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On the connection of uncertainty principles for functions on the circle and on the real line

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ABSTRACT. An uncertainty principle for 2π-periodic functions and the classical Heisenberg uncertainty principle are shown to be linked by a limit process. Dependent on a parameter, a function on the real line generates periodic functions either by periodization or sampling. It is proven that under certain smoothness conditions, the periodic uncertainty products of the generated functions converge to the real-line uncertainty product of the original function if the parameter tends to infinity. These results are used to find asymptotically optimal sequences for the periodic uncertainty principle, based either on Theta functions or trigonometric polynomials obtained by sampling B-splines.

1. Introduction

The Heisenberg uncertainty principle has been studied in all sorts of variations and settings (cf. e.g. [5]). The aim of this paper is to contribute to these studies by investigating the connection between the basic mathematical formulation of the Heisenberg principle for functions on the real line and a much more recent uncertainty principle for 2π-periodic functions. From a physicists point of view such uncertainty relations were discussed by Breitenberger in [1]. The first mathematical formulation of the periodic

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uncertainty principle in question dates back to Narcowich and Ward [9]. While the best lower bound 1/2 for the real line is attained if and only if the given function is a Gaussian, the situation is different in the periodic case, as the best constant (again 1/2) is not attained. This fact, together with the proof that a certain function sequence based on Theta functions is asymptotically optimal, was established by Prestin and Quak in [12]. A unifying operator theoretical approach to both uncertainty principles was given by Selig in [18].

Functions on the real line can be easily used to generate periodic functions. Two obvious ways of doing so are either to periodize the original function, or to sample its function values at a certain rate and to use these samples as the Fourier coefficients for a periodic function. Then it is natural to ask whether the uncertainty products on the line and on the circle can be related to each other using these processes. We proceed here by defining for a given function f and real parameter $a > 0$ the two 2π-periodic functions

$$f_a^{per}(t) = \sqrt{a} \sum_{k=-\infty}^{\infty} f(a(t + 2\pi k))$$

and

$$f_a^{sam}(t) = \sum_{k=-\infty}^{\infty} f\left(\frac{k}{a}\right) e^{ikt}.$$

The two approaches are clearly linked by Poisson's summation formula. As our main results, we establish that the periodic uncertainty products $U_{2\pi}$ for these functions converge to the uncertainty product $U_{\mathbb{R}}$ of the generating function f, i.e.

$$\lim_{a\to\infty} U_{2\pi}(f_a^{per}) \ = \ U_{\mathbb{R}}(f)$$

and

$$\lim_{a\to\infty} U_{2\pi}(f_a^{sam}) \ = \ U_{\mathbb{R}}(f),$$

given certain smoothness conditions on f or its Fourier transform $\hat{f}$, respectively. As applications of our results, we will consider f to be either a Gaussian or a centralized B-spline and then determine various function sequences being asymptotically optimal for the periodic uncertainty principle. These are based on Theta functions if f is a Gaussian, and on trigonometric polynomials obtained by sampling B-splines.

The paper is organized as follows. We list the needed mathematical formulations for the Heisenberg uncertainty principle and for the periodic one in Sections 2 and 3, respectively. In Section 4 we provide auxiliary results for the dilation and periodization tools. The main limit theorem for periodized functions, namely Theorem 3, is established in Section 5, while the main one for functions obtained from sampling is Theorem 6. It is proven in Section 6, alongside corollaries that state some sufficient conditions on the generating function f to guarantee the desired limit behaviour. Corollary 4 is of special interest as it is used in Section 7 in order to generate asymptotically optimal trigonometric polynomials.

2. Uncertainty principle for the real line

Let $L^2(\mathbb{R})$ denote the space of complex-valued square-integrable functions on the real line with inner product $\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}dx$ and norm $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\int_{-\infty}^{\infty} |f(x)|^2 dx}$. The Fourier transform of a function $f \in L^2(\mathbb{R})$ is then defined as

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x)e^{-ix\xi}\, dx$$

with the understanding that $\int_{-\infty}^{\infty} = \lim_{n\to\infty} \int_{-n}^{n}$. We will use the notations $AC(\mathbb{R}),\ AC_{loc}(\mathbb{R})$,
$C_0(\mathbb{R})$ and $L^1(\mathbb{R})$ as usual (see for example [2]). Important characteristics of a function and its Fourier transform are provided by their expectations and variances in the following sense.

Definition 1. Let $f \in L^2(\mathbb{R}),\ f \neq 0$, such that

$$(\cdot f), (\cdot \hat{f}) \in L^2(\mathbb{R}). \tag{2.1}$$

We define the time and frequency centers

$$\begin{aligned} x_0 = x_0(f) &:= \frac{1}{\|f\|^2}\int_{-\infty}^{\infty} x|f(x)|^2\, dx, \\ \xi_0 = \xi_0(f) &:= \frac{1}{\|\hat{f}\|^2}\int_{-\infty}^{\infty} \xi|\hat{f}(\xi)|^2\, d\xi, \end{aligned}$$

and the time and frequency variances

$$\begin{aligned} \Delta x(f) &:= \frac{1}{\|f\|^2}\int_{-\infty}^{\infty} (x-x_0)^2|f(x)|^2\, dx = \frac{\|(\cdot - x_0)f\|^2}{\|f\|^2}, \\ \Delta \xi(f) &:= \frac{1}{\|\hat{f}\|^2}\int_{-\infty}^{\infty} (\xi-\xi_0)^2|\hat{f}(\xi)|^2\, d\xi = \frac{\|(\cdot - \xi_0)\hat{f}\|^2}{\|\hat{f}\|^2}. \end{aligned}$$

The fundamental result about the correspondence of time and frequency variances can then be formulated as follows.

Theorem 1.

(Heisenberg uncertainty principle) For a function $f \in L^2(\mathbb{R})$ satisfying (2.1) with time and frequency uncertainties $\sqrt{\Delta x(f)}$ and $\sqrt{\Delta\xi(f)}$, let the uncertainty product be given as

$$U_{\mathbb{R}}(f) \;:=\; \sqrt{\Delta x(f)\Delta\xi(f)} \;=\; \frac{\|(\cdot - x_0)f\|\;\|(\cdot-\xi_0)\hat{f}\|}{\|f\|\;\|\hat{f}\|}.$$

Then

$$U_{\mathbb{R}}(f) \;\geq\; \frac{1}{2},$$

with equality iff f is a Gaussian function of the form ce^{-ax^2+bx}, where $a > 0,\; b, c \in \mathbb{C},\; c \neq 0$.

