Multivariate Trigonometric Wavelet Decompositions

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Abstract

Decomposition and reconstruction algorithms for nested spaces of multivariate trigonometric polynomials are described. The scaling functions of these spaces are chosen as fundamental polynomials of Lagrange interpolation. Their properties are crucial for the approach presented here. In particular, FFT–algorithms can be used efficiently.

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1 Introduction

Signal and image processing is nowadays mathematically examined in terms of wavelet analysis. We are interested here in decomposing multivariate periodic functions into trigonometric wavelet parts. In particular, the wavelets are trigonometric polynomials built from certain de la Vallée Poussin means of the Dirichlet kernel. So we did not start by periodizing a wavelet on the real line but, used directly, the properties of the corresponding trigonometric kernels. A first attempt in this direction has been done by C. K. Chui and H. N. Mhaskar in [1]. They used Fourier sums of the Haar scaling functions and wavelets to construct a univariate wavelet decomposition of $L_2(\mathbb{T})$. Then every wavelet space consists of trigonometric polynomials with frequencies from a certain dyadic band. The disadvantages of that approach are the lack of interpolatory properties of the scaling funtions and wavelets and their large oscillations based on the unboundedness of the Fourier sums with respect to the uniform norm. To overcome these difficulties we follow the ideas of A. A. Privalov ([2]), where he used a special de la Vallée Poussin mean of the Fourier sum to construct a polynomial basis of $C(\mathbb{T})$ with a certain order of growth of their degrees. A wavelet analysis for the whole scale of possible de la Vallée Poussin means with the corresponding two-scale relations and decomposition formulas is outlined in [3] and [4]. Moreover, a unified introduction to periodic univariate wavelets based on calculations with Fourier coefficients is due to G. Plonka and M. Tasche ([5]).

Here, we consider bivariate trigonometric polynomials on the torus based on tensor products of the univariate scaling functions and wavelets. Then, the general multivariate case can be handled analogously. As usual, we obtain, in the bivariate case, three different wavelet spaces on every level. For the decomposition and reconstruction of a given function, signal or image, it is important to know the basis transformations between the different levels of sample and wavelet spaces. Therefore, the main part of this paper deals with the investigation of the corresponding matrices. The block circulant structure of these matrices allows to apply the Fast Fourier Transform. Starting with the function values on a certain regular grid with $c2^j \times c2^j$ points, the algorithms need $\mathcal{O}(j2^{2j})$ operations to decompose a function into its wavelet parts. For a fixed number of interpolation points, one can choose with the different de la Vallée Poussin means between better localization and lower degree of the polynomials.

2 Definitions

Let us briefly recall the definitions of the univariate scaling functions as described in [2],[4].

Definition 2.1 *For $\ell \in \mathbb{N}$, the Dirichlet kernel $D_\ell \in T_\ell$ is defined as*

$$D_\ell = \frac{1}{2} + \sum_{k=1}^{\ell} \cos kx = \begin{cases} \frac{\sin(\ell+\frac{1}{2})x}{2\sin\frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ \ell + \frac{1}{2} & \text{for } x \in 2\pi\mathbb{Z}. \end{cases}$$

Here, and in the following, T_ℓ denotes the linear space of trigonometric polynomials of degree at most ℓ. Furthermore, for $N, M \in \mathbb{N}, N > M$, the de la Vallée Poussin means φ_N^M are given as

$$\begin{aligned} \varphi_N^M(x) &= \frac{1}{2MN} \sum_{\ell=N-M}^{N+M-1} D_\ell(x) \\ &= \begin{cases} \frac{\sin Nx \, \sin Mx}{4NM\sin^2\frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ 1 & \text{for } x \in 2\pi\mathbb{Z}. \end{cases} \end{aligned}$$

For $j \in \mathbb{N}_0$, the univariate scaling function $\phi_{j,0}$ is defined as

$$\phi_{j,0} = \varphi_{N_j}^{M_j},$$

with

$$\begin{aligned} N_j &= c \cdot 2^j, \\ M_j &= \begin{cases} 2^{j-\lambda} & \text{for } j \geq \lambda, \\ 1 & \text{for } j < \lambda, \end{cases} \end{aligned} \tag{1}$$

where $\lambda \in \mathbb{N}_0$ and $c \in \mathbb{N}$ fulfil the condition

$$c \geq \begin{cases} 3 & \text{for } \lambda = 0, \\ 2 & \text{for } \lambda = 1, \\ 1 & \text{for } \lambda \geq 2. \end{cases} \tag{2}$$

The choice of λ and c corresponds to different de la Vallée Poussin means. The most interesting ones are the limit cases $\lambda = 0, c = 3$ and $\lambda = \infty$ with arbitrary c. While the first one describes an ordinary de la Vallée Poussin mean, the second one is the usual modified Dirichlet kernel. For all possible N_j and M_j the crucial interpolatory property of $\phi_{j,0}$ is

$$\phi_{j,0}\left(\frac{k\pi}{N_j}\right) = \delta_{k,0}, \quad k = 0, 1, \ldots, 2N_j - 1. \tag{3}$$

Let us denote the translates of $\phi_{j,0}$ by $\phi_{j,k}(x) = \phi_{j,0}(x - \frac{k\pi}{N_j}), k = 0, 1, \ldots, 2N_j - 1$.

