &= \sup_{x\in[0,2\pi)} \|K_\mu(x,\cdot)\|_1\,. \end{aligned} \tag{4.3}$$

Now we distinguish 3 cases.

Case I. Let $\mu \le 2N_\lambda$: For $\mu \in \{1,2\}$, it is trivial that $\|K_\mu(x,\cdot)\|_1 = 1$. For $k = 2,\dots,N_\lambda$, one easily obtains with (3.4) and (4.1)

$$\begin{aligned}\|K_{2k-1}(x,\cdot)\|_1 &= \frac{1}{2\pi}\int_0^{2\pi}\left|1+2\sum_{m=1}^{k-1}\cos m(\xi-x)\right|d\xi \\ &= \|\varphi^1_{k-1}(\cdot-x)-\cos k(\cdot-x)\|_1 \\ &\le \|\varphi^1_{k-1}\|_1+\frac{2}{\pi} \\ &< 2.3+\frac{4}{\pi^2}\ \ln k\end{aligned}$$

and analogously

$$\begin{aligned}\|K_{2k}(x,\cdot)\|_1 &= \frac{1}{2\pi}\int_0^{2\pi}\left|1+2\sum_{m=1}^{k-1}\cos m(\xi-x)+2\cos k\xi\cos kx\right|d\xi \\ &= \|\varphi^1_{k-1}(\cdot-x)+\cos k(\cdot+x)\|_1 \\ &\le \|\varphi^1_{k-1}\|_1+\frac{2}{\pi} \\ &< 2.3+\frac{4}{\pi^2}\ \ln k.\end{aligned}$$

Hence, the projection operator is bounded for all indices $\mu \le 2N_\lambda = 3\cdot 2^{\lambda+1}$. As $\lambda < \log_2 \frac{8}{3\varepsilon}$, we have

$$\begin{aligned}\sup_{x\in[0,2\pi)}\|K_\mu(x,\cdot)\|_1 &\le 2.3+\frac{4}{\pi^2}\ \ln\frac{\mu}{2} \le 2.3+\frac{4}{\pi^2}\ \ln(3\cdot 2^\lambda) \\ &< 2.3+\frac{4}{\pi^2}\ \ln\frac{8}{\varepsilon} < 3.2+\frac{4}{\pi^2}\ \ln\frac{1}{\varepsilon}.\end{aligned}$$

Therewith, the assertion is proved for $\mu \le 2N_\lambda$.

Case II. Let $\mu = 2N_j^{\lambda,p}$: We consider the projection operator $S_{2N_j^{\lambda,p}}$. Since the range of the orthogonal projection operator is independent of the choice of its basis, for every real orthonormal basis $\{s_{j,k}^{\lambda,p} \;:\; k=0,\ldots,2N_j^{\lambda,p}-1\}$ of the range of $S_{2N_j^{\lambda,p}}$ it holds that

$$K_{2N_j^{\lambda,p}}(x,\xi) \quad = \quad \sum_{k=0}^{2N_j^{\lambda,p}-1} s_k(\xi)\; s_k(x)\,.$$

By (4.3), we have that

$$\|S_{2N_j^{\lambda,p}}\|_{C\to C} \quad = \quad \sup_{x\in[0,2\pi)} \|K_{2N_j^{\lambda,p}}(x,\cdot)\|_1 \quad = \quad \sup_{x\in[0,2\pi)} \left\| \sum_{k=0}^{2N_j^{\lambda,p}-1} s_k(\cdot)\; s_k(x) \right\|_1 .$$

Using Lemma 3.4 and the functions $\theta_{j,k}^{\lambda,p}$ defined therein, we choose $s_0 \;=\; 1$, and

$$\begin{aligned} s_k &= \sqrt{2}\;\cos k\cdot && (k=1,\ldots,N_j-M_j)\,,\\ s_{N_j+k} &= \theta_{j-1,N_{j-\lambda-1}+k}^{\lambda,2^\lambda-1} && (k=-M_j+1,\ldots,M_j-1)\,,\\ s_{2N_j-k} &= \sqrt{2}\;\sin k\cdot && (k=1,\ldots,N_j-M_j)\,, \end{aligned}$$

if $p=0$, or

$$\begin{aligned} s_k &= \sqrt{2}\,T_{j+1}^{b(\lambda+1,2^\lambda+p-1)}(\cos k\cdot) && (k=1,\ldots,N_j^{\lambda,p}-M_j)\,,\\ s_{N_j^{\lambda,p}+k} &= \theta_{j,N_{j-\lambda}+k}^{\lambda,p-1} && (k=-M_j+1,\ldots,M_j-1)\,,\\ s_{2N_j^{\lambda,p}-k} &= \sqrt{2}\,T_{j+1}^{b(\lambda+1,2^\lambda+p-1)}(\sin k\cdot) && (k=1,\ldots,N_j^{\lambda,p}-M_j)\,, \end{aligned}$$