For a proof of the uncertainty principle in this particular formulation, see [4, Theorem 3.5].

As a motivation for the corresponding definition in the periodic case, we will sketch how the frequency variance, and thus the uncertainty product as well, can be expressed without the use of the Fourier transform. Let us first make an auxiliary statement.

Lemma 1.

Let $f \in L^2(\mathbb{R})$ and $(\cdot f) \in L^2(\mathbb{R})$. Then $\hat{f}$ is absolutely continuous on any bounded interval and

$$(\hat{f})'(\xi) \;=\; -\,i\;\mathcal{F}(\cdot f)(\xi) \quad a.e., \tag{2.2}$$

hence we have $(\hat{f})' \in L^2(\mathbb{R})$. If furthermore $\mathcal{F}(\cdot f) \in L^1(\mathbb{R})$, then $\hat{f}$ is absolutely continuous on $\mathbb{R}$.

Proof. The assertions of this lemma are quite well-known and the proof is standard, hence omitted. $\square$

Lemma 2.

If $f \in L^2(\mathbb{R})$ satisfies (2.1) then $f' \in L^2(\mathbb{R})$ and the variances of f can be expressed as

$$\Delta x(f) \;=\; \frac{\|\cdot f\|^2}{\|f\|^2} - \frac{\langle \cdot f, f\rangle^2}{\|f\|^4}, \tag{2.3}$$

$$\Delta\xi(f) \;=\; \frac{\|f'\|^2}{\|f\|^2} + \frac{\langle f', f\rangle^2}{\|f\|^4}. \tag{2.4}$$

Proof. Since $(\cdot f) \in L^2(\mathbb{R})$, the time variance is well defined and

$$\begin{aligned}\Delta x(f) \;&=\; \frac{1}{\|f\|^2}\left(\int_{-\infty}^{\infty}(x-x_0)^2|f(x)|^2\,dx\right) \;=\; \frac{\int_{-\infty}^{\infty}x^2|f(x)|^2\,dx}{\|f\|^2} - x_0^2\\ &=\; \frac{\|\cdot f\|^2}{\|f\|^2} - \frac{\left(\int_{-\infty}^{\infty}x|f(x)|^2\,dx\right)^2}{\|f\|^4} \;=\; \frac{\|\cdot f\|^2}{\|f\|^2} - \frac{\langle\cdot f, f\rangle^2}{\|f\|^4}.\end{aligned}$$

Application of Parseval's identity together with (2.2) yields

$$\Delta x(f) = \frac{\|(\hat{f})'\|^2}{\|\hat{f}\|^2} - \frac{\langle i(\hat{f})', \hat{f}\rangle^2}{\|\hat{f}\|^4} = \frac{\|(\hat{f})'\|^2}{\|\hat{f}\|^2} + \frac{\langle (\hat{f})', \hat{f}\rangle^2}{\|\hat{f}\|^4}. \qquad (2.5)$$

Now, let $f^-(x) := f(-x)$ be the reflection of f (about the origin). As $\hat{f}^- = \mathcal{F}(f^-) = (\hat{f})^-$, we set $g := \frac{1}{2\pi}\hat{f}^-$ so that $\hat{g} = f$ almost everywhere. Since $(\cdot g) \in L^2(\mathbb{R})$, Lemma 1 yields $(\hat{g})' = f' \in L^2(\mathbb{R})$. Using (2.5) for g, we obtain

$$\Delta x(g) = \frac{\|f'\|^2}{\|f\|^2} + \frac{\langle f', f\rangle^2}{\|f\|^4},$$

which gives (2.4) since $\Delta x(g) = \Delta x(\hat{f}^-) = \Delta\xi(f)$. □

3. Uncertainty principle for the circle

Let $L^2_{2\pi}$ denote the space of complex-valued square-integrable 2π-periodic functions with inner product $\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\overline{g(t)}dt$ and norm $\|f\| = \sqrt{\langle f, f\rangle} = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(t)|^2 dt}$, and $AC_{2\pi}$ its subspace of absolutely continuous functions. The following notions of variances for a 2π-periodic function are given both in terms of the function f and of its Fourier coefficients.

Definition 2. For a nonzero function $f \in L^2_{2\pi}$ represented by its Fourier series $f = \sum_{s=-\infty}^{\infty} c_s e^{is\cdot}$, we define the first trigonometric moment

$$\tau(f) := \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{it}|f(t)|^2 dt = \sum_{s=-\infty}^{\infty} c_s\overline{c_{s+1}},$$

the angular variance

$$\mathrm{var}_A(f) := \frac{\|f\|^4 - |\tau(f)|^2}{|\tau(f)|^2} = \left|\frac{\sum_{s=-\infty}^{\infty}|c_s|^2}{\sum_{s=-\infty}^{\infty} c_s\overline{c_{s+1}}}\right|^2 - 1,$$

(possibly $\mathrm{var}_A(f) = \infty$ if $\tau(f) = 0$) and, for $f \in AC_{2\pi}$ with $f' \in L^2_{2\pi}$, the frequency variance

$$\mathrm{var}_F(f) := \frac{\|f'\|^2}{\|f\|^2} + \frac{\langle f', f\rangle^2}{\|f\|^4} = \frac{\sum_{s=-\infty}^{\infty} s^2|c_s|^2}{\sum_{s=-\infty}^{\infty}|c_s|^2} - \left(\frac{\sum_{s=-\infty}^{\infty} s|c_s|^2}{\sum_{s=-\infty}^{\infty}|c_s|^2}\right)^2.$$

The following theorem establishes the lower bound for the product of the two variances. The angular variance can be infinite or arbitrarily close to zero but not zero for $f \in L^2_{2\pi}$ whereas the frequency variance is finite for all functions for which it is defined. It attains zero iff $f = c \cdot e^{ikx}$, $c \in \mathbb{C}$, $k \in \mathbb{Z}$, and for these monomials the product would be "$\infty \cdot 0$" and hence not be defined. This case has to be excluded.

Theorem 2.