Definition 2.2 *For $j \in \mathbb{N}_0$, $K=(k_1,k_2)$, $k_1,k_2=0,1,\ldots,2N_j-1$, the bivariate scaling functions $\Phi_{j,K}:\mathbb{T}^2\to\mathbb{R}$ are defined as*

$$\Phi_{j,K}(x,y)=\phi_{j,k_1}(x)\cdot\phi_{j,k_2}(y).$$

Analogously, these functions fulfil interpolatory conditions

$$\Phi_{j,K}\left(\frac{m\pi}{N_j},\frac{n\pi}{N_j}\right)=\delta_{k_1,m}\cdot\delta_{k_2,n},\qquad \text{for all}\quad k_1,k_2,m,n=0,1,\ldots,2N_j-1.$$

For $j\in\mathbb{N}_0$, the sample spaces V_j are defined by

$$V_j=\operatorname{span}\{\Phi_{j,K}:K=(k_1,k_2),\ k_1,k_2=0,1,\ldots,2N_j-1\}.$$

Because of the interpolatory properties of the functions $\Phi_{j,K}$, one obtains immediately

$$\dim V_j=4N_j^2.$$

For $j\in\mathbb{N}_0$, the interpolation operator L_j mapping into the sample space V_j is defined as

$$L_jf(x,y)=\sum_{k_1=0}^{2N_j-1}\sum_{k_2=0}^{2N_j-1}f\left(\frac{k_1\pi}{N_j},\frac{k_2\pi}{N_j}\right)\Phi_{j,(k_1,k_2)}(x,y).$$

The following properties of the operator L_j can be easily deduced from the univariate case (see references [2], [6], Chap. 10):

(i) $\qquad L_jf\in T^2_{N_j+M_j-1},\qquad$ where $T_l^2:=T_l\otimes T_l$,

(ii) $\qquad L_jf\left(\frac{m\pi}{N_j},\frac{n\pi}{N_j}\right)=f\left(\frac{m\pi}{N_j},\frac{n\pi}{N_j}\right)\quad$ for all $\quad m,n\in\mathbb{Z}$,

(iii) $\qquad L_jf=f\qquad$ for all $\quad f\in T^2_{N_j-M_j}\cup V_j,$,

hence,

(iv) $\qquad T^2_{N_j-M_j}\subset V_j\subset T^2_{N_j+M_j-1}$.

Property (iv) and our special choice of M_j and N_j (see (1), (2)) guarantee the imbedding

$$V_j\subset V_{j+1}. \tag{4}$$

Hence, we obtain a chain of finite dimensional subspaces V_j of the Hilbert space $L_2(\mathbb{T}^2)$ with

$$clos_{L_2(\mathbb{T}^2)}\bigcup_{j\in\mathbb{N}_0}V_j=L_2(\mathbb{T}^2).$$

Now let us consider for $j\in\mathbb{N}_0$ the univariate wavelet functions

$$\psi_{j,m}(x)=2\phi_{j+1,2m+1}(x)-\phi_{j,m}\left(x-\frac{\pi}{2N_j}\right),$$

which fulfil the following interpolatory properties:

$$\psi_{j,m}\left(\frac{(2\ell+1)\pi}{N_{j+1}}\right)=\delta_{\ell,m}\qquad\text{for all}\quad \ell\in\mathbb{Z}, \tag{5}$$

$$\psi_{j,m}\left(\frac{\ell\pi}{N_j}\right)=-\phi_{j,m}\left(\frac{(2\ell-1)\pi}{N_{j+1}}\right)\qquad\text{for all}\quad \ell\in\mathbb{Z}. \tag{6}$$

In 1991 A. A. Privalov proved the orthogonality between the univariate scaling functions and the wavelets from the same level j.

Theorem 2.1 [2] *For all $m, n = 0, \ldots, 2N_j - 1$, it holds that*

$$\frac{1}{2\pi} \int_0^{2\pi} \phi_{j,m}(x) \psi_{j,n}(x) \, dx = 0.$$

In $L_2(\mathbb{T}^2)$ we use the standard inner product

$$\langle f, g \rangle = \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} f(x,y) g(x,y) \, dx dy \, .$$

Hence, the tensor product structure of our scaling functions $\Phi_{j,K}$ yields straightforward the corresponding wavelets and wavelet spaces.

Definition 2.3 *For $j \in \mathbb{N}_0$, $s = 1, 2, 3$, the wavelet spaces W_j^s are defined by*

$$W_j^s = span\{\Psi_{j,K}^s : K = (k_1, k_2), \; k_1, k_2 = 0, 1, \ldots, 2N_j - 1\},$$

where

$$\begin{aligned} \Psi_{j,K}^1(x,y) &= \phi_{j,k_1}(x) \cdot \psi_{j,k_2}(y), \\ \Psi_{j,K}^2(x,y) &= \psi_{j,k_1}(x) \cdot \phi_{j,k_2}(y) \end{aligned}$$

and

$$\Psi_{j,K}^3(x,y) = \psi_{j,k_1}(x) \cdot \psi_{j,k_2}(y).$$

From the interpolatory properties of the functions $\phi_{j,k}$ and $\psi_{j,k}$, we obtain

$$\dim W_j^s = 4N_j^2.$$

Theorem 2.2 *For the above defined sample spaces V_j and wavelet spaces W_j, it holds that*

$$V_{j+1} = V_j \oplus^\perp W_j^1 \oplus^\perp W_j^2 \oplus^\perp W_j^3,$$

where $\oplus^\perp$ denotes orthogonal summation.

Proof: From (4) and from the the definition of V_j and W_j^s, it is clear that every space on the right-hand side is a subspace of V_{j+1}. From Theorem 2.1, it follows the orthogonality of the univariate functions $\phi_{j,m}$ and $\psi_{j,n}$ and so the pairwise orthogonality of the spaces V_j and W_j^s, $s = 1, 2, 3$. Hence, the summation is really an orthogonal summation and the sum itself is a subspace of V_{j+1}. In order to obtain the equality, we consider

$$\begin{aligned} \dim V_{j+1} &= 4N_{j+1}^2 = 4 \cdot 4N_j^2 \\ &= \dim V_j + \sum_{s=1}^{3} \dim W_j^s \end{aligned}$$

which completes the proof. ■

To illustrate the different de la Vallée Poussin cases, we present the following pictures.