if $p\in\{1,\ldots,2^\lambda-1\}$. Since all summands in the kernel $K_{2N_j^{\lambda,p}}(x,\xi)$ have the same shift, we can take both of the shift operators $T_{j+1}^{b(\lambda+1,2^\lambda+p-1)}$, acting on ξ and x, respectively, in front of the sum and finally neglect them totally when we integrate over ξ and take the supremum over x. Hence, for $p=1,\ldots,2^\lambda-1$, we insert the inverse shift operators and consider suitably

$$\|S_{2N_j^{\lambda,p}}\|_{C\to C} = \sup_{x\in[0,2\pi)} \left\| \sum_{k=0}^{2N_j^{\lambda,p}-1} T_{j+1}^{-b(\lambda+1,2^\lambda+p-1)} s_k(\cdot)\; T_{j+1}^{-b(\lambda+1,2^\lambda+p-1)} s_k(x) \right\|_1 .$$

For the sake of simplicity, we set $N = N_j^{\lambda,p}$ and $M = M_j$. Using trigonometric formulas we obtain, for all $p=0,\ldots,2^\lambda-1$,

$$\begin{aligned} \|S_{2N}\|_{C\to C} \quad = \quad & \sup_{x\in[0,2\pi)} \frac{1}{2\pi}\int_0^{2\pi} \Bigg| 1 + 2\sum_{k=1}^{N-M} \cos k(\xi-x) + 2\cos N\xi\cos Nx \\ & + \sum_{k=1}^{M-1} \Big(\tfrac{(M+k)^2}{M^2+k^2}\cos(N-k)(\xi-x) + \tfrac{(M-k)^2}{M^2+k^2}\cos(N+k)(\xi-x) \Big) \\ & + \sum_{k=1}^{M-1} \tfrac{M^2-k^2}{M^2+k^2}\; r_k(x,\xi) \Bigg|\, d\xi \end{aligned}$$

with remainder terms

$$\begin{aligned} r_k(x,\xi) &= \cos(N(\xi+x)-k(\xi-x))+\cos(N(\xi+x)+k(\xi-x)) \\ &= 2\ \cos N(\xi+x)\ \cos k(\xi-x)\,. \end{aligned}$$

Decomposing $2\cos N\xi\cos Nx = \cos N(\xi-x)+\cos N(\xi+x)$, we can write

$$\begin{aligned} \|S_{2N}\|_{C\to C} &= \sup_{x\in[0,2\pi)} \|Q_N^M(\cdot-x)+\cos N(\cdot+x)\ R_M(\cdot-x)\|_1 \\ &\le \sup_{x\in[0,2\pi)} \|Q_N^M(\cdot-x)\|_1 + \sup_{x\in[0,2\pi)} \|R_M(\cdot-x)\|_1 \\ &= \|Q_N^M\|_1\ + \|R_M\|_1\,, \end{aligned}$$

with

$$Q_N^M(x) \ := \ 1+2\sum_{k=1}^{N-M}\cos kx + \sum_{k=-M+1}^{M-1}\tfrac{(M-k)^2}{M^2+k^2}\cos(N+k)x$$

and

$$R_M(x) \ := \ 1+2\sum_{k=1}^{M-1}\tfrac{M^2-k^2}{M^2+k^2}\cos kx\,.$$

For the $L^1_{2\pi}$ norm of a polynomial $p_n \in \mathbb{T}_n$, we use an inequality from [Ti], Chap. 4,

$$\text{(4.4)}\qquad \frac{1}{2\pi}\int_0^{2\pi}|p_n(\xi)|\,d\xi \ \le\ \sup_x \frac{1}{m}\sum_{s=0}^{m-1}\left|p_n\left(x-\tfrac{2\pi s}{m}\right)\right| \qquad (m\in\mathbb{N})\,.$$

Applying this estimate to R_M of degree $M-1$ with $m=2M$ we obtain from Lemma 4.2,

$$\text{(4.5)}\qquad \|R_M\|_1 \ \le\ \frac{1}{2M}\left\|\sum_{s=0}^{2M-1}\left|R_M(\cdot-\tfrac{s\pi}{M})\right|\right\|_\infty \ \le\ 2.1\,.$$