(Uncertainty principle for the circle) For a function $f \in AC_{2\pi}$ with $f' \in L^2_{2\pi}$ where f is not of the form ce^{ikx} for any $c \in \mathbb{C}$, $k \in \mathbb{Z}$, with angular and frequency uncertainties $\sqrt{\mathrm{var}_A(f)}$ and $\sqrt{\mathrm{var}_F(f)}$, let the uncertainty product be given as

$$U_{2\pi}(f) \;:=\; \sqrt{\mathrm{var}_A(f)\;\; \mathrm{var}_F(f)}.$$

Then it holds

$$U_{2\pi}(f) \;>\; \frac{1}{2}.$$

The lower bound is not attained by any function, but is best possible.

This formulation first appears in [9], while it is established in [12] that the bound is best possible. In Section 7 we will give examples for asymptotically optimal functions.

The condition $f \in AC_{2\pi}$, and thus the restriction of the frequency variance to absolutely continuous functions, is necessary since otherwise one could come up with the following counterexample. Let $f = \chi^{per}_{[0,1]}$ be the characteristic function of the interval $[0,1]$, 2π-periodically extended to $\mathbb{R}$. Clearly $f' = 0$ almost everywhere and thus $\mathrm{var}_F(f) = 0$. Furthermore $|\tau(f)| > 0$ and thus $U_{2\pi}(f) = 0$.

We remark that we sometimes use the same inner product and norm symbols for the real line and the circle if it is clear from the context which type of function - and thus norm - is meant. Note that with this stipulation the expression for the frequency variance on the circle is the same as the one for the real line given in (2.4). We conclude this section by deriving for later use also a different way to represent the angular variance.

Lemma 3.

For a function $f \in L^2_{2\pi}$, define the two terms

$$A(f) \;:=\; \frac{1}{4\pi}\int_{-\pi}^{\pi} \left|e^{it}-1\right|^2 |f(t)|^2\, dt \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(t/2)\; |f(t)|^2\, dt,$$

and

$$B(f) \;:=\; \frac{1}{4\pi}\int_{-\pi}^{\pi} (e^{it}-1)(e^{-it}+1)|f(t)|^2\, dt \;=\; i\frac{1}{2\pi}\int_{-\pi}^{\pi} \sin t\; |f(t)|^2\, dt.$$

If $\tau(f) \neq 0$, then the angular variance can be written as

$$\begin{aligned}\operatorname{var}_A(f) &= \frac{\|f\|^4 - (\|f\|^2 - A(f))^2 + B^2(f)}{(\|f\|^2 - A(f))^2 - B^2(f)} \\ &= \frac{2A(f)\|f\|^2 - A^2(f) + B^2(f)}{(\|f\|^2 - A(f))^2 - B^2(f)}.\end{aligned}$$

Proof. First we rewrite the trigonometric moment $\tau(f)$. We have

$$\begin{aligned}2\|f\|^2 - 2\Re\tau(f) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} (2 - e^{it} - e^{-it})|f(t)|^2 \, dt \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |e^{it} - 1|^2 |f(t)|^2 \, dt,\end{aligned}$$

and thus for the real part of $\tau(f)$

$$\Re\tau(f) = \|f\|^2 - A(f).$$

Furthermore, consider the imaginary part of $\tau(f)$

$$\Im\tau(f) = \frac{1}{2i}(\tau(f) - \overline{\tau(f)}) = \frac{-i}{4\pi}\int_{-\pi}^{\pi} (e^{it} - 1)(e^{-it} + 1)|f(t)|^2 \, dt,$$

which gives

$$\Im\tau(f) = -iB(f).$$

Altogether, we obtain

$$|\tau(f)|^2 = (\Re\tau(f))^2 + (\Im\tau(f))^2 = \left(\|f\|^2 - A(f)\right)^2 - B^2(f).$$

By inserting this into the definition of the angular variance, we get the proposition of the lemma. $\square$

4. Dilation and periodization

Our main objective is the investigation of the relationship between the uncertainty product on the line and on the circle. For convenience, we consider the following class of functions with certain decay conditions.

Definition 3. A non-zero function $f \in L^2(\mathbb{R})$ is called admissible if $f \in AC_{loc}(\mathbb{R})$, $f' \in L^2(\mathbb{R})$ and if f satisfies the conditions

$$|f(x)| \leq \frac{C}{|x|^\gamma} \quad \text{for all } x \in \mathbb{R} \setminus \{0\}, \tag{4.1}$$

$$|f'(x)| \leq \frac{D}{|x|^\beta} \quad \text{a.e.}, \tag{4.2}$$

for constants $C, D > 0$ and $\beta > 1,\ \gamma > \frac{3}{2}$.

First, we summarize some properties of admissible functions that will be needed later on.

Lemma 4.
Let f be an admissible function. Then f satisfies (2.1) and, additionally,

$$\begin{aligned} f \in AC(\mathbb{R}) \cap L^1(\mathbb{R}) \quad &\textit{with} \quad f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R}), \\ \hat{f} \in AC_{loc}(\mathbb{R}) \cap L^1(\mathbb{R}) \quad &\textit{with} \quad (\hat{f})' \in L^2(\mathbb{R}). \end{aligned}$$

Proof. The properties of f are immediate consequences of the admissibility conditions. Since $f \in AC_{loc}(\mathbb{R}),\ f' \in L^2(\mathbb{R})$ and (4.2) yield $f' \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, we have $\mathcal{F}(f') \in C_0(\mathbb{R}) \cap L^2(\mathbb{R})$ and therefore $\mathcal{F}(f')(\xi) = i\xi\hat{f}(\xi)$ for all $\xi \in \mathbb{R}$, which implies $(\cdot\hat{f}) \in L^2(\mathbb{R})$. Now Lemma 1 can be used to derive the remaining properties of $\hat{f}$. $\square$

For an admissible function f and a parameter $a > 0$, we define the admissible function f_a by

$$f_a(x) \ := \ \sqrt{a}\, f(ax).$$

We use the notations $\langle f, g\rangle_{L^2(-\pi,\pi)} := \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} f(x)\overline{g(x)}dx$ and $\|g\|^2_{L^2(-\pi,\pi)} := \frac{1}{2\pi}\int\limits_{-\pi}^{\pi} |g(x)|^2 dx$ if the function g is originally defined on the whole real line and is not 2π-periodic. Note that the computation of the terms $A(g)$ and $B(g)$ defined in Lemma 3 is possible even for a function g that is not 2π-periodic. Further, in the following we write for short $f_a' = (f_a)'\ (\neq (f')_a)$.