Figure 1 includes scaling functions $\Phi_{1,(6,6)}$, whereas Figure 2 shows $\Psi^1_{1,(6,6)}$ and Figure 3 $\Psi^3_{1,(6,6)}$. The upper pictures a) are for the parameters $c = 3, \lambda = 0$ and the lower pictures b) are with $c = 3, \lambda = \infty$ (compare (1),(2)). I.e., a) corresponds to the ordinary de la Vallée Poussin case and b) reflects the pure Fourier case.

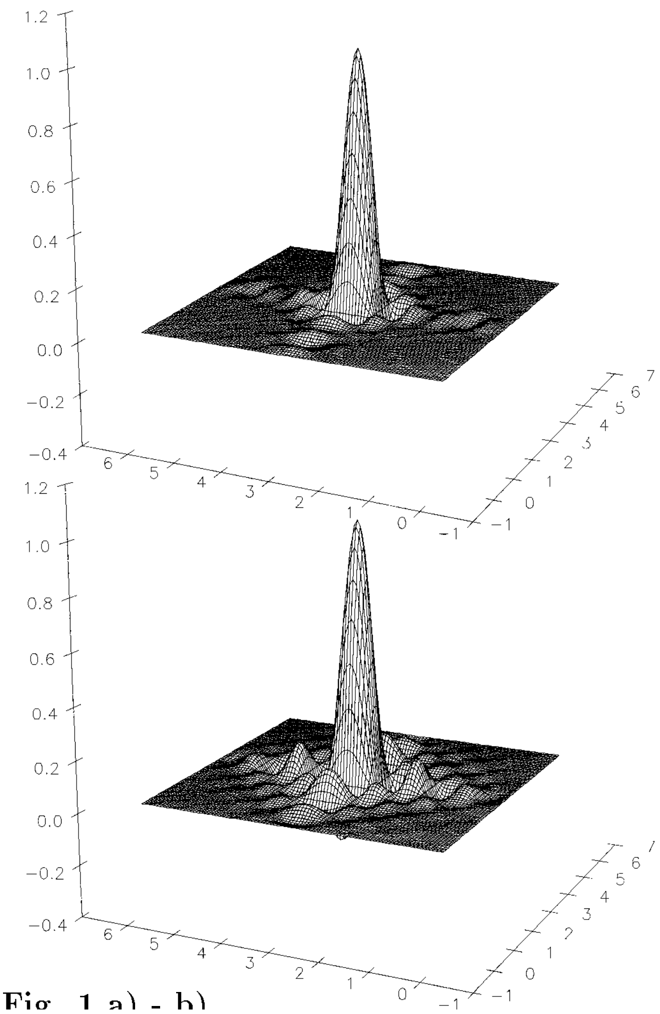

Fig. 1 a) - b)

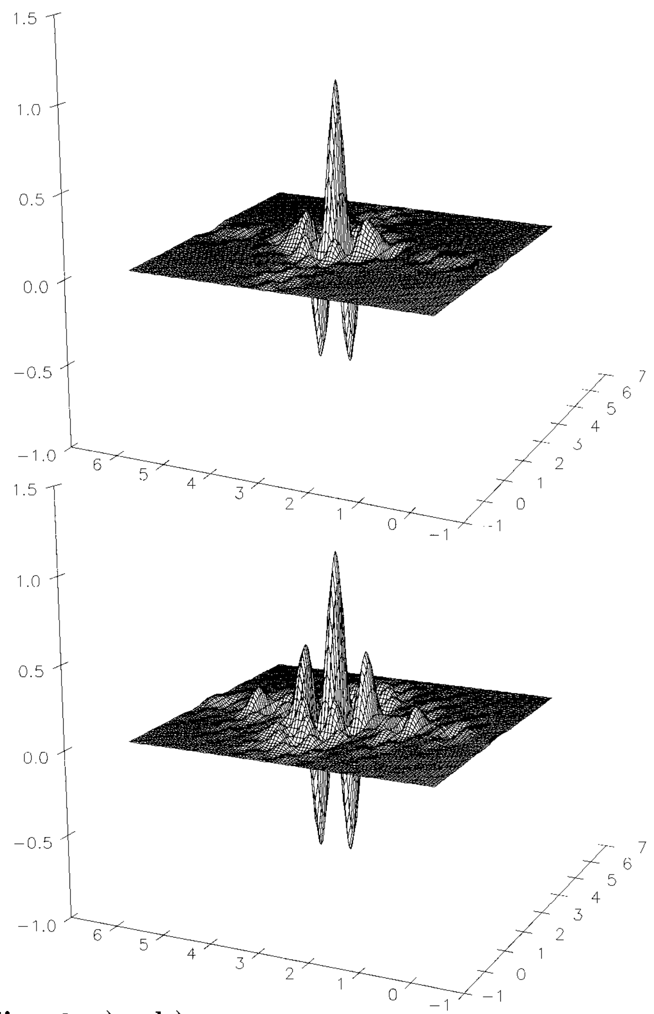

Fig. 2 a) - b)

