Asymptotic formulas for the frame coefficients generated by Laguerre and Hermite type polynomials

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Abstract

Polynomial frames based on orthogonal decompositions with respect to weights of Laguerre and Hermite type are considered. Asymptotic formulas for the coefficients from expansions on such frames are presented. The ability to detect singularities of all order is studied in detail.

1 Introduction

Let I be an open real interval, $f : I \to I\!R$, and $r \geq 0$ be an integer. In this paper, an inner point $x_0 \in I$ will be called a singularity of f of order r if the

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$(r-1)-$st derivative, $f^{(r-1)}$, is absolutely continuous in a neighborhood of x_0 and $f^{(r)}$ is continuous in this neighborhood, except for a jump discontinuity at x_0. In many applications, for example, image and data compression, prediction of time series, and antenna technology, one needs to determine the localization of the singularities of a function of various order.

The theory of wavelets provides many popular tools for the solution of this problem (see for example [1]). But this theory has a few disadvantages, we point out two of them which were motivated to study trigonometric and orthogonal polynomials to determine singularities. To apply wavelet analysis, one needs either to know or to compute the wavelet coefficients of the target function. However, in some applications, the information available about the function consists of the coefficients of the function in some orthogonal polynomial expansion (see for example [2], [3]). Since the underlying function is not smooth, it is not appropriate first to compute an approximation to the function, and then to use this approximation to compute wavelet coefficients. Also, most of the compactly supported wavelets have the inherent limitation that they cannot simultaneously detect singularities of all orders.

A different construction of polynomial wavelets and bases is discussed in [4]. In [5], a general construction of polynomial frames based on quadrature rules to span previously studied polynomial spaces is presented. In the case of Jacobi polynomials, those frames are capable of simultaneously detecting singularities of all order of a given function.

In this paper in contrast to the previous ones we consider infinite real intervals $(0,\infty)$, $(-\infty,\infty)$ and orthogonal polynomials of Laguerre and Hermite type defined on them, respectively. Starting from the general construction of the frames in [5], we examine the localization properties of our frames and discuss how they can be used to detect singularities. For this reason we obtain an asymptotic representation for the frame coefficients in the vicinity of the singularities.

The paper is organized as follows. In Section 2 we follow [5] to introduce the polynomial frames we are going to study. In Section 3 we state and prove our main results. In Theorem 2 we obtain asymptotic formulas for the frame coefficients based on Laguerre polynomials and Theorem 3 gives the corresponding result for polynomials of Hermite type. Section 4 is devoted to the analysis of the ability of such frames to detect singularities. Theorem 4 and 5 show that the order of magnitude of frame coefficients can be distinguished between subintervals outside some vicinity of singularity and values near points of discontinuity. In Section 5 we present some numerical results, which show the ability to detect singularities knowing Hermite or Laguerre coefficients.

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2 Polynomial frames

In this paper we will consider two types of weight functions:

1) Laguerre type: $\omega_\alpha(x) := e^{-x}x^\alpha$, for $x \in (0,\infty)$ and $\alpha > -1$
and
2) Hermite type: $\omega(x) := e^{-x^2}$, for $x \in \mathbb{R}$.
It is well known that there exist unique sequences of orthogonal polynomials $\{L_k^\alpha\}_{k=0}^\infty$, $\{H_k\}_{k=0}^\infty$ defined by

$$\int_0^\infty L_n^\alpha(x)L_m^\alpha(x)\omega_\alpha(x)dx = \Gamma(\alpha+1)\binom{n+\alpha}{n}\delta_{n,m} \quad \text{for} \quad \alpha > -1 \tag{1}$$

and

$$\int_{-\infty}^\infty H_n(x)H_m(x)\omega(x)dx = \pi^{\frac{1}{2}}2^n n!\delta_{n,m}, \tag{2}$$

respectively. From the definition it follows that analogues of Rodrigues' formula are

$$e^{-x}x^\alpha L_n^\alpha(x) = \frac{1}{n}\Big(\frac{d}{dx}\Big)^n(e^{-x}x^{n+\alpha}),$$

$$e^{-x^2}H_n(x) = (-1)^n\Big(\frac{d}{dx}\Big)^n e^{-x^2}.$$

In the rest of this section polynomials p_k refer to L_k^α or H_k and weights $\tilde{\omega}$ are equal to ω_α or ω. By Π_n we denote as usual the subspaces of all algebraic polynomials of degree at most n.

As in [5] let $G = \{g_{k,l}\}_{k=0,\ldots,l,l=1,2,\ldots}$ be a triangular matrix, and for integer $l \geq 0$, let

$$K_l(G;x,t) := \sum_{k=0}^{l} g_{k,l}p_k(x)p_k(t).$$

We now define the frame operator by

$$\tau_l(G; f, x) := \int f(t) K_l(G; x, t)\tilde{\omega}(t)dt,$$

whenever the integral is well defined, and write

$$\tau_{l,k,m}(G; f) := \int f(t) K_l(G; t, x_{k,m})\tilde{\omega}(t)dt.$$

We will say that G is a scaling matrix if $g_{k,l} \neq 0, \quad k = 0, ..., l,$ and for $0 < s < N$, that G is a $(s-)$ partial frame matrix if $g_{k,2l} = 0, \ 0 \leq k < s; g_{k,2l} \neq 0, \ s \leq k \leq 2l$ (if $s = l$ we will call G just frame matrix). For a triangular matrix G, the matrix $G^{[\sigma]}$ is defined for integer σ by

$$g^{[\sigma]}_{k,l} = g^{\sigma}_{k,l}.$$

To explain the impact of the frame operators τ_l we include a representation proved in [5].

Theorem 1 *([5],Theorem 1). If G is a scaling or partial frame matrix, then for every $P \in W_n := \{P \in V_{n+1} : \int P(t)R(t)\tilde{\omega}(t)dt = 0, \ R \in V_n\}, \quad V_n := \Pi_N, N = 2^n$ and $m \geq 2N + 1$,*

$$P(x) = \sum_{k=1}^{m} \lambda_{k,m}\tau_{2N,k,m}(G, P) K_{2N}(G^{[-1]}; x, x_{k,m}).$$

Moreover, if G is a frame matrix, then $K_{2N}(G^{[-1]}; \cdot, x_{k,m}) \in W_n$.