From the form (3.4) of the de la Vallée Poussin mean we deduce

$$Q_N^M \ = \ \varphi_N^M + \varrho_N^M$$

with

$$\begin{aligned} \varrho_N^M(x) &:= \sum_{k=-M+1}^{M-1}\left(\tfrac{(M-k)^2}{M^2+k^2}-\tfrac{M-k}{M}\right)\cos(N+k)x \\ &= \tfrac{2}{M}\ \sin Nx\sum_{k=1}^{M-1}k\ \tfrac{M^2-k^2}{M^2+k^2}\sin kx \\ &= -\tfrac{1}{M}\ \sin Nx\ R_M'(x)\,. \end{aligned}$$

By Bernstein's inequality, $\|p_n'\|_1 \leq n\|p_n\|_1$, for all $p_n \in \mathbb{T}_n$, we further get

$$\|\varrho_N^M\|_1 \leq \frac{1}{M}\|R_M'\|_1 \leq \|R_M\|_1 < 2.1 .$$

The $L^1_{2\pi}$ norm of $\varphi_N^M = \varphi_{N_j^{\lambda,p}}^{M_j}$ comes by iteration of Lemma 4.1,

$$\|\varphi_{N_j^{\lambda,p}}^{M_j}\|_1 = \|\varphi^1_{N_j^{\lambda,p}/M_j}\|_1 = \|\varphi^1_{3(2^\lambda+p)}\|_1 < 1.6+\frac{4}{\pi^2}\ln(3\cdot 2^{\lambda+1}) < 2.8+\frac{4}{\pi^2}\ln\frac{1}{\varepsilon}.$$

Altogether, we obtain

$$\begin{aligned} \|S_{2N_j^{\lambda,p}}\|_{C\to C} &\leq \|Q_{N_j^{\lambda,p}}^{M_j}\|_1 + \|R_{M_j}\|_1 \\ &\leq \|\varphi_{N_j^{\lambda,p}}^{M_j}\|_1 + 2\|R_{M_j}\|_1 < \frac{4}{\pi^2}\ln\frac{1}{\varepsilon} + 2.8 + 4.2 . \end{aligned} \tag{4.6}$$

Case III. Let $\mu > 2N_\lambda$ be arbitrary: For every $\mu > 2N_\lambda$ we have exactly one triple (j, p, s) of numbers $j \geq \lambda$, $p \in \{0, \ldots, 2^\lambda - 1\}$ and $s \in \{1, \ldots, 2N_{j-\lambda}\}$ such that $\mu = 2N_j^{\lambda,p} + s$. We decompose the orthogonal projection S_μ into the operator $S_{2N_j^{\lambda,p}}$ and a remainder,

$$S_\mu f = S_{2N_j^{\lambda,p}} f + \sum_{k=2N_j^{\lambda,p}+1}^{\mu} \langle f, t_k\rangle\, t_k .$$

Estimating the norm of the remainder operator we obtain

$$\begin{aligned} \sup_{\|f\|_\infty=1} \sup_{x\in[0,2\pi)} \left| \sum_{k=2N_j^{\lambda,p}+1}^{\mu} \langle f, t_k\rangle\, t_k(x)\right| &= \sup_{x\in[0,2\pi)} \frac{1}{2\pi}\int_0^{2\pi} \left|\sum_{k=2N_j^{\lambda,p}+1}^{\mu} t_k(\xi)\, t_k(x)\right| d\xi \\ &\leq \sup_{x\in[0,2\pi)} \sum_{k=2N_j^{\lambda,p}+1}^{2N_j^{\lambda,p+1}} |t_k(x)|\, \|t_k\|_1 \\ &= \left\|t_{2N_j^{\lambda,p}+1}\right\|_1 \left\| \sum_{s=1}^{2N_{j-\lambda}} \left|t_{2N_j^{\lambda,p}+s}\right| \right\|_\infty \\ &\leq \frac{1}{2N_{j-\lambda}} \left\| \sum_{s=1}^{2N_{j-\lambda}} \left|t_{2N_j^{\lambda,p}+s}\right| \right\|_\infty^2 , \end{aligned}$$

where the last inequality is obtained from (4.4) with $m = 2N_{j-\lambda}$. Applying Lemma 4.3 and adding together with (4.6) we finally get, for all $\mu > 2N_\lambda$,

$$\begin{aligned} \|S_\mu\|_{C\to C} &\leq \|S_{2N_j^{\lambda,p}}\|_{C\to C} + \sup_{\|f\|_\infty=1} \sup_{x\in[0,2\pi)} \left| \sum_{k=2N_j^{\lambda,p}+1}^{\mu} \langle f, t_k\rangle\, t_k(x)\right| \\ &< 7 + \frac{4}{\pi^2}\ln\frac{1}{\varepsilon} + (2.8)^2 , \end{aligned}$$

which concludes the proof. □

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References

[CMW] R. R. Coifman, Y. Meyer, M. V. Wickerhauser, Size properties of wavelet packets, in: *Wavelets and their applications* (M. B. Ruskai, G. Beylkin, R. Coifman, I. Daubechies, S. Mallat, Y. Meyer, L. Raphael, eds.), Jones and Bartlett, Boston, 1992, 453–470.