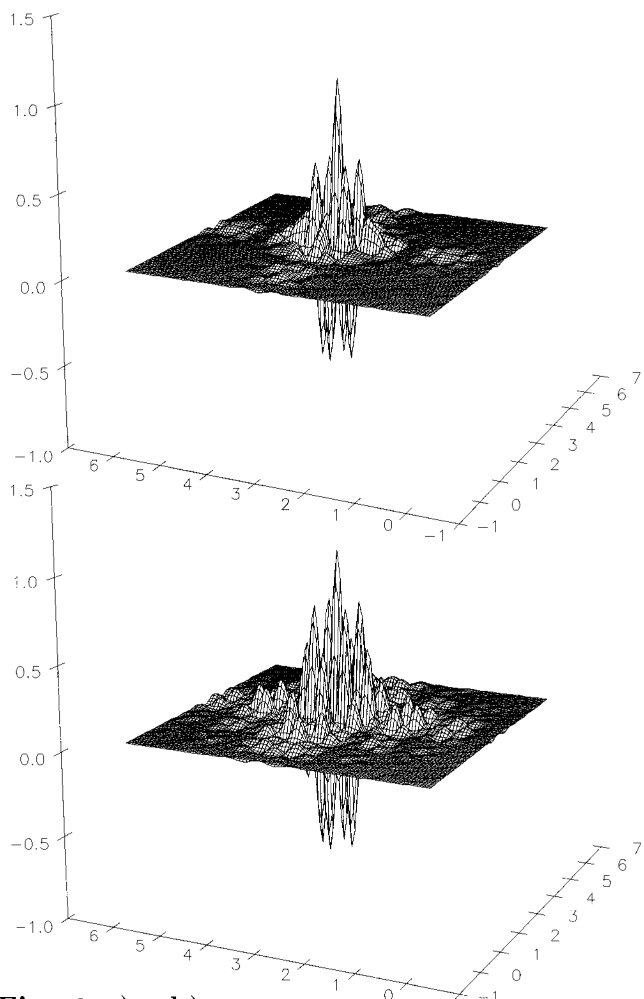

Fig. 3 a) - b)

3 Two-scale Relations

This section is devoted to the computation of the two-scale sequences for reconstruction.

As $V_j \subset V_{j+1}$, there exist coefficients $\varrho_{j,K,R}$ such that

$$\Phi_{j,K} = \sum_R \varrho_{j,K,R} \Phi_{j+1,R}, \qquad \text{where } R = (r_1, r_2).$$

Property (iii) of the interpolation operator introduced in Section 2 implies that the functions $\Phi_{j,K}$ are reproduced by L_{j+1}, i.e.,

$$\Phi_{j,K} = \sum_{r_1,r_2=0}^{2N_{j+1}-1} \Phi_{j,K}\left(\frac{r_1\pi}{N_{j+1}}, \frac{r_2\pi}{N_{j+1}}\right) \Phi_{j+1,R}. \tag{7}$$

Thus, the coefficients

$$\varrho_{j,K,R} = \Phi_{j,K}\left(\frac{r_1\pi}{2N_j}, \frac{r_2\pi}{2N_j}\right)$$

can be obtained from knot evaluation of the scaling function. Further, since $W_j^s \subset V_{j+1}$, $s = 1, 2, 3$, there have to exist coefficients $\sigma_{j,K,R}^s$ such that

$$\Psi_{j,K}^s = \sum_R \sigma_{j,K,R}^s \Phi_{j+1,R}. \tag{8}$$

Analogously to (7), we conclude

$$\sigma_{j,K,R} = \Psi_{j,K}^s\left(\frac{r_1\pi}{2N_j}, \frac{r_2\pi}{2N_j}\right).$$

From the interpolatory properties (3), (5) and (6) of $\phi_{j,k}$ and $\psi_{j,k}$, it is natural to distinguish between even and odd translates.

With the notation $|K| = k_1 \cdot 2N_j + k_2$ the vectors

$$\boldsymbol{\Phi}_j = (\Phi_{j,K})_{|K|=0}^{4N_j^2-1} \qquad \text{and} \qquad \boldsymbol{\Psi}_j^s = \left(\Psi_{j,K}^s\right)_{|K|=0}^{4N_j^2-1}, \; s = 1, 2, 3,$$

are uniquely determined. To reorder the components with odd and even indices we introduce the permutation matrix

$$\boldsymbol{P}_j = \left(p_{j,|K|,|L|}\right)_{|K|,|L|=0}^{16N_j^2-1},$$

where

$$p_{j,|K|,|L|} = (\delta_{2k_1,\ell_1} + \delta_{2k_1,\ell_1+4N_j-1})(\delta_{2k_2,\ell_2} + \delta_{2k_2,\ell_2+4N_j-1}),$$
$$\text{for } k_1, k_2, \ell_1, \ell_2 = 0, 1, \ldots, 4N_j - 1.$$

Furthermore, let $\boldsymbol{F}_j$ be the Fourier matrix of order $2N_j$

$$\boldsymbol{F}_j = \frac{1}{\sqrt{2N_j}} \left(e^{\frac{k\ell\pi i}{N_j}}\right)_{k,\ell=0}^{2N_j-1}$$

and $\boldsymbol{K}_j$ be the knot evaluation matrix of the univariate scaling functions

$$\boldsymbol{K}_j = \left(\phi_{j,k}\left(\frac{(2\ell+1)\pi}{2N_j}\right)\right)_{k,\ell=0}^{2N_j-1}.$$

Then the two-scale relations in matrix form are given in the following way.

Theorem 3.1 *For $j \in \mathbb{N}_0$, it holds that*

$$\begin{pmatrix} \boldsymbol{\Phi}_j \\ \hline \boldsymbol{\Psi}_j^1 \\ \hline \boldsymbol{\Psi}_j^2 \\ \hline \boldsymbol{\Psi}_j^3 \end{pmatrix} = \boldsymbol{C}_j \boldsymbol{P}_j \boldsymbol{\Phi}_{j+1}$$

with the reconstruction matrix

$$\boldsymbol{C}_j = \left(\begin{array}{c|c|c|c} \boldsymbol{I}_j \otimes \boldsymbol{I}_j & \boldsymbol{I}_j \otimes \boldsymbol{K}_j & \boldsymbol{K}_j \otimes \boldsymbol{I}_j & \boldsymbol{K}_j \otimes \boldsymbol{K}_j \\ \hline -\boldsymbol{I}_j \otimes \boldsymbol{K}_j^T & \boldsymbol{I}_j \otimes \boldsymbol{I}_j & -\boldsymbol{K}_j \otimes \boldsymbol{K}_j^T & \boldsymbol{K}_j \otimes \boldsymbol{I}_j \\ \hline -\boldsymbol{K}_j^T \otimes \boldsymbol{I}_j & -\boldsymbol{K}_j^T \otimes \boldsymbol{K}_j & \boldsymbol{I}_j \otimes \boldsymbol{I}_j & \boldsymbol{I}_j \otimes \boldsymbol{K}_j \\ \hline \boldsymbol{K}_j^T \otimes \boldsymbol{K}_j^T & -\boldsymbol{K}_j^T \otimes \boldsymbol{I}_j & -\boldsymbol{I}_j \otimes \boldsymbol{K}_j^T & \boldsymbol{I}_j \otimes \boldsymbol{I}_j \end{array} \right).$$