If $f \in L^2_{\tilde{\omega}}$ and $n \geq 0$ is an integer, the orthogonal projection of f onto W_n is given by

$$\sum_{k=N+1}^{2N} a_k(f) p_k,$$

where $a_k(f) := \int f p_k \tilde{\omega} dt, \quad k = 0, 1, ...,$ whenever the integrals are well defined. Further, if G is a frame matrix then we can use Theorem 1 to obtain for the frame decomposition (convergent in $L^2_{\tilde{\omega}}$)

$$f = \sum_{r=0}^{2} a_r(f) p_r + \sum_{n=1}^{\infty}\sum_{k=1}^{m} \sqrt{\tilde{\omega}(x_{k,m})}\tau_{2N,k,m}(G; f)\Psi_{n,k,m},$$

where $m \geq 2N + 1, \ n = 0, 1, ...$ and the frame elements $\Psi_{n,k,m}$ are defined for integer $k = 1, ..., m, \ n = 0, 1, ...,$ by

$$\Psi_{n,k,m}(x) := \lambda_{k,m} K_{2N}(G^{[-1]}; x, x_{k,m})(\tilde{\omega}(x_{k,m}))^{-1/2}.$$

3 Asymptotics for frame coefficients of functions with discontinuous derivatives

Let $r \geq 0$ be an integer, and $f : [a,b] \to I\!R$ has a singularity of order r at some point $y \in (a,b)$ ($a = 0,\ \ b = \infty$ in Laguerre case and $a = -\infty,\ \ b = \infty$ in Hermite case). In this section, we examine the behavior of the frame coefficients $\sqrt{\tilde{\omega}(x_{k,m})}\tau_{2N,k,m}(G,f)$. For this purpose one important role plays the truncated power function defined by

$$x_+^r := \begin{cases} x^r, & \text{if } x \geq 0 \\ 0, & \text{otherwise.} \end{cases}$$

In fact, if f has a singularity at y with jump equal to d, then the function $f - \frac{d}{r!}(\cdot - y)_+^r$ is r times continuously differentiable at y. Repeating this process, most functions of practical interest can be written in the form

$$f(x) = \varphi(x) + \sum_{r=0}^{R} \frac{1}{r!} \sum_{k=1}^{k_r} d_{r,k}(x - y_{r,k})_+^r,$$

where φ is R times continuously differentiable on $[-1,1]$, $d_{r,k} \in I\!R$, and $y_{r,k} \in (-1,1)$. For functions of this form,

$$\tau_{2N}(G;f) = \tau_{2N}(G;\varphi) + \sum_{r=0}^{R} \frac{1}{r!} \sum_{k=1}^{k_r} d_{r,k}\tau_{2N}(G;(\cdot - y_{r,k})_+^r).$$

Our main goal is to obtain asymptotic formulas for the $\tau_{2N}(G;(\cdot - y)_+^r)$ in case of frame elements based on Laguerre and Hermite expansions which are similar to representations in [1], Theorem 2 for Jacobi expansions.

Let $g : [0,2\pi] \to I\!R$. The matrix G defined by

$$g_{k,l} := g\Big(\frac{2\pi k}{l+1}\Big), \qquad k = 0,\ldots,l, \quad l = 0,1,\ldots,$$

will be called the matrix *generated by* g. If $g(0) = g(2\pi) = 0$, then G is a partial frame matrix. In this case, we assume that g is extended to the whole real line as a $2\pi-$periodic function, and the coefficients $g_{k,l}$ are defined for all integer k using this extended function.

Let $q \geq 0$ be an integer, and let BV_0^q denote the class consisting of all $2\pi-$periodic functions h, which can be expressed in the form

$$h(x) = \frac{1}{(q-1)!}\int_0^x (x-t)_+^{q-1} h^{(q)}(t)dt, \quad \text{for all } x \in [0,2\pi],$$

where $h^{(q)}$ is a function having a bounded variation on $[0, 2\pi]$. We observe that if $h \in BV_0^q$, then $h^{(k)}(0) = h^{(k)}(2\pi) = 0, \quad k = 0, \ldots, q-1$.

3.1 The Laguerre case

Here we prove the following asymptotic representations of the frame coefficients for the Laguerre polynomial setting.

Theorem 2 *Let $\alpha > -1$ and r, q, p be nonnegative integers. Let ϵ_1 and ϵ_2 be fixed positive numbers, $\epsilon_1 < \epsilon_2$. Suppose that $g \in BV_0^q$, and for some $\kappa > 0$, $g(t) = 0$ for $t \in [0, \kappa]$. Let G be the matrix generated by g. Then there exists a sequence of functions $\Phi_\nu := \Phi_\nu(r, x, y)$ and $\Psi_\nu := \Psi_\nu(r, x, y)$ which are regular for $x, y > 0$ such that uniformly for $x, y \in [\epsilon_1, \epsilon_2]$ we have*

$$\begin{aligned}
&\left(\frac{2N+1}{2\pi}\right)^{r/2+1} \omega_\alpha^{1/2}(x)\tau_{2N}(G; (\cdot - y)_+^r, x) \\
&= \frac{(-1)^{r+1} r! \omega_{\alpha+r+\frac{1}{2}}^{1/2}(y) x^{-1/4}}{\pi} \left\{ \sum_{\nu=0}^{q} \left(\frac{2\pi}{2N+1}\right)^{\nu/2} \Phi_\nu(x, y) \right. \\
&\times \int_\kappa^{2\pi} g(t) t^{-r/2-1-\nu/2} \cos\left(\sqrt{\frac{2N+1}{2\pi}} 2(\sqrt{x} - \sqrt{y})\sqrt{t} + \frac{(r+1)\pi}{2}\right) dt \\
&+ \sum_{\nu=0}^{q} \left(\frac{2\pi}{2N+1}\right)^{\nu/2} \Psi_\nu(x, y) \\
&\times \left. \int_\kappa^{2\pi} g(t) t^{-r/2-1-\nu/2} \sin\left(\sqrt{\frac{2N+1}{2\pi}} 2(\sqrt{x} - \sqrt{y})\sqrt{t} + \frac{(r+1)\pi}{2}\right) dt \right\} \\
&+ O\left(N^{-\frac{q}{2}-1}\right) \qquad \textit{as } N \longrightarrow \infty.
\end{aligned}$$

The proof of this theorem involves certain facts concerning Laguerre polynomials. Let us recall them.