Lemma 5.
Let f be an admissible function and $a > 0$. Then

$$\begin{aligned} \lim_{a\to\infty} \|f_a\|^2_{L^2(-\pi,\pi)} \quad &= \quad \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{R})}, \\ \lim_{a\to\infty} \frac{1}{a^2}\|f_a'\|^2_{L^2(-\pi,\pi)} \quad &= \quad \frac{1}{2\pi}\|f'\|^2_{L^2(\mathbb{R})} \end{aligned}$$

and

$$\lim_{a\to\infty} \frac{1}{a}\langle f_a', f_a\rangle_{L^2(-\pi,\pi)} \quad = \quad \frac{1}{2\pi}\langle f', f\rangle_{L^2(\mathbb{R})}.$$

Furthermore,

$$\lim_{a\to\infty} 2a^2 A(f_a) \quad = \quad \frac{1}{2\pi}\|\cdot f\|^2_{L^2(\mathbb{R})}$$

and

$$\lim_{a\to\infty} aB(f_a) \;=\; \frac{i}{2\pi}\langle \cdot f, f\rangle_{L^2(\mathbb{R})},$$

with the terms A and B as introduced in Lemma 3.

Proof. Immediately, we obtain

$$\begin{aligned}\lim_{a\to\infty} \|f_a\|^2_{L^2(-\pi,\pi)} &= \lim_{a\to\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} a|f(ax)|^2\,dx \\ &= \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a\pi}^{a\pi} |f(x)|^2\,dx \;=\; \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{R})}\end{aligned}$$

and

$$\begin{aligned}\lim_{a\to\infty} \frac{1}{a^2}\|f_a'\|^2_{L^2(-\pi,\pi)} &= \lim_{a\to\infty} \frac{1}{2\pi a^2}\int_{-\pi}^{\pi} a^3|f'(ax)|^2\,dx \\ &= \frac{1}{2\pi}\lim_{a\to\infty}\int_{-a\pi}^{a\pi} |f'(x)|^2\,dx \;=\; \frac{1}{2\pi}\|f'\|^2_{L^2(\mathbb{R})},\end{aligned}$$

and similarly

$$\begin{aligned}\lim_{a\to\infty} \frac{1}{a}\langle f_a', f_a\rangle_{L^2(-\pi,\pi)} &= \lim_{a\to\infty} \frac{1}{2\pi a}\int_{-\pi}^{\pi} a^2 f'(ax)\overline{f(ax)}\,dx \\ &= \frac{1}{2\pi}\langle f', f\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

Now we consider the terms from Lemma 3. Note that for all $x\in\mathbb{R}$, it holds

$$(2a\sin(x/2a))^2 \;\le\; x^2 \quad\text{and}\quad \lim_{a\to\infty} 2a\sin(x/2a) \;=\; x.$$

Lebesgue's dominated convergence theorem yields

$$\begin{aligned}\lim_{a\to\infty} 2a^2 A(f_a) &= \lim_{a\to\infty} \frac{4a^2}{2\pi}\int_{-\pi}^{\pi} \sin^2(x/2)\; a|f(ax)|^2\,dx \\ &= \lim_{a\to\infty} \frac{1}{2\pi}\int_{-a\pi}^{a\pi} (2a\sin(x/2a))^2\;|f(x)|^2\,dx \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} x^2|f(x)|^2\,dx \;=\; \frac{1}{2\pi}\|\cdot f\|^2_{L^2(\mathbb{R})}.\end{aligned}$$

Next, since

$$\begin{aligned}|a(e^{ix/a}-1)| \;=\; |2a\sin(x/2a)| \;&\le\; |x|, \qquad \lim_{a\to\infty} a(e^{ix/a}-1) \;=\; ix, \\ |e^{-ix/a}+1| \;&\le\; 2, \qquad \lim_{a\to\infty} e^{-ix/a}+1 \;=\; 2,\end{aligned}$$

we can apply Lebesgue's dominated convergence theorem again and obtain

$$\begin{aligned}\lim_{a\to\infty} aB(f_a) &= \lim_{a\to\infty} \frac{a}{4\pi}\int_{-\pi}^{\pi}(e^{ix}-1)(e^{-ix}+1)\ a\,|f(ax)|^2\,dx \\ &= \lim_{a\to\infty}\frac{1}{4\pi}\int_{-a\pi}^{a\pi} a(e^{ix/a}-1)(e^{-ix/a}+1)|f(x)|^2\,dx \\ &= \frac{i}{2\pi}\int_{-\infty}^{\infty} x|f(x)|^2\,dx \;=\; \frac{i}{2\pi}\langle\cdot f, f\rangle_{L^2(\mathbb{R})}.\end{aligned}$$

□

We will consider two different, but related ways of generating a periodic function from one that is given on the whole real line, namely periodizing and sampling. By making these processes dependent on dilation by a real positive parameter a, we create two families of periodic functions from one given function on the real line.

For a function $f \in L^1(\mathbb{R})$ and a parameter $a > 0$, we can generate a function of period 2π as

$$f_a^{per}(t) := \sqrt{a}\sum_{k=-\infty}^{\infty} f(a(t+2\pi k)) = \sum_{k=-\infty}^{\infty} f_a(t+2\pi k). \tag{4.3}$$

By the theorem of B. Levi (cf. [8, Chap. 2 (18)]) f_a^{per} converges almost everywhere.

Lemma 6.
Let f be admissible and $a > 0$. Then $f_a^{per} \in AC_{2\pi}$ and $(f_a^{per})' \in L^2_{2\pi}$.

Proof. For $-\pi \le x \le \pi$ and any $a > 0$, we obtain the following estimate, using the decay condition (4.1) for $\gamma > 1$,

$$\begin{aligned}&\left|\sum_{k=-\infty}^{\infty} f_a(x+2\pi k) - \sum_{k=-N}^{N} f_a(x+2\pi k)\right| = \left|\sum_{|k|>N}\sqrt{a}f(a(x+2\pi k))\right| \\ &\le \sum_{|k|>N}\sqrt{a}|f(a(x+2\pi k))| \le 2\sum_{k=N+1}^{\infty}\frac{C\sqrt{a}}{(a(x+2\pi k))^\gamma} \\ &\le 2Ca^{1/2-\gamma}\pi^{-\gamma}\sum_{k=N+1}^{\infty}(2k-1)^{-\gamma} \quad\to 0 \quad (N\to\infty).\end{aligned}$$

Since the last expression is independent of x, the convergence is uniform and therefore also pointwise. As f and thus also f_a are continuous, we get that f_a^{per} is continuous.