Here, $\boldsymbol{I}_j \in \mathbb{R}^{2N_j \times 2N_j}$ denotes the identity matrix and the Kronecker product of two $n \times n$ matrices $\boldsymbol{A} = (a_{k,\ell})$ and $\boldsymbol{B}$ is given by the $n^2 \times n^2$ matrix

$$\boldsymbol{A} \otimes \boldsymbol{B} = (a_{k,\ell}\boldsymbol{B})_{k,\ell}\,.$$

While $\boldsymbol{K}_j$ is circulant, the blocks of $\boldsymbol{C}_j$ are block circulants with circulant blocks.
Proof: The desired result is only the reformulation of (7) and (8) in matrix notation, where we used the interpolatory conditions (3) and (5) to simplify

$$\Phi_{j,K}\left(\frac{r_1\pi}{2N_j}, \frac{r_2\pi}{2N_j}\right) = 0 \quad \text{if} \quad (r_1, r_2) \neq K \quad \text{and} \quad (r_1 \text{ or } r_2 \text{ even }),$$

$$\Psi^1_{j,K}\left(\frac{r_1\pi}{2N_j}, \frac{r_2\pi}{2N_j}\right) = 0 \quad \text{if} \quad (r_1, r_2) \neq K \quad \text{and} \quad (r_1 \text{ even or } r_2 \text{ odd })$$

and analogously for $\Psi^2_{j,K}$ and $\Psi^3_{j,K}$. ■

4 Riesz Stability

For further usage, we have to consider inner products of scaling functions on the same level. Let us define the Gramian

$$\boldsymbol{G}_j = \left(\langle \phi_{j,k}, \phi_{j,\ell}\rangle\right)_{k,\ell=0}^{2N_j-1}.$$

From the definition of the scaling functions, we know

$$\boldsymbol{G}_j = \operatorname{circ}(\langle \phi_{j,0}, \phi_{j,0}\rangle, \langle \phi_{j,0}, \phi_{j,1}\rangle, \ldots, \langle \phi_{j,0}, \phi_{j,2N_j-1}\rangle).$$

and one can see easily [4] that $\boldsymbol{G}_j$ is symmetric and positive definite with

$$\boldsymbol{G}_j = \overline{\boldsymbol{F}}_j \boldsymbol{\Gamma}_j \boldsymbol{F}_j,$$

where

$$\mathbf{\Gamma}_j = \mathrm{diag}(\gamma_{j,\ell})_{\ell=0}^{2N_j-1}.$$

In [4], it is proved that the eigenvalues $\gamma_{j,\ell}$ are given by

$$\begin{aligned}
\gamma_{j,\ell} &= \sum_{k=0}^{2N_j-1} \langle \phi_{j,0}, \phi_{j,k} \rangle \cdot e^{\frac{k\ell\pi i}{N_j}} \\
&= \frac{1}{4N_j} \begin{cases} 2 & \text{if } 0 \leq \ell \leq N_j - M_j,\ N_j + M_j \leq \ell \leq 2N_j - 1, \\ \left(1 + \left(\frac{N_j-\ell}{M_j}\right)^2\right) & \text{if } N_j - M_j \leq \ell \leq N_j + M_j. \end{cases}
\end{aligned} \tag{9}$$

Now we are able to prove the Riesz stability of the scaling function basis in V_j.

Theorem 4.1 *For arbitrary sequences $\boldsymbol{\alpha} \in \mathbb{R}^{4N_j^2}$, it holds that*

$$\frac{1}{2}\left(\sum_{k_1,k_2=0}^{2N_j-1} \alpha_K^2\right)^{\frac{1}{2}} \leq \left\| \sum_{k_1,k_2=0}^{2N_j-1} 2N_j \alpha_K \Phi_{j,K} \right\|_{L_2(\mathbb{T}^2)} \leq \left(\sum_{k_1,k_2=0}^{2N_j-1} \alpha_K^2\right)^{\frac{1}{2}}.$$

Proof: It holds

$$\boldsymbol{G}_j \otimes \boldsymbol{G}_j = \left(\langle \Phi_{j,K}, \Phi_{j,L} \rangle\right)_{|K|,|L|=0}^{4N_j^2-1}.$$

The Kronecker product $\boldsymbol{G}_j \otimes \boldsymbol{G}_j$ is circulant, symmetric and positive definite with the eigenvalues $\Gamma_{j,L} = \gamma_{j,\ell_1}\gamma_{j,\ell_2}$ and the corresponding orthonormal eigenvectors $\boldsymbol{U}_{j,L} \in \mathbb{R}^{4N_j^2}$ Then, we can write every $\boldsymbol{\alpha} \in \mathbb{R}^{4N_j^2}$ in the form