Lemma 1 *Let $k, r+1$ be positive integers with $k \geq r+1$.*
(a) Let $\alpha > -1$. Then for the Laguerre polynomials L_k^α we have

$$\int_t^\infty (x-t)^r L_k^\alpha(x) e^{-x} x^\alpha dx = \frac{(-1)^{r+1} r! (n-r-1)!}{n!} e^{-t} t^{\alpha+r+1} L_{k-r-1}^{\alpha+r+1}(t).$$

(b) For the Hermite polynomials we have

$$\int_t^\infty (x-t)^r H_k(x) e^{-x^2} dx = r! e^{-t^2} H_{k-r-1}(t).$$

Proof. From Rodrigues' formula we have

$$\int_t^\infty L_n^\alpha(x) e^{-x} x^\alpha dx = \frac{e^{-t} t^{\alpha+1}}{n} L_{n-1}^{\alpha+1}(t).$$

Integration by parts r times yields part (a). The second part (b) can be proved the same way. □

Lemma 2 *Let k, r be integers and let $h_k = \Gamma(\alpha+1)\binom{n+\alpha}{n}$ be the normalizing coefficients in (1). Then*

$$(h_k)^{-1} \frac{(k-r-1)!}{k!} = \frac{\Gamma(k-r)}{\Gamma(k+\alpha+1)} \sim k^{-\alpha-r-1} \left(\sum_{\ell=0}^{q} d_\ell k^{-\ell} + O(k^{-q-1}) \right), \tag{3}$$

where

$$d_\ell = (-1)^\ell \frac{B_\ell^{(-r-\alpha)}(-r)}{\ell!} \frac{\Gamma(\alpha+r+\ell+1)}{\Gamma(\alpha+r+1)}$$

and $B_\ell^a(-r)$ are the Bernoulli numbers.

Proof. The assertion follows from Watson's lemma (see [8], p. 67). □

To obtain asymptotic formulas instead of exact expressions we will use the following result.

Lemma 3 *(Perron's generalization of Fejer's formula, see [2] p. 198) Let ϵ_1 and ϵ_2 be fixed positive numbers, $\epsilon_1 < \epsilon_2$. Moreover, let α be an arbitrary real number. Then we have for $x > 0$*

$$\begin{aligned} L_n^{(\alpha)}(x) \\ &= \pi^{-\frac{1}{2}} e^{x/2} x^{-\alpha/2-\frac{1}{4}} n^{\alpha/2-\frac{1}{4}} \Big(\cos\left\{ 2(nx)^{\frac{1}{2}} - \alpha\pi/2 - \pi/4 \right\} \sum_{\nu=0}^{p-1} A_\nu(x) n^{-\nu/2} \\ &+ \sin\left\{ 2(nx)^{\frac{1}{2}} - \alpha\pi/2 - \pi/4 \right\} \sum_{\nu=0}^{p-1} B_\nu(x) n^{-\nu/2} + O(n^{-p/2}) \Big), \end{aligned}$$

where $A_\nu(x)$ and $B_\nu(x)$ are certain functions of x independent of n and regular for $x > 0$. The bound for the remainder holds uniformly in $[\epsilon_1, \epsilon_2]$. We notice that $A_0(x) = 1$ and $B_0(x) = 0$.

Next, we present a result which plays a crucial role in proving Theorem 2. The proof of this lemma is based on the well known Euler-McLaurin summation formula which we recall first.

Lemma 4 *(Euler-McLaurin summation formula) Let the function $\psi \in BV_0^q$ with $q \geq 1$ be given. Then*

$$\sum_{k=1}^{n} \psi\left(\frac{2\pi(k-1)}{n}\right) = \frac{n}{2\pi}\int_0^{2\pi} \psi(t)dt + \sum_{k=0}^{q-2} A_k \left(\frac{2\pi}{n}\right)^k \left(\psi^{(k)}(2\pi) - \psi^{(k)}(0)\right)$$
$$-\frac{n}{2\pi}\sum_{i=1}^{n}\int_0^{\frac{2\pi}{n}} \psi^{(q)}\left(\frac{2\pi i}{n} - z\right)\phi_q(z)dz,$$

where $A_k = \frac{B_k}{k!}$, B_k is a Bernoulli number and ϕ_q is a polynomial of the form

$$\phi_q(x) = \frac{x^q}{q!} - \frac{\frac{2\pi}{n}x^{q-1}}{(q-1)!} + \sum_{k=2}^{q-1} \frac{A_k \frac{2\pi}{n}^k x^{q-k}}{(q-k)!}. \qquad (4)$$

Lemma 5 *Let $h \in BV_0^q$ be given. For $n \to \infty$ we have*

$$\left|\frac{1}{n}\sum_{k=0}^{n-1} h\left(\frac{2\pi k}{n}\right)\cos(\sqrt{k}) - \frac{1}{2\pi}\int_0^{2\pi} h(t)\cos\left(\sqrt{\frac{nt}{2\pi}}\right)dt\right| = O\left(\frac{1}{n^{q/2+1}}\right),$$

where the constant involved in the $O-$term depends upon h and q.

Proof. For $q = 0$ the Lemma is a simple error estimate for the quadrature sum. For $q > 0$ we apply the Euler-McLaurin formula from Lemma 4 and obtain

$$
\begin{aligned}
&\left|\frac{1}{n}\sum_{k=0}^{n-1} h\left(\frac{2\pi k}{n}\right)\cos(\sqrt{k}) - \frac{1}{2\pi}\int_0^{2\pi} h(t)\cos\left(\sqrt{\frac{nt}{2\pi}}\right)dt\right| \\
&= \left|-\frac{1}{2\pi}\sum_{i=1}^{n}\int_{\frac{2\pi(i-1)}{n}}^{\frac{2\pi i}{n}} \left(h(t)\cos\sqrt{\frac{nt}{2\pi}}\right)^{(q)} \phi_q\left(\frac{2\pi i}{n}-t\right)dt\right| \\
&= \frac{1}{2q\pi}\left|\sum_{i=1}^{n}\int_{\frac{2\pi(i-1)}{n}}^{\frac{2\pi i}{n}} \left(h(t)\cos\sqrt{\frac{nt}{2\pi}}\right)^{(q)} d\phi_{q+1}\left(\frac{2\pi i}{n}-t\right)\right| \\
&= \frac{1}{2q\pi}\left|\sum_{i=1}^{n}\int_{\frac{2\pi(i-1)}{n}}^{\frac{2\pi i}{n}} \phi_{q+1}\left(\frac{2\pi i}{n}-t\right) d\left(h(t)\cos\sqrt{\frac{nt}{2\pi}}\right)^{(q)}\right| \\
&\leq \frac{1}{2q\pi}R\sum_{i=1}^{n} M_i = \frac{1}{2q\pi}RM,
\end{aligned}
$$