Since $f' \in L^1(\mathbb{R})$ implies $(f_a)' \in L^1(\mathbb{R})$, we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} |f_a'(x+2\pi k)|\ dx &= \sum_{k=-\infty}^{\infty} \int_{(2k-1)\pi}^{(2k+1)\pi} |f_a'(x)|\ dx \\ &= \int_{-\infty}^{\infty} |f_a'(x)|\ dx\ <\ \infty.\end{aligned}$$

By the theorem of B. Levi the function $g(x) = \sum_{k=-\infty}^{\infty} f_a'(x+2\pi k)$ converges (absolutely) almost everywhere and $g \in L^1(-\pi,\pi)$. Now, using B. Levi's theorem one more time and applying the fundamental theorem of calculus for the Lebesgue integral, we obtain

$$\int_{-\pi}^{x} g(v)dv\ =\ \int_{-\pi}^{x} \sum_{k=-\infty}^{\infty} f_a'(v+2\pi k)dv\ =\ \sum_{k=-\infty}^{\infty} \int_{-\pi}^{x} f_a'(v+2\pi k)dv$$

$$=\ \sum_{k=-\infty}^{\infty} \big(f_a\big(x+2\pi k) - f_a((2k-1)\pi)\big)\ =\ f_a^{per}(x) - f_a^{per}(-\pi).$$

This implies that f_a^{per} is absolutely continuous and $(f_a^{per})' = g$ almost everywhere. Furthermore,

$$\begin{aligned}\|(f_a^{per})'\|_{L^2_{2\pi}} &= \left\| \sum_{k=-\infty}^{\infty} (f_a)'(\cdot + 2\pi k) \right\|_{L^2(-\pi,\pi)} \\ &\leq \|f_a'\|_{L^2(-\pi,\pi)} + \sum_{k\neq 0} \|f_a'(\cdot + 2\pi k)\|_{L^2(-\pi,\pi)}.\end{aligned}$$

Since for $k \neq 0$

$$\|f_a'(\cdot+2\pi k)\|^2_{L^2(-\pi,\pi)}\ =\ \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_a'(x+2\pi k)|^2 dx$$

$$=\ \frac{a^3}{2\pi}\int_{-\pi}^{\pi} |f'(ax+a2\pi k)|^2 dx\ =\ \frac{a^2}{2\pi}\int_{a(2k-1)\pi}^{a(2k+1)\pi} |f'(x)|^2 dx$$

$$\leq\ \frac{a^2}{2\pi}\int_{a(2k-1)\pi}^{a(2k+1)\pi} \frac{D^2}{|x|^{2\beta}} dx\ \leq\ \frac{a^3 D^2}{(a(2|k|-1)\pi)^{2\beta}}\ =\ \frac{D^2 a^{3-2\beta}}{\pi^{2\beta}}(2|k|-1)^{-2\beta},$$

we find that

$$\sum_{k\neq 0} \|f_a'(\cdot+2\pi k)\|_{L^2(-\pi,\pi)}\ \leq\ \frac{2Da^{3/2-\beta}}{\pi^\beta}\sum_{k=1}^{\infty}(2k-1)^{-\beta}\ <\ \infty$$

and thus $(f_a^{per})' \in L^2_{2\pi}$. $\square$

For the function f_a, Poisson's summation formula holds in the following form.

Corollary 1.
Let f be admissible and $a > 0$. Then

$$f_a^{per}(t) \;=\; \frac{1}{2\pi\sqrt{a}} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{k}{a}\right) e^{ikt}, \quad t \in \mathbb{R}.$$

Proof. Due to Lemma 6 the Fourier series of f_a^{per} converges everywhere to the function value. The Fourier coefficients of f_a^{per} are given by

$$c_k(f_a^{per}) \;=\; \frac{1}{2\pi}\hat{f}_a(k) \;=\; \frac{1}{2\pi\sqrt{a}}\hat{f}\left(\frac{k}{a}\right).$$

□

5. Main limit result

The following result states conditions for the original function f on the real line, which guarantee the desired limit behaviour of the uncertainty product for the functions f_a^{per} obtained by periodization as defined in (4.3).

Theorem 3.
If f is admissible then

$$\lim_{a\to\infty} \frac{1}{a^2}\,\mathrm{var}_F(f_a^{per}) \;=\; \Delta\xi(f)$$

and

$$\lim_{a\to\infty} a^2\,\mathrm{var}_A(f_a^{per}) \;=\; \Delta x(f),$$

yielding altogether

$$\lim_{a\to\infty} U_{2\pi}(f_a^{per}) \;=\; U_{\mathbb{R}}(f) \;<\; \infty.$$

In order to prove this theorem, we consider separately the limit behaviour of the two variances occurring in the periodic uncertainty principle.

Theorem 4.
If f is admissible then

$$\lim_{a\to\infty} \frac{1}{a^2}\,\mathrm{var}_F(f_a^{per}) \;=\; \Delta\xi(f).$$

Proof. With the decay condition (4.1), we estimate the following term for $k \neq 0$ using that $\min(|2k+1|, |2k-1|) \geq |2k|-1$

$$\begin{aligned}
\|f_a(\cdot + 2\pi k)\|^2_{L^2(-\pi,\pi)} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |f_a(x+2\pi k)|^2 \, dx \\
= \frac{1}{2\pi}\int_{a(2k-1)\pi}^{a(2k+1)\pi} |f(x)|^2 \, dx &\leq \frac{1}{2\pi}\int_{a(2k-1)\pi}^{a(2k+1)\pi} \frac{C^2}{|x|^{2\gamma}} \, dx \\
\leq \frac{aC^2}{(a(2|k|-1)\pi)^{2\gamma}} &= \frac{C^2}{\pi^{2\gamma}} a^{1-2\gamma}(2|k|-1)^{-2\gamma}.
\end{aligned}$$

Thus

$$\begin{aligned}
\sum_{k\neq 0} \|f_a(\cdot + 2\pi k)\|_{L^2(-\pi,\pi)} &\leq \frac{C}{\pi^\gamma}\, a^{1/2-\gamma}\; 2\sum_{k=1}^{\infty}(2k-1)^{-\gamma} \\
&= \frac{2C}{\pi^\gamma}\, S(\gamma)\, a^{1/2-\gamma} \;\to\; 0 \quad (a\to\infty) \quad (5.1)
\end{aligned}$$

where $S(\gamma) = \sum_{k=1}^{\infty}(2k-1)^{-\gamma} < \infty$ since $\gamma > 3/2$.