$$\boldsymbol{\alpha} = \sum_L \beta_L \boldsymbol{U}_{j,L}.$$

From

$$\sum_K \alpha_K^2 = \sum_K \beta_K^2$$

and

$$\begin{aligned}
\left\| \sum_{k_1,k_2=0}^{2N_j-1} 2N_j \alpha_K \Phi_{j,K} \right\|_{L_2(\mathbb{T}^2)}^2 &= 4N_j^2 \left\langle \sum_{k_1,k_2=0}^{2N_j-1} \alpha_K \Phi_{j,K}, \sum_{\ell_1,\ell_2=0}^{2N_j-1} \alpha_L \Phi_{j,L} \right\rangle \\
&= 4N_j^2 \sum_{k_1,k_2=0}^{2N_j-1} \sum_{\ell_1,\ell_2=0}^{2N_j-1} \alpha_K \alpha_L \langle \Phi_{j,K}, \Phi_{j,L} \rangle \\
&= 4N_j^2 \left((\boldsymbol{G}_j \otimes \boldsymbol{G}_j)\boldsymbol{\alpha}, \boldsymbol{\alpha}\right) \\
&= 4N_j^2 \sum_{k_1,k_2=0}^{2N_j-1} \sum_{\ell_1,\ell_2=0}^{2N_j-1} \beta_K \beta_L \left((\boldsymbol{G}_j \otimes \boldsymbol{G}_j)\boldsymbol{U}_{j,K}, \boldsymbol{U}_{j,L}\right) \\
&= 4N_j^2 \sum_{k_1,k_2=0}^{2N_j-1} \sum_{\ell_1,\ell_2=0}^{2N_j-1} \beta_K \beta_L \left(\Gamma_{j,K}\boldsymbol{U}_{j,K}, \boldsymbol{U}_{j,L}\right) \\
&= 4N_j^2 \sum_{k_1,k_2=0}^{2N_j-1} \beta_K^2 \gamma_{j,k_1}\gamma_{j,k_2}
\end{aligned}$$

we obtain with (9) the desired Riesz constants in the theorem as

$$\begin{aligned}
2N_j \left(\min_\ell \gamma_{j,\ell}\right) &= \frac{1}{2}, \\
2N_j \left(\max_\ell \gamma_{j,\ell}\right) &= 1.
\end{aligned}$$

■

5 Decomposition Relations

To state the counterpart of Theorem 3.1, i.e. to write the scaling functions $\Phi_{j+1,K} \in V_{j+1}$ in the corresponding scaling function and wavelet basis of $V_j \oplus^\perp W_j^1 \oplus^\perp W_j^2 \oplus^\perp W_j^3$, we need some preliminaries.

Since $\boldsymbol{G}_j$ is circulant, its inverse is given by

$$\boldsymbol{G}_j^{-1} = \overline{\boldsymbol{F}}_j \boldsymbol{\Gamma}_j^{-1} \boldsymbol{F}_j = \overline{\boldsymbol{F}}_j \mathrm{diag} \left(\frac{1}{\gamma_{j,\ell}} \right)_{\ell=0}^{2N_j-1} \boldsymbol{F}_j.$$

For the knot evaluation matrix $\boldsymbol{K}_j$ and the matrix of inner products $\boldsymbol{G}_j$ there exist the following helpful relation.

Lemma 5.1 *For $j \in \mathbb{N}_0$, it holds that*

$$\boldsymbol{I}_j + \boldsymbol{K}_j \boldsymbol{K}_j^T = 4N_j \boldsymbol{G}_j. \tag{10}$$

Proof: Because all matrices involved are circulant, it suffices to show that the equality holds for the first row. Then we use the well-known quadrature formula for trigonometric polynomials with equidistant nodes (see e.g. [6], Chap. 10) to conclude

$$\begin{aligned}
(4N_j \boldsymbol{G}_j)_{0,r} &= 4N_j \langle \phi_{j,0}, \phi_{j,r} \rangle = \frac{4N_j}{2\pi} \int_0^{2\pi} \phi_{j,0}(x) \phi_{j,r}(x)\, dx \\
&= \sum_{k=0}^{4N_j-1} \phi_{j,0} \left(\frac{k\pi}{2N_j} \right) \phi_{j,r} \left(\frac{k\pi}{2N_j} \right) \\
&= \delta_{0,r} + \sum_{s=0}^{2N_j-1} \phi_{j,0} \left(\frac{(2s+1)\pi}{2N_j} \right) \phi_{j,r} \left(\frac{(2s+1)\pi}{2N_j} \right) \\
&= \left(\boldsymbol{I}_j + \boldsymbol{K}_j \boldsymbol{K}_j^T \right)_{0,r},
\end{aligned}$$

which proves the assertion. ■

Now we can summarize our results in the following theorem.

Theorem 5.2 *For $j \in \mathbb{N}_0$, it holds that*

$$\boldsymbol{P}_j \boldsymbol{\Phi}_{j+1} = \boldsymbol{D}_j \begin{pmatrix} \boldsymbol{\Phi}_j \\ \hline \boldsymbol{\Psi}_j^1 \\ \hline \boldsymbol{\Psi}_j^2 \\ \hline \boldsymbol{\Psi}_j^3 \end{pmatrix}$$

with the decomposition matrix

$$16N_j^2\boldsymbol{D}_j = 16N_j^2(\boldsymbol{I}_4 \otimes \boldsymbol{G}_j^{-1} \otimes \boldsymbol{I}_j)\boldsymbol{C}_j^T(\boldsymbol{I}_4 \otimes \boldsymbol{I}_j \otimes \boldsymbol{G}_j^{-1}) =$$

$$\left(\begin{array}{c|c|c|c}
\boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1} & -\boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1}\boldsymbol{K}_j & -\boldsymbol{G}_j^{-1}\boldsymbol{K}_j \otimes \boldsymbol{G}_j^{-1} & \boldsymbol{G}_j^{-1}\boldsymbol{K}_j \otimes \boldsymbol{G}_j^{-1}\boldsymbol{K}_j \\ \hline
\boldsymbol{G}_j^{-1} \otimes \boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} & \boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1} & -\boldsymbol{G}_j^{-1}\boldsymbol{K}_j \otimes \boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} & -\boldsymbol{G}_j^{-1}\boldsymbol{K}_j \otimes \boldsymbol{G}_j^{-1} \\ \hline
\boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1} & -\boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1}\boldsymbol{K}_j & \boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1} & -\boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1}\boldsymbol{K}_j \\ \hline
\boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} \otimes \boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} & \boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1} & \boldsymbol{G}_j^{-1} \otimes \boldsymbol{K}_j^T\boldsymbol{G}_j^{-1} & \boldsymbol{G}_j^{-1} \otimes \boldsymbol{G}_j^{-1}
\end{array}\right)$$

where $\boldsymbol{I}_4 \in \mathbb{R}^{4\times 4}$ *denotes the identity matrix.*