where $R = \max(|\phi_{q+1}(x)|)$ for $x \in [0, \frac{2\pi}{n}]$ and $M_i,\ M$ are the total variations of the function $\left(h(t)\cos\sqrt{\frac{nt}{2\pi}}\right)^{(q)}$ on the intervals $[\frac{2\pi(i-1)}{n}, \frac{2\pi i}{n}],\ [0, 2\pi]$ respectively. From (4) we obtain

$$
R \leq \left(\frac{2\pi}{n}\right)^{q+1}\frac{1}{(q+1)!} + \left(\frac{2\pi}{n}\right)^{q+1}\frac{1}{q!} + \sum_{k=2}^{q}\left(\frac{2\pi}{n}\right)^{q+1}\frac{A_k}{(q+1-k)!}
$$

$$
= \left(\frac{2\pi}{n}\right)^{q+1}\Big(\frac{1}{(q+1)!} + \frac{1}{(q)!} + \sum_{k=2}^{q}\frac{A_k}{(q+1-k)!}\Big) = \frac{1}{n^{q+1}}C_1,
$$

where C_1 depends on q only. Now we observe for the total variation on $[0, 2\pi]$

$$
\begin{aligned}
\mathrm{V}_0^{2\pi}\left[\left(h(\cdot)\cos\sqrt{\frac{n\cdot}{2\pi}}\right)^{(q)}\right] &= \mathrm{V}_0^{2\pi}\left[\sum_{k=0}^{q}\binom{q}{k}h^{(k)}(\cdot)\Big(\cos\sqrt{\frac{n\cdot}{2\pi}}\Big)^{(q-k)}\right] \\
&\leq \sum_{k=0}^{q}\binom{q}{k}\mathrm{V}_0^{2\pi}\Big[h^{(k)}(\cdot)\Big(\cos\sqrt{\frac{n\cdot}{2\pi}}\Big)^{(q-k)}\Big].
\end{aligned}
$$

At $t = 0$ the function $\left(\cos\sqrt{\frac{nt}{2\pi}}\right)^{(q-k)}$ has a pole of degree $(q-k)/2$ and the function $h^{(k)}(t)$ has a zero of degree $q-k$. Therefore, the functions

$h^{(k)}(t)\Big(\cos\sqrt{\frac{nt}{2\pi}}\Big)^{(q-k)}$ for $k = 1, \dots, q$ are continuous and have a bounded variation on $[0, 2\pi]$. Hence,

$$
\begin{aligned}
&\sum_{k=0}^{q}\binom{q}{k}\mathrm{V}_0^{2\pi}\Big[h^{(k)}(\cdot)\Big(\cos\sqrt{\frac{n\cdot}{2\pi}}\Big)^{(q-k)}\Big] \\
&\quad\le \mathrm{V}_0^{2\pi}[h^{(q)}] + \binom{q}{q/2}\sum_{k=0}^{q-1}\int_0^{2\pi}\Big|\Big(h^{(k)}(t)\Big(\cos\sqrt{\frac{nt}{2\pi}}\Big)^{(q-k)}\Big)'\Big|dt \\
&\quad\le \mathrm{V}_0^{2\pi}[h^{(q)}] + 4\pi q\binom{q}{q/2}\Big(\frac{n}{2\pi}\Big)^{q/2}\max_{k=0,\dots,q}\max_{t\in[0,2\pi]}|h^{(k)}(t)|.
\end{aligned}
$$

Thus, we have

$$RM \le \frac{1}{n^{q+1}}C_1\Big(\mathrm{V}_0^{2\pi}[h^{(q)}] + C_2 n^{q/2}\max_{k=0,\dots,q}\max_{t\in[0,2\pi]}|h^{(k)}(t)|\Big),$$

which completes the proof of Lemma 5. □

Furthermore we need the following series representation.

Lemma 6 *Let k, r be integers. Then*

$$\cos((k-r-1)^{1/2}) = \sum_{m=0}^{\infty}\Big(\frac{\rho_m(r)\cos(k^{1/2})}{k^m} + \frac{\varrho_m(r) sin(k^{1/2})}{k^{\frac{m+1}{2}}}\Big),$$

$$\sin((k-r-1)^{1/2}) = \sum_{m=0}^{\infty}\Big(\frac{\rho_m(r)\sin(k^{1/2})}{k^m} + \frac{\varrho_m(r) cos(k^{1/2})}{k^{\frac{m+1}{2}}}\Big),$$

where $\rho_m(r)$ and $\varrho_m(r)$ are polynomials of degree m.

Proof. The above functions are the real and imaginary part of

$$\exp(i(k-r-1)^{1/2})$$

which can be rewritten as

$$
\begin{aligned}
&\exp\left(ik^{1/2}\sum_{n=0}^{\infty}\binom{n}{1/2}(-1)^n\Big(\frac{r+1}{k}\Big)^n\right) \\
&\quad= \exp\left(ik^{1/2}\right)\left(1 - \frac{(r+1)i}{2k^{1/2}} - \frac{(r+1)^2}{k} + i\rho(r)\frac{1}{k^{3/2}} + \cdots\right),
\end{aligned}
$$

where ρ is a polynomial of r. □

Proof of Theorem 2. We start from the representation

$$\tau_{2N}(G;(\cdot - y)_+^r, x) = \sum_{k=0}^{2N} g_{k,2N}\{h_k\}^{-1} p_k(x) \int_y^\infty (x-y)^r p_k(x)\omega_\alpha(x)dx.$$

Using Lemma 1 we transform this formula for the case of $p_k(x) := L_k^\alpha(x)$ to

$$\begin{aligned}
&\tau_{2N}(G;(\cdot - y)_+^r, x)\\
&= \sum_{k=0}^{2N} g_{k,2N}\{h_k\}^{-1} L_k^\alpha(x) \frac{(-1)^{r+1} r!(k-r-1)!}{k!} e^{-t} t^{\alpha+r+1} L_{k-r-1}^{\alpha+r+1}(y)\\
&= (-1)^{r+1} r!\omega_{\alpha+r+1}(y) \sum_{k=0}^{2N} g_{k,2N} \frac{(h_k)^{-1}(k-r-1)!}{k!} L_k^\alpha(x) L_{k-r-1}^{\alpha+r+1}(y). \quad (5)
\end{aligned}$$