By Lemma 6 $(f_a^{per})' \in L^2_{2\pi}$ and analogously to the above estimations we obtain

$$\frac{1}{a^2}\sum_{k\neq 0} \left\|f_a'(\cdot + 2\pi k)\right\|_{L^2(-\pi,\pi)} \leq \frac{2D}{\pi^\beta}\, S(\beta)\, a^{1/2-\beta} \;\to\; 0 \quad (a\to\infty). \quad (5.2)$$

Now we consider

$$\begin{aligned}
&\left| \|f_a^{per}\|_{L^2_{2\pi}} - \sqrt{\frac{1}{2\pi}}\|f\|_{L^2(\mathbb{R})} \right| \\
&\leq \left| \|f_a^{per}\|_{L^2_{2\pi}} - \|f_a\|_{L^2(-\pi,\pi)} \right| + \left| \|f_a\|_{L^2(-\pi,\pi)} - \sqrt{\frac{1}{2\pi}}\|f\|_{L^2(\mathbb{R})} \right|, \quad (5.3)
\end{aligned}$$

where the first term can be estimated as follows

$$\begin{aligned}
&\left| \|f_a^{per}\|_{L^2_{2\pi}} - \|f_a\|_{L^2(-\pi,\pi)} \right| \\
&= \left| \left\| f_a + \sum_{k\neq 0} f_a(\cdot + 2\pi k) \right\|_{L^2(-\pi,\pi)} - \|f_a\|_{L^2(-\pi,\pi)} \right| \\
&\leq \left\| \sum_{k\neq 0} f_a(\cdot + 2\pi k) \right\|_{L^2(-\pi,\pi)} \leq \sum_{k\neq 0} \|f_a(\cdot + 2\pi k)\|_{L^2(-\pi,\pi)}.
\end{aligned}$$

Using (5.1) as well as Lemma 5 for the last term in (5.3) establishes

$$\lim_{a\to\infty} \|f_a^{per}\|^2_{L^2_{2\pi}} = \frac{1}{2\pi}\|f\|^2_{L^2(\mathbb{R})}. \quad (5.4)$$

Analogously we obtain with (5.2) that

$$\lim_{a\to\infty} \frac{1}{a^2}\|(f_a^{per})'\|_{L^2_{2\pi}}^2 \;=\; \frac{1}{2\pi}\|f'\|_{L^2(\mathbb{R})}^2. \tag{5.5}$$

Furthermore, recalling again $f_a' = (f_a)'$ we have

$$\begin{aligned}
\frac{1}{a}\langle (f_a^{per})', f_a^{per}\rangle_{L^2_{2\pi}} &= \frac{1}{a}\left\langle \sum_{k=-\infty}^{\infty} f_a'(\cdot+2\pi k), \sum_{l=-\infty}^{\infty} f_a(\cdot+2\pi l)\right\rangle_{L^2(-\pi,\pi)} \\
&= \frac{1}{a}\langle f_a', f_a\rangle_{L^2(-\pi,\pi)} + \frac{1}{a}\sum_{l\neq 0}\langle f_a', f_a(\cdot+2\pi l)\rangle_{L^2(-\pi,\pi)} \\
&+ \frac{1}{a}\sum_{k\neq 0}\sum_{l=-\infty}^{\infty}\langle f_a'(\cdot+2\pi k), f_a(\cdot+2\pi l)\rangle_{L^2(-\pi,\pi)}.
\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we estimate the last two sums as

$$\begin{aligned}
&\left|\frac{1}{a}\sum_{l\neq 0}\langle f_a', f_a(\cdot+2\pi l)\rangle_{L^2(-\pi,\pi)}\right| \\
&\leq \underbrace{\frac{1}{a}\|f_a'\|_{L^2(-\pi,\pi)}}_{\to \|f'\|_{L^2(\mathbb{R})}/\sqrt{2\pi}} \sum_{l\neq 0}\|f_a(\cdot+2\pi l)\|_{L^2(-\pi,\pi)} \;\to\; 0 \quad (a\to\infty)
\end{aligned}$$

and

$$\begin{aligned}
&\left|\frac{1}{a}\sum_{k\neq 0}\sum_{l=-\infty}^{\infty}\langle f_a'(\cdot+2\pi k), f_a(\cdot+2\pi l)\rangle_{L^2(-\pi,\pi)}\right| \\
&\leq \sum_{k\neq 0}\frac{1}{a}\|f_a'(\cdot+2\pi k)\|_{L^2(-\pi,\pi)} \underbrace{\sum_{l=-\infty}^{\infty}\|f_a(\cdot+2\pi l)\|_{L^2(-\pi,\pi)}}_{\to \|f\|_{L^2(\mathbb{R})}/\sqrt{2\pi}} \;\to\; 0
\end{aligned}$$

as $a\to\infty$, where we used (5.1), (5.2) and Lemma 5 one more time. Thus

$$\lim_{a\to\infty} \frac{1}{a}\langle (f_a^{per})', f_a^{per}\rangle_{L^2_{2\pi}} \;=\; \frac{1}{2\pi}\langle f', f\rangle_{L^2(\mathbb{R})}. \tag{5.6}$$

Putting together (5.4), (5.5) and (5.6) finally yields with (2.4)

$$\lim_{a\to\infty}\frac{1}{a^2}\operatorname{var}_F(f_a^{per}) = \lim_{a\to\infty}\left(\frac{\frac{1}{a^2}\|(f_a^{per})'\|_{L^2_{2\pi}}^2}{\|f_a^{per}\|_{L^2_{2\pi}}^2} + \frac{\frac{1}{a^2}\langle (f_a^{per})', f_a^{per}\rangle_{L^2_{2\pi}}^2}{\|f_a^{per}\|_{L^2_{2\pi}}^4}\right) = \Delta\xi(f).$$

□

Corollary 2.