Proof: To prove the theorem, it suffices to show that $\boldsymbol{C}_j\boldsymbol{D}_j = \boldsymbol{I} \in \mathbb{R}^{4N_{j+1}^2 \times 4N_{j+1}^2}$. Only by applying the rules of computation for the Kronecker product (see e.g. [7], Chap. 2) we get

$$16N_j^2\boldsymbol{C}_j\boldsymbol{D}_j = \boldsymbol{I}_4 \otimes \boldsymbol{H}_j$$

with

$$\boldsymbol{H}_j = \left(\boldsymbol{G}_j^{-1} + \boldsymbol{K}_j\boldsymbol{K}_j^T\boldsymbol{G}_j^{-1}\right) \otimes \left(\boldsymbol{G}_j^{-1} + \boldsymbol{K}_j\boldsymbol{K}_j^T\boldsymbol{G}_j^{-1}\right).$$

Multiplying (10) in Lemma 5.1 by $\boldsymbol{G}_j^{-1}$ and using the result one obtains

$$\boldsymbol{H}_j = 16N_j^2\,\boldsymbol{I}_j \otimes \boldsymbol{I}_j$$

such that

$$\boldsymbol{C}_j\boldsymbol{D}_j = \boldsymbol{I}_4 \otimes \boldsymbol{I}_j \otimes \boldsymbol{I}_j = \boldsymbol{I}$$

what proves the Theorem. ■

6 Algorithms

Now we want to describe how to use the two-scale and decomposition relations, respectively, for a given function. As $V_{j+1} = V_j \oplus^\perp W_j^1 \oplus^\perp W_j^2 \oplus^\perp W_j^3$, a function $f_{j+1} \in V_{j+1}$ can be written uniquely as

$$f_{j+1} = f_j + g_j^1 + g_j^2 + g_j^3 \quad \text{with} \quad f_j \in V_j \quad \text{and} \quad g_j^s \in W_j^s, s = 1,2,3.$$

Using the basis functions of these spaces, one obtains

$$f_{j+1} = \sum_{k_1,k_2=0}^{2N_{j+1}-1} c_K^{j+1}\Phi_{j+1,K},$$

$$f_j = \sum_{k_1,k_2=0}^{2N_j-1} c_K^j\Phi_{j,K} \quad \text{and} \quad g_j^s = \sum_{k_1,k_2=0}^{2N_j-1} d_K^{s,j}\Psi_{j,K}^s \quad \text{for } s = 1,2,3.$$

Denoting $\boldsymbol{c}_j = (c_K^j)_{|K|=0}^{4N_j^2-1}$ and $\boldsymbol{d}_j^s = (d_K^{s,j})_{|K|=0}^{4N_j^2-1}$, $s = 1,2,3$, respectively, one gets

$$f_{j+1} = \boldsymbol{\Phi}_{j+1}^T\boldsymbol{c}_{j+1}, \quad f_j = \boldsymbol{\Phi}_j^T\boldsymbol{c}_j \quad \text{and} \quad g_j^s = \left(\boldsymbol{\Psi}_j^s\right)^T\boldsymbol{d}_j^s \quad \text{for } s = 1,2,3.$$

As $\boldsymbol{P}_j^T = \boldsymbol{P}_j^{-1}$, we can write

$$f_{j+1} = \left(\boldsymbol{P}_j \boldsymbol{\Phi}_{j+1}\right)^T \boldsymbol{P}_j \boldsymbol{c}_{j+1}.$$

On the other hand, it holds that

$$\begin{aligned} f_j + g_j^1 + g_j^2 + g_j^3 &= \left(\left(\boldsymbol{\Phi}_j\right)^T, \left(\boldsymbol{\Psi}_j^1\right)^T, \left(\boldsymbol{\Psi}_j^2\right)^T, \left(\boldsymbol{\Psi}_j^3\right)^T\right) \left(\left(\boldsymbol{c}_j\right)^T, \left(\boldsymbol{d}_j^1\right)^T, \left(\boldsymbol{d}_j^2\right)^T, \left(\boldsymbol{d}_j^3\right)^T\right)^T \\ &= \left(\boldsymbol{P}_j \boldsymbol{\Phi}_{j+1}\right)^T \boldsymbol{C}_j^T \left(\left(\boldsymbol{c}_j\right)^T, \left(\boldsymbol{d}_j^1\right)^T, \left(\boldsymbol{d}_j^2\right)^T, \left(\boldsymbol{d}_j^3\right)^T\right)^T. \end{aligned}$$

Comparing coefficients leads to one step of the reconstruction algorithm in matrix form

$$\boldsymbol{P}_j \boldsymbol{c}_{j+1} = \boldsymbol{C}_j^T \left(\left(\boldsymbol{c}_j\right)^T, \left(\boldsymbol{d}_j^1\right)^T, \left(\boldsymbol{d}_j^2\right)^T, \left(\boldsymbol{d}_j^3\right)^T\right)^T.$$