Applying Lemma 3 we can write

$$\begin{aligned}
&L_k^{(\alpha)}(x)\\
&= \pi^{-\frac{1}{2}} e^{x/2} x^{-\alpha/2-\frac{1}{4}} \cos\{2(kx)^{\frac{1}{2}} - \alpha\pi/2 - \pi/4\} \sum_{j=0}^q A_j(x) k^{\alpha/2-j/2-1/4}\\
&+ \pi^{-\frac{1}{2}} e^{x/2} x^{-\alpha/2-\frac{1}{4}} \sin\{2(kx)^{\frac{1}{2}} - \alpha\pi/2 - \pi/4\} \sum_{j=0}^q B_j(x) k^{\alpha/2-j/2-1/4}\\
&+ O(k^{\alpha/2-(q+1)/2-1/4})
\end{aligned}$$

and

$$\begin{aligned}
&L_{k-r-1}^{(\alpha+r+1)}(y)\\
&= \pi^{-\frac{1}{2}} e^{y/2} y^{-(\alpha+r)/2-\frac{3}{4}} \cos\{2((k-r-1)y)^{\frac{1}{2}} - (\alpha+r)\pi/2 - 3\pi/4\}\\
&\times \sum_{\nu=0}^q \sum_{l=0}^{q/2} d_l \tilde{A}_\nu(y) k^{(\alpha+r)/2+\frac{1}{4}-\nu/2-l}\\
&+ \pi^{-\frac{1}{2}} e^{y/2} y^{-(\alpha+r)/2-\frac{3}{4}} \sin\{2((k-r-1)y)^{\frac{1}{2}} - (\alpha+r)\pi/2 - 3\pi/4\}\\
&\times \sum_{\nu=0}^q \sum_{l=0}^{q/2} d_l \tilde{B}_\nu(y) k^{(\alpha+r)/2+\frac{1}{4}-\nu/2-l} + O(k^{(\alpha+r)/2+\frac{1}{4}-(q+1)/2}).
\end{aligned}$$

Thus, using Lemma 6 and after collecting cosine and sine terms together and also relabeling indices we obtain

$$
\begin{aligned}
&L_{k-r-1}^{(\alpha+r+1)}(y) \\
&= \pi^{-\frac{1}{2}} e^{y/2} y^{-(\alpha+r)/2-\frac{3}{4}} \cos\{2(k)y)^{\frac{1}{2}} - (\alpha+r)\pi/2 - 3\pi/4\} \\
&\times \sum_{\nu=0}^{q} \tilde{A}_\nu(y)\tilde{\rho}_\nu(r) k^{(\alpha+r)/2+\frac{1}{4}-\nu/2} \\
&+ \pi^{-\frac{1}{2}} e^{y/2} y^{-(\alpha+r)/2-\frac{3}{4}} \sin\{2(ky)^{\frac{1}{2}} - (\alpha+r)\pi/2 - 3\pi/4\} \\
&\times \sum_{\nu=0}^{q} \tilde{B}_\nu(y)\tilde{\varrho}_\nu(r) k^{(\alpha+r)/2+\frac{1}{4}-\nu/2} + O(k^{(\alpha+r)/2+\frac{1}{4}-(q+1)/2}).
\end{aligned}
$$

Therefore, after rearranging the double sum, again relabeling the indices, collecting cosine, sine terms and terms of order $O(k^{\alpha+r/2-(q+1)/2})$ together we conclude

$$
\begin{aligned}
&L_k^{(\alpha)}(x) L_{k-r-1}^{(\alpha+r+1)}(y) = \pi^{-1} e^{x/2} x^{-\alpha/2-\frac{1}{4}} e^{y/2} y^{-(\alpha+r)/2-\frac{3}{4}} \\
&\times \sum_{\nu=0}^{q} k^{\alpha+\frac{r}{2}-\frac{\nu}{2}} \Big\{ (\phi_\nu(x,y) - \eta_\nu(x,y)) \cos(2k^{\frac{1}{2}}(x^{\frac{1}{2}} + y^{\frac{1}{2}}) + \psi) \\
&\quad + (\phi_\nu(x,y) + \eta_\nu(x,y)) \cos(2k^{1/2}(x^{1/2} - y^{1/2}) + \phi) \\
&\quad + (\chi_\nu(x,y) + \psi_\nu(x,y)) \sin(2k^{1/2}(x^{1/2} + y^{1/2}) + \psi) \\
&\quad + (\psi_\nu(x,y) - \chi_\nu(x,y)) \sin(2k^{1/2}(x^{1/2} - y^{1/2}) + \phi) \Big\} \\
&+ O(k^{\alpha+r/2-(q+1)/2}),
\end{aligned} \tag{6}
$$

where $\psi = -\alpha\pi - \pi - \frac{r\pi}{2}$ and $\phi = \frac{\pi+\pi r}{2}$ and $\phi_\nu(x,y),\ \chi_\nu(x,y),\ \psi_\nu(x,y),\ \eta_\nu(x,y)$ are continuous functions depending also on α and r. Substituting from (6) and (3) into (5) and simplifying as before, we get

$$
\begin{aligned}
\tau_{2N}(G;(\cdot-y)_+^r,x) &= \frac{(-1)^{r+1} r! \omega^{1/2}_{\alpha+r+\frac{1}{2}}(y)\omega^{-1/2}_{\alpha+\frac{1}{2}}(x)}{\pi} \\
\times & \sum_{\nu=0}^{q}\sum_{k=0}^{2N} k^{-r/2-\nu/2-1}\Big\{(\tilde{\phi}_\nu(x,y)-\tilde{\eta}_\nu(x,y))\cos(2k^{1/2}(x^{1/2}+y^{1/2})+\psi) \\
& +(\tilde{\phi}_\nu(x,y)+\tilde{\eta}_\nu(x,y))\cos(2k^{1/2}(x^{1/2}-y^{1/2})+\phi) \\
& +(\tilde{\chi}_\nu(x,y)+\tilde{\psi}_\nu(x,y))\sin(2k^{1/2}(x^{1/2}+y^{1/2})+\psi) \\
& +(\tilde{\psi}_\nu(x,y)-\tilde{\chi}_\nu(x,y))\sin(2k^{1/2}(x^{1/2}-y^{1/2})+\phi)\Big\} \\
+ & \ O(k^{-r/2-(q+1)/2-1}).
\end{aligned} \tag{7}
$$