Let f be admissible with compact support. Then there exists a constant $a_0 > 0$ such that

$$\frac{1}{a^2}\,\mathrm{var}_F(f_a^{per}) \;=\; \Delta\xi(f) \quad \text{for all } \; a \geq a_0.$$

Proof. If a is large enough then $f_a^{per} = f_a$ on $[-\pi,\pi]$ and $f_a(x) = 0$ for $x \notin [-\pi,\pi]$ because of the compact support. From this the assertion follows immediately. $\square$

Now we will consider the angular variance of the periodized functions, which may be infinite for certain $a > 0$, but exhibits the following limit behaviour.

Theorem 5.

Let f be admissible. Then there exists an $a_1 > 0$, such that $\mathrm{var}_A(f_a^{per}) < \infty$ for all $a > a_1$ and it holds

$$\lim_{a\to\infty} a^2\,\mathrm{var}_A(f_a^{per}) \;=\; \Delta x(f).$$

Proof. We use the representations for the angular variance from Lemma 3 and consider $A(f_a^{per})$ first. Since $\sin^2(x/2) \geq 0$ and $\sin^2(x/2) = 0$ only on a set of measure zero, $\sqrt{A(\cdot)}$ is actually a weighted L^2-norm. Thus we can proceed analogously as in the proof of (5.4).

Using the decay condition (4.1) on f, we obtain that

$$\begin{aligned} A(f_a(\cdot+2\pi k)) &= \frac{1}{\pi}\int_{-\pi}^{\pi} \sin^2(x/2)\;|f_a(x+2\pi k)|^2\;dx \\ &\leq \frac{1}{\pi}\int_{-\pi}^{\pi} |f_a(x+2\pi k)|^2\;dx \;=\; 2\|f_a(\cdot+2\pi k)\|^2_{L^2(-\pi,\pi)} \end{aligned}$$

and consequently as in (5.1)

$$\sqrt{2}a\sum_{k\neq 0}\sqrt{A(f_a(\cdot+2\pi k))} \;\leq\; \frac{4C}{\pi^\gamma}\;S(\gamma)\;a^{3/2-\gamma} \;\to\; 0 \quad (a\to\infty). \tag{5.7}$$

Since the triangle inequality for the weighted norm implies

$$\sqrt{2}a\left|\sqrt{A(f_a^{per})} - \sqrt{A(f_a)}\right| \;\leq\; \sqrt{2}a\sum_{k\neq 0}\sqrt{A(f_a(\cdot+2\pi k))},$$

we get with Lemma 5 that

$$\lim_{a\to\infty} 2a^2 A(f_a^{per}) \;=\; \frac{1}{2\pi}\|\cdot f\|^2_{L^2(\mathbb{R})}. \tag{5.8}$$

As far as $B(f_a^{per})$ is concerned, note that $B(\cdot)$ cannot be used directly in a norm. In light of Lemma 5, it is sufficient to prove

$$\lim_{a\to\infty} a(B(f_a^{per}) - B(f_a)) \;=\; 0. \tag{5.9}$$

We split the corresponding integral into two parts and set

$$B_1(g) := \frac{1}{2\pi}\int_0^{\pi} \sin x \;|g(x)|^2\; dx, \quad B_2(g) := \frac{1}{2\pi}\int_{-\pi}^{0} (-\sin x)\;|g(x)|^2\; dx,$$

so that

$$B(g) \;=\; i(B_1(g) - B_2(g)),$$

further reducing the problem to proving

$$\lim_{a\to\infty} a(B_j(f_a^{per}) - B_j(f_a)) \;=\; 0, \quad j = 1, 2. \tag{5.10}$$

Since $\sqrt{B_1(\cdot)}$ and $\sqrt{B_2(\cdot)}$ are weighted L^2-norms on $L^2(0,\pi)$ and $L^2(-\pi,0)$, respectively, we can proceed as for the $A(f_a^{per})$ terms and obtain, for $j = 1, 2$, analogously to (5.7) and (5.1)

$$\begin{aligned}\sqrt{a}\sum_{k\neq 0}\sqrt{B_j(f_a(\cdot + 2\pi k))} \;&\le\; \sqrt{a}\sum_{k\neq 0}\|f_a(\cdot + 2\pi k)\|_{L^2(-\pi,\pi)}\\ &\le\; \frac{2C}{\pi^{\gamma}}\, S(\gamma)\; a^{1-\gamma} \;\to\; 0 \quad (a\to\infty),\end{aligned}$$

and thus (5.10). This implies (5.9) and with Lemma 5 the limit

$$\lim_{a\to\infty} aB(f_a^{per}) \;=\; \frac{i}{2\pi}\langle \cdot f, f\rangle_{L^2(\mathbb{R})}. \tag{5.11}$$

Combining (5.4), (5.8) and (5.11) with Lemma 3 yields

$$\begin{aligned}\lim_{a\to\infty}|\tau(f_a^{per})|^2 \;&=\; \lim_{a\to\infty}\Big((\|f_a^{per}\|^2_{L^2_{2\pi}} - A(f_a^{per}))^2 - B^2(f_a^{per})\Big)\\ &=\; \frac{1}{(2\pi)^2}\|f\|^4_{L^2(\mathbb{R})} \;>\; 0,\end{aligned}$$

from which the existence of a suitable a_1 follows. Using (2.3) we obtain

$$\begin{aligned}&\lim_{a\to\infty} a^2\,\mathrm{var}_A(f_a^{per})\\ &=\; \lim_{a\to\infty}\frac{2a^2A(f_a^{per})\|f_a^{per}\|^2_{L^2_{2\pi}} + a^2B^2(f_a^{per}) - a^2A^2(f_a^{per})}{(\|f_a^{per}\|^2_{L^2_{2\pi}} - A(f_a^{per}))^2 - B^2(f_a^{per})}\\ &=\; \frac{\|\cdot f\|^2_{L^2(\mathbb{R})}\;\|f\|^2_{L^2(\mathbb{R})} - \langle \cdot f, f\rangle^2_{L^2(\mathbb{R})}}{\|f\|^4_{L^2(\mathbb{R})}} \;=\; \Delta x(f).\end{aligned}$$

This completes the proof. □

The combination of Theorems 4 and 5 provides the general limit result of Theorem 3 for the periodic uncertainty products.