Multiplying by the inverse $(\boldsymbol{C}_j^T)^{-1} = \boldsymbol{D}_j^T$ gives the matrix representation of one step of the decomposition algorithm

$$\left(\left(\boldsymbol{c}_j\right)^T, \left(\boldsymbol{d}_j^1\right)^T, \left(\boldsymbol{d}_j^2\right)^T, \left(\boldsymbol{d}_j^3\right)^T\right)^T = \boldsymbol{D}_j^T \boldsymbol{P}_j \boldsymbol{c}_{j+1}.$$

Computing reconstruction or decomposition sequences, respectively, we have to take into consideration the fact that the blocks of the reconstruction matrix $\boldsymbol{C}_j$ as well as the ones of the decomposition matrix $\boldsymbol{D}_j$ are block circulants with circulant blocks. For every matrix $\boldsymbol{A}_j \in \mathbb{R}^{4N_j^2 \times 4N_j^2}$ with such a special structure, it holds that ([7], Chap. 5)

$$\boldsymbol{A}_j = \overline{(\boldsymbol{F}_j \otimes \boldsymbol{F}_j)} \boldsymbol{B}_j \left(\boldsymbol{F}_j \otimes \boldsymbol{F}_j\right)$$

with a diagonal matrix $\boldsymbol{B}_j$. That's why the multiplication of such $\boldsymbol{A}_j$ by a vector can be realized by Tensor product discrete Fourier transforms of length $2N_j \times 2N_j$. Because of the special choice of N_j (see (1)) we can compute reconstruction or decomposition sequences for the level j with $\mathcal{O}(j2^{2j})$ essential operations by using methods of Fast Fourier Transform ([8]).

Let us finish with the decomposition of the quadratic Box-spline

$$f(x, y) = B^{1,1,2}(x - \frac{3}{2}, y - \frac{3}{2})$$

into its wavelet parts. To do that, we first interpolate f at 384×384 points to obtain $L_6 f$ with $c = 3$ and $\lambda = 0$. Then we use our algorithms to compute the orthogonal projections $g_5^s(L_6 f)$, $s = 1, 2, 3$ into the wavelet spaces W_5^s, which detect the discontinuities of the first derivatives of f.

The result can be seen in the following pictures.

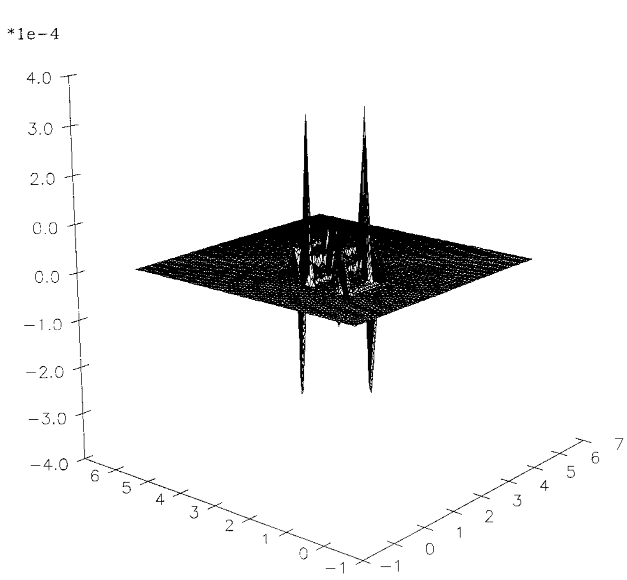

Fig. 4 $g_5^1(L_6 f)$.

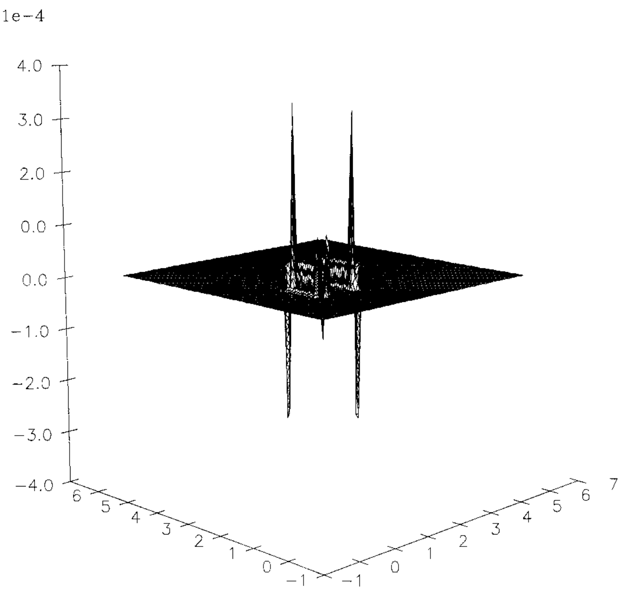

Fig. 5 $g_5^2(L_6 f)$.

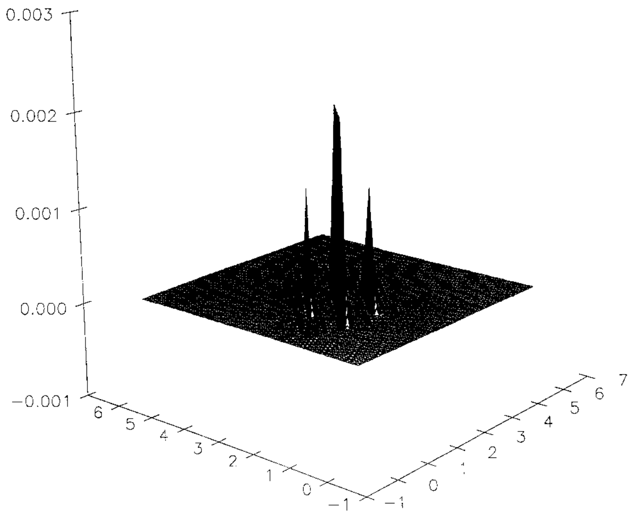

Fig. 6 $g_5^3(L_6 f)$.

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