Using Lemma 3, we transform the inner sum over k into

$$
\begin{aligned}
& \sum_{k=0}^{2N} g_{k,2N} k^{-r/2-\nu/2-1}\cos(2k^{1/2}(x^{1/2}-y^{1/2})+\phi) \\
&= \left(\frac{2\pi}{2N+1}\right)^{r/2+1+\nu/2}\sum_{k=0}^{2N} g_{k,2N}\left(\frac{2\pi k}{2N+1}\right)^{-r/2-\nu/2-1} \\
&\quad \times\cos(2k^{1/2}(x^{1/2}-y^{1/2})+\phi) \\
&= \left(\frac{2\pi}{2N+1}\right)^{r/2+1+\nu/2} \\
&\quad \times\Big\{\int_0^{2\pi} g(t)t^{-r/2-1-\nu/2}\cos\left(\sqrt{\frac{2N+1}{2\pi}}2(\sqrt{x}-\sqrt{y})\sqrt{t}+\phi\right)dt \\
&\quad +O\Big(\frac{1}{(2N+1)^{q/2+1}}\Big)\Big\}.
\end{aligned} \tag{8}
$$

Similarly, we handle all other analogous terms. Since the functions $g(t)t^{-r/2-1-\nu/2}$ are all in BV_0^q, a repeated integration by parts gives

$$
\begin{aligned}
& \sum_{k=0}^{2N} g_{k,2N} k^{-r/2-1-\nu/2}\cos(2k^{1/2}(x^{1/2}+y^{1/2})+\psi) \\
= & \left(\frac{2\pi}{2N+1}\right)^{r/2+1+\nu/2}\Big\{O\Big(\frac{1}{\sqrt{N}(\sqrt{x}+\sqrt{y})}\Big)^q+O\Big(\frac{1}{(2N+1)^{q/2+1}}\Big)\Big\},
\end{aligned} \tag{9}
$$

and similarly

$$\sum_{k=0}^{2N} g_{k,2N} k^{-r/2-1-\nu/2} \sin(2k^{1/2}(x^{1/2}+y^{1/2})+\psi)$$

$$= \left(\frac{2\pi}{2N+1}\right)^{r/2+1+\nu/2} \Big\{ O\Big(\frac{1}{\sqrt{N}(\sqrt{x}+\sqrt{y})}\Big)^{q} + O\Big(\frac{1}{(2N+1)^{q/2+1}}\Big)\Big\} \quad (10)$$

Substituting from (9), (10), (8) into (7), we obtain

$$\tau_{2N}(G;(\cdot-y)_+^r,x) = \frac{(-1)^{r+1} r! \omega^{1/2}_{\alpha+r+\frac{1}{2}}(y)\omega^{-1/2}_{\alpha+\frac{1}{2}}(x)}{\pi}$$

$$\times \quad \left(\frac{2\pi}{2N+1}\right)^{r/2+1} \left(\sum_{\nu=0}^{q} \left(\frac{2\pi}{2N+1}\right)^{\nu/2} (\tilde{\phi}_\nu(x,y)+\tilde{\eta}_\nu(x,y)) \right.$$

$$\times \int_{\kappa}^{2\pi} g(t) t^{-r/2-1-\nu/2} \cos(\sqrt{\frac{2N+1}{2\pi}} 2(\sqrt{x}-\sqrt{y})\sqrt{t} + (r+1)\pi/2) dt$$

$$+ \sum_{\nu=0}^{q} (\frac{2\pi}{2N+1})^{\nu/2} (\tilde{\psi}_\nu(x,y)+\tilde{\chi}_\nu(x,y))$$

$$\left. \times \int_{\kappa}^{2\pi} g(t) t^{-r/2-1-\nu/2} \sin(\sqrt{\frac{2N+1}{2\pi}} 2(\sqrt{x}-\sqrt{y})\sqrt{t} + (r+1)\pi/2) dt \right)$$

$$+ \quad O\Big(N^{-r/2-2-q/2}\Big),$$

which completes the proof of Theorem 2. □

3.2 The Hermite case

In this section we obtain the asymptotic formula for the frame coefficients in case of frame elements constructed on Hermite polynomials. Let us emphasize here thatin this case one can detect singularities on arbitrary points $y \in \mathbb{R}$.

Essentially we use the close relation of Hermite polynomials to those of Laguerre.

Lemma 7 *(see [6], p.106) Hermite polynomials can be entirely reduced to Laguerre polynomials with the parameters $\alpha = \pm\frac{1}{2}$, namely*

$$H_{2m}(x) = (-1)^m 2^{2m} m! L_m^{(-\frac{1}{2})}(x^2), \qquad H_{2m+1}(x) = (-1)^m 2^{2m+1} m! x L_m^{(\frac{1}{2})}(x^2).$$

Theorem 3 *Let r, q be nonnegative integers and let ℓ be defined by $r = 2\ell$ or $r = 2\ell + 1$. Let ϵ_1 and ϵ_2 be fixed real numbers, $\epsilon_1 < \epsilon_2$. Suppose that $g \in BV_0^q$, and for some $\kappa > 0$, $g(t) = 0$ for $t \in [0, \kappa]$. Let G be the matrix generated by g. Then there exist sequences of functions $\tilde{\Phi}_\nu := \tilde{\Phi}_\nu(r, x, y)$, $\tilde{\Psi}_\nu := \tilde{\Psi}_\nu(r, x, y)$ which are regular for $x, y \in I\!R$ such that uniformly for $x, y \in [\epsilon_1, \epsilon_2]$*

$$
\begin{aligned}
&\left(\frac{2N+1}{2\pi}\right)^{\ell+1} \omega^{1/2}(x)\tau_{2N}(G; (\cdot - y)_+^r, x) \\
= \quad & \frac{(-1)^{\ell+1}2^{-2\ell-1}e^{-y^2/2}}{\pi} \left\{ \sum_{\nu=0}^{q} \left(\frac{2\pi}{2N+1}\right)^{\nu/2} \tilde{\Phi}_\nu(x, y) \right. \\
\times \quad & \int_\kappa^{2\pi} g(t)t^{-\nu/2-\ell-1} \cos\left(\sqrt{\frac{2N+1}{2\pi}} 2(x-y)\sqrt{t}\right) dt \\
+ \quad & \sum_{\nu=0}^{q} \left(\frac{2\pi}{2N+1}\right)^{\nu/2} \tilde{\Psi}_\nu(x, y) \\
\times \quad & \left. \int_\kappa^{2\pi} g(t)t^{-\nu/2-\ell-1} \sin\left(\sqrt{\frac{2N+1}{2\pi}} 2(x-y)\sqrt{t}\right) dt \right\} \\
+ \quad & O\Big(N^{-q/2-1}\Big) \quad as \quad N \to \infty.
\end{aligned}
$$