[CW] R. R. Coifman, M. V. Wickerhauser, Wavelets and adapted waveform analysis. A toolkit for signal processing and numerical analysis, in: *Different perspectives on wavelets* (I. Daubechies, ed.), Amer. Math. Soc., Providence, 1993, 119–153.

[Fa] G. Faber, Über die interpolatorische Darstellung stetiger Funktionen, *Jber. Deutsch. Math. Verein.* **23** (1914), 192–210.

[Fi] G. M. Fichtenholz, *Differential- und Integralrechnung III*, Deutscher Verlag der Wissenschaften, Berlin, 1970.

[KLT] Y. M. Koh, S. L. Lee, H. H. Tan, Periodic orthogonal splines and wavelets, *Appl. Comput. Harmon. Anal.* **2**(3) (1995), 201–218.

[LS] R. A. Lorentz, A. A. Sahakian, Orthogonal trigonometric Schauder bases of optimal degree for $C(0, 2\pi)$, *J. Fourier Anal. Appl.* **1** (1994), 103–112.

[PT2] G. Plonka, M. Tasche, A unified approach to periodic wavelets, in: *Wavelets: theory, algorithms, and applications* (C.K. Chui, L. Montefusco, L. Puccio, eds.), Academic Press, San Diego, 1994, 137–151.

[Pre] J. Prestin, Best approximation in Lipschitz spaces, in: *Coll. Math. Soc. Jan. Bolyai* 49: Alfred Haar Memorial Conf. (J. Szabados, K. Tandori, eds.) Budapest Jan. Bolyai Math. Soc., 1987, and North Holland Publ. Comp., Amsterdam-Oxford-New York, 1987, 753–759.

[PP] J. Prestin, S. Prößdorf, Error estimates in generalized trigonometric Hölder-Zygmund norms, *Z. Anal. Anwendungen* **9**(4) (1990), 343–349.

[PS1] J. Prestin, K. Selig, Interpolatory and orthonormal trigonometric wavelets, in: *Signal and image representation in Combined Spaces* (J. Zeevi, R. Coifman, eds.), Academic Press, San Diego, 1998, 201–255.

[PS2] J. Prestin, K. Selig, On the Gram matrix of translates of de la Vallée Poussin kernels, *Rostock. Math. Kolloq.* **49** (1995), 105–114.

[Pri1] A. A. Privalov, On the growth of degrees of polynomial bases and approximation of trigonometric projectors, *Mat. Zametki* **42**(2) (1987), 207–214; Engl. Transl.: *Math. Notes* **42**(2) (1987), 619–623.

[Pri2] A. A. Privalov, Growth of degrees of polynomial bases, *Mat. Zametki* **48**(4) (1990), 69–78; Engl. Transl.: *Math. Notes* **48**(4) (1990), 1017–1024.

[Pri3] A. A. Privalov, On an orthogonal trigonometric basis, *Mat. Sb.* **182**(3) (1991), 384–394; Engl. Transl.: *Math. USSR Sb.* **72**(2) (1992), 363–372.

[Prö] S. Prößdorf, Zur Konvergenz der Fourierreihen hölderstetiger Funktionen, *Math. Nachr.* **69** (1975), 7–14.

[Se1] K. Selig, Trigonometric wavelets for time–frequency–analysis, in: *Approximation theory, wavelets and applications* (S. P. Singh, ed.), Kluwer Academic Publ., Dordrecht, 1995, 453–464.

[Se2] K. Selig, Trigonometric wavelets and the uncertainty principle, in: *Approximation theory* (M. W. Müller, M. Felten, D. H. Mache, eds.), Akademie Verlag, Berlin, 1995, 293–304.

[Se3] K. Selig, *Periodische Wavelet-Packets und eine gradoptimale Schauderbasis*, Thesis, Univ. Rostock 1997, Shaker Verlag, Aachen, 1998.

[Ti] A. F. Timan, *Theory of approximation of functions of a real variable*, Pergamon Press, Oxford, 1963.

[Zy] A. Zygmund, *Trigonometric series*, Cambridge University Press, Cambridge, 1968.

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