Proof. As in the proof of Theorem 2 we start from the representation

$$
\tau_{2N}(G; (\cdot - y)_+^r, x) = \sum_{k=0}^{2N} g_{k,2N}\{h_k\}^{-1} p_k(x) \int_y^\infty (x-y)^r p_k(x)\omega(x)dx.
$$

Using Lemma 1 we transform this formula with $p_k(x) := H_k(x)$ into

$$
\tau_{2N}(G, (\cdot - y)^r, x) = \sum_{k=0}^{2N} g_{k,2N} r!(\sqrt{\pi}2^k k!)^{-1} H_k(x) H_{k-r-1}(y) e^{-y^2}. \qquad (11)
$$

We will prove the theorem for even $r = 2\ell$, $\ell = 0, 1, 2, \ldots$. The case of r odd gives the same result and requires a similar way of proof. We point out that for $r = 2\ell$ the expression $k - r - 1$ is odd if k is even and even if k is odd. This parity relation allows us to divide the sum in the right side of (11) into an even and odd part and to apply Lemma 7 which gives

$$
\begin{aligned}
&\tau_{2N}(G,(\cdot-y)^r,x)\\
&= \sum_{n=0}^{N} \frac{g_{2n,2N}(2\ell)!}{\sqrt{\pi}}(-1)^{l+1}\frac{2^{2n}2^{-2l-1}n!(n-\ell-1)!}{(2n)!}L_n^{(-\frac{1}{2})}(x^2)L_{n-l-1}^{(\frac{1}{2})}(y^2)e^{-y^2}y\\
&+ \sum_{n=0}^{N-1} \frac{g_{2n+1,2N}(2\ell)!}{\sqrt{\pi}}(-1)^{l}\frac{2^{2n}2^{-2l}n!(n-\ell)!}{(2n+1)!}L_n^{(\frac{1}{2})}(x^2)L_{n-l}^{(-\frac{1}{2})}(y^2)e^{-y^2}x.
\end{aligned}
$$

Replacing the Laguerre polynomials by their asymptotic representation with the help of Lemma 3 and simplifying as in the proof of Theorem 2, the last expression can be transformed into an sum consisting of eight terms. For the further analysis it is sufficient to consider two of them. For the six remaining terms the same procedure can be applied. Thus, let us consider the following expression

$$
\begin{aligned}
S &= \frac{(-1)^{l+1}2^{-2l-1}e^{-y^2}e^{x^2}}{\pi^{3/2}}\sum_{n=0}^{N} g_{2n,2N}\frac{2^{2n}n!(n-l-1)!}{(2n)!}\\
&\quad\times\Big\{\cos\{2(x+y)n^{\frac{1}{2}}\}\sum_{\nu=0}^{q}\phi_\nu(x,y)n^{-\frac{1}{2}-\nu/2}+O(k^{-\frac{1}{2}-(q+1)/2})\Big\}\\
&\quad+\frac{(-1)^{l}2^{-2l}e^{-y^2}e^{x^2}}{\pi^{3/2}}\sum_{n=0}^{N-1} g_{2n+1,2N}\frac{2^{2n}n!(n-l)!}{(2n+1)!}\\
&\quad\times\Big\{\cos\{2(x+y)n^{\frac{1}{2}}\}\sum_{\nu=0}^{q}\tilde{\phi}_\nu(x,y)n^{-\frac{1}{2}-\nu/2}+O(k^{-\frac{1}{2}-(q+1)/2})\Big\}. \quad (12)
\end{aligned}
$$

To obtain an asymptotic representation for $\frac{2^{2n}n!(n-l)!}{(2n+1)!}$ we expand it as a power series. Analogously to Lemma 2 we use Legendre's multiplication formula

$$
\Gamma(n+\frac{1}{2}) = \sqrt{\pi}2^{-2n}\frac{(2n)!}{(n)!}, \quad n=0,1,2,\ldots
$$

and again a corollary of Watson's lemma (cf. [8], p. 67)

$$
\frac{\Gamma(z+a)}{\Gamma(z+b)} \sim z^{a-b}\sum_{n=0}^{\infty}(-1)^n\frac{B_n^{(a-b+1)}(a)}{n!}\frac{\Gamma(b-a+n)}{\Gamma(b-a)}z^{-n}.
$$

Summarizing

$$\frac{2^{2n}n!(n-l-1)!}{(2n)!} = \frac{\sqrt{\pi}\Gamma(n-l)}{\Gamma(n+\frac{1}{2})} = \sqrt{\pi}n^{-l-1/2}\sum_{k=0}^{q} c_k n^{-k} + O(n^{-q-l-3/2})$$

we can put this approximation into the expression (12) which finally leads to

$$\begin{aligned} S &= \frac{(-1)^{l+1}2^{-2l-1}e^{-y^2}e^{x^2}}{\pi} \\ &\quad \times\Big(\sum_{\nu=0}^{q}\psi_\nu(x,y)\sum_{n=0}^{N} g_{2n,2N}\cos\{2(x+y)n^{\frac{1}{2}}\}n^{-\nu/2-l-1} \\ &\quad +\sum_{\nu=0}^{q}\tilde{\psi}_\nu(x,y)\sum_{n=0}^{N-1} g_{2n+1,2N}\cos\{2(x+y)n^{\frac{1}{2}}\}n^{-\nu/2-l-1}\Big) \\ &\quad +O(N^{-(q+1)/2-l-1}). \end{aligned}$$

To complete the proof one needs to apply quadrature formulas similar to Lemma 3 in the same way as in Theorem 2. □

4 Detection of singularities

In this section we present a corollary of Theorem 2 to highlight some interesting aspects of the behavior of $\tau_{2N}(\omega_\alpha, G; (\cdot - y)_+^r, x)$ near and away from the singularity y. In the Laguerre case we obtain the following result.

Theorem 4 *With the same notation as in Theorem 2, there exist constants $a, b \in I\!R$ and a positive constant c, (all possibly depending on y), such that if $\sqrt{\frac{2N+1}{2\pi}}(\sqrt{x}-\sqrt{y}) \in (a,b)$, then*

$$\Big(\frac{2N+1}{2\pi}\Big)^{r/2+1}|\tau_{2N}(G; (\cdot - y)_+^r, x)| \geq c. \tag{13}$$

If $x \neq y$, then

$$\Big(\frac{2N+1}{2\pi}\Big)^{r/2+1}|\tau_{2N}(G; (\cdot - y)_+^r, x)| \leq \frac{c_1}{(N|\sqrt{x}-\sqrt{y}|)^{(q+1)/2}}. \tag{14}$$

Proof. To prove (13) we apply Theorem 2 with $q = 0$. The function

$$z \to \int_{\kappa}^{2\pi} \frac{g(t)}{t^{r/2+1+\nu/2}} \cos\left(zt + \frac{(r+1)\pi}{2}\right) dt$$

is an entire function, and hence, nonzero on some interval $[a, b]$. For $|\sqrt{x} - \sqrt{y}| \leq 1/\sqrt{N}$, the numbers a, b can be chosen independent of x. In light of this observation, Theorem 2 leads to (13). Since, the functions $g(t)t^{-r/2-1-\nu}$, $\nu = 0, \ldots, q$ are all in BV_0^q, a repeated integration by parts shows that

$$\int_{\kappa}^{2\pi} g(t)t^{-r/2-1-\nu/2} \cos\Big(\sqrt{\frac{2N+1}{2\pi}} 2(\sqrt{x} - \sqrt{y})\sqrt{t} + \frac{(r+1)\pi}{2}\Big) dt$$
$$= O\Big((N|\sqrt{x} - \sqrt{y}|)^{-r/2-1-\nu/2}\Big)$$

and similarly

$$\int_{\kappa}^{2\pi} g(t)t^{-r/2-1-\nu/2} \sin\Big(\sqrt{\frac{2N+1}{2\pi}} 2(\sqrt{x} - \sqrt{y})\sqrt{t} + \frac{(r+1)\pi}{2}\Big) dt$$
$$= O\Big((N|\sqrt{x} - \sqrt{y}|)^{-r/2-1-\nu/2}\Big).$$

Therefore, Theorem 2 implies (14). □.

For the Hermite case the corresponding result has the following form.

Theorem 5 *With the same notation as in Theorem 3, there exist constants* $a, b \in I\!R$ *and a positive constant* c*, (all possibly depending on* y*), such that if* $\sqrt{\frac{2N+1}{2\pi}}(\sqrt{x} - \sqrt{y}) \in (a, b)$*, then*

$$\Big(\frac{2N+1}{2\pi}\Big)^{l+1} |\tau_{2N}(G; (\cdot - y)_+^r, x)| \geq c.$$

If $x \neq y$*, then*

$$\Big(\frac{2N+1}{2\pi}\Big)^{l+1} |\tau_{2N}(G; (\cdot - y)_+^r, x)| \leq \frac{c_1}{(N|\sqrt{x} - \sqrt{y}|)^{(q+1)/2}}.$$

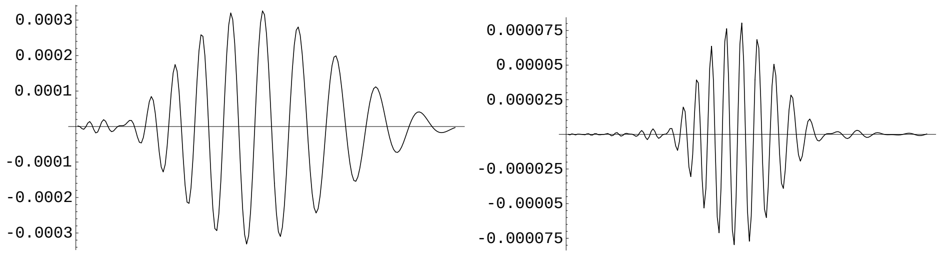

Figure 1: (a)$\tau_{100}(G_2, (\cdot - 10)^3, x)$ on $(0, 20]$ (b)$\tau_{400}(G_2, (\cdot - 10)^3, x)$ on $(0, 20]$.

5 Numerical experiments

In this section, we present the results of some numerical computations. As a test case, we are interested in detecting the singularity of the function

$$f(x) := (x - 10)_+^3. \tag{15}$$

In our experiments, we consider $\alpha = 0$ in case of Laguerre type of weight function. The matrix $G := G_q$ is generated by

$$g(x) = \begin{cases} (x-\pi)^q (x-2\pi)^q & \text{for } \; x \in [\pi, 2\pi], \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to see that $g \in BV_0^q$.

Figure 1 shows the behavior of $\tau_{100}(G_2; f)$ on $(0, 20]$ and $\tau_{400}(G_2; f)$ on $(0, 20]$. One can see that for larger values of N one obtains better localization of the transform near 10.

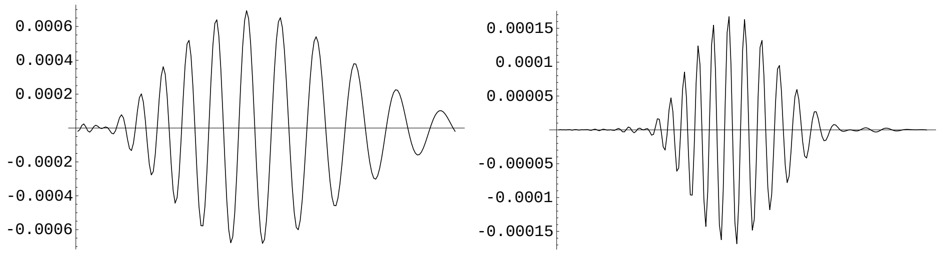

Figure 2: (a)$\tau_{100}(G_3, (\cdot - 10)^3, x)$ on $(0, 20]$ (b)$\tau_{400}(G_4, (\cdot - 10)^3, x)$ on $(0, 20]$.

Comparing Figure 1 with Figure 2 we can see that localization is not improved with $q = 3$. The explanation of this fact lies in the nature of constants in (13) and (14). They grow with q and for better decay one needs larger N.

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