6. Limit results for sampling

Let $f \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ be such that for a parameter $a > 0$ the sequence $\left(f\left(\frac{k}{a}\right)\right)_{k\in\mathbb{Z}}$ is square-summable. In that case we define

$$f_a^{sam} \; := \; \sum_{k=-\infty}^{\infty} f\left(\frac{k}{a}\right) e^{ik\cdot} \quad \in L^2_{2\pi}.$$

For an admissible function g and any $a > 0$, Corollary 1 states that

$$g_a^{per}(t) \; = \; \frac{1}{2\pi\sqrt{a}} \, (\hat{g})_a^{sam}. \tag{6.1}$$

Theorem 6.
Let $f \in L^2(\mathbb{R}) \cap C_0(\mathbb{R})$ be such that $\hat{f}$ is admissible. Then

$$\begin{aligned} \lim_{a\to\infty} a^2 \operatorname{var}_A(f_a^{sam}) \; &= \; \Delta x(\hat{f}) \; = \; \Delta\xi(f), \\ \lim_{a\to\infty} \frac{1}{a^2} \operatorname{var}_F(f_a^{sam}) \; &= \; \Delta\xi(\hat{f}) \; = \; \Delta x(f) \end{aligned}$$

and

$$\lim_{a\to\infty} U_{2\pi}(f_a^{sam}) \; = \; U_{\mathbb{R}}(f) \; < \; \infty.$$

Proof. Using again the reflection f^-, we set $g := \frac{1}{2\pi}\hat{f}^-$ and find that g is admissible with $\hat{g} = f$ for all $x \in \mathbb{R}$, since $\hat{g}$ is also continuous by Lemma 1. Using (6.1), we obtain

$$g_a^{per} \; = \; \frac{1}{2\pi\sqrt{a}}(\hat{g})_a^{sam} \; = \; \frac{1}{2\pi\sqrt{a}} f_a^{sam}.$$

Since by Lemma 6 it holds $g_a^{per} \in L^2_{2\pi}$, we can conclude that $\left(f\left(\frac{k}{a}\right)\right)_{k\in\mathbb{Z}}$ is square-summable and, hence, that f_a^{sam} is well defined for any $a > 0$. The variances in Definition 2 are invariant under multiplication of the function by a constant and exist for f_a^{sam} due to Lemma 6. So the application of Theorem 3 yields

$$\lim_{a\to\infty} \frac{1}{a^2} \operatorname{var}_F(f_a^{sam}) \; = \; \lim_{a\to\infty} \frac{1}{a^2} \operatorname{var}_F(g_a^{per}) \; = \; \Delta\xi(g) \; = \; \Delta x(f)$$

and

$$\lim_{a\to\infty} a^2 \operatorname{var}_A(f_a^{sam}) \; = \; \lim_{a\to\infty} a^2 \operatorname{var}_A(g_a^{per}) \; = \; \Delta x(g) \; = \; \Delta\xi(f).$$

□

Now we investigate some sufficient conditions on f such that the prerequisites of Theorem 6 hold. At first, we need the following lemma which is a generalization of Lemma 3 in [6].

Lemma 7.

Let $p = 1$ or $p = 2$. If $f \in L^p(\mathbb{R}) \cap AC(\mathbb{R})$ and $f' \in BV(\mathbb{R})$, then $\hat{f} \in L^1(\mathbb{R})$, and furthermore

$$|\hat{f}(\xi)| \;\leq\; \frac{V(f')}{\xi^2} \quad \begin{cases} \textit{everywhere} & \textit{if} \;\; p = 1, \\ \textit{a.e.} & \textit{if} \;\; p = 2, \end{cases}$$

where $V(f')$ denotes the total variation of f' on $\mathbb{R}$.

Proof. Since $f \in L^p(\mathbb{R})$, the Fourier transform of f exists. As $f \in AC(\mathbb{R})$, we can apply the fundamental theorem of calculus for the Lebesgue integral and obtain

$$\int_0^1 f'(x-t)\; dt \;=\; f(x) - f(x-1). \tag{6.2}$$

The left-hand side of (6.2) can be expressed as $(\chi_{[0,1]} * f')(x)$, where $\chi_{[0,1]}$ is the characteristic function of the interval $[0,1]$ and $*$ is the convolution. We have that

$$\hat{\chi}_{[0,1]}(\xi) \;=\; \frac{1 - e^{-i\xi}}{i\xi}.$$

As $f' \in L^1(\mathbb{R})$, we can take the Fourier transform on both sides of (6.2)

$$\frac{1 - e^{-i\xi}}{i\xi} \mathcal{F}(f')(\xi) \;=\; \left(1 - e^{-i\xi}\right) \hat{f}(\xi)$$

giving

$$i\xi\; \hat{f}(\xi) \;=\; \mathcal{F}(f')(\xi) \quad \begin{cases} \text{everywhere} & \text{if} \;\; p = 1, \\ \text{a.e.} & \text{if} \;\; p = 2. \end{cases}$$

Since $f' \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, it holds $\lim_{x\to\pm\infty} f'(x) = 0$. We use integration by parts for Stieltjes integrals and get

$$\begin{aligned} \left|\mathcal{F}(f')(\xi)\right| \;&=\; \left|\int_{-\infty}^{\infty} f'(x) e^{-i\xi x}\; dx\right| \;=\; \frac{1}{\xi}\left|\lim_{b\to\infty} \int_{-b}^{b} f'(x) de^{-i\xi x}\right| \\ &=\; \frac{1}{\xi}\left|\lim_{b\to\infty} \left(f'(b)e^{-i\xi b} - f'(-b)e^{i\xi b} - \int_{-b}^{b} e^{-i\xi x}\; df'(x)\right)\right| \\ &=\; \frac{1}{\xi}\left|\int_{-\infty}^{\infty} e^{-i\xi x}\; df'(x)\right| \;\leq\; \frac{1}{\xi} V(f'). \end{aligned}$$

Note that $\hat{f} \in L^1(\mathbb{R})$ follows easily by splitting the domain of integration into the intervals $[-1,1]$ and $\mathbb{R}\backslash[-1,1]$. $\square$

It remains to specify explicit classes of functions which satisfy the condition set in Theorem 6. We can state the following result.

Corollary 3.
Let $f \in L^2(\mathbb{R}) \cap AC(\mathbb{R})$ with $(\cdot f) \in L^2(\mathbb{R}) \cap AC(\mathbb{R})$. If additionally $f', (\cdot f)' \in BV(\mathbb{R})$, then $\hat{f}$ is admissible and the limit results of Theorem 6 hold.

Proof. As shown in the proof of Lemma 1, we have $f \in L^1(\mathbb{R})$. Therefore Lemma 7 for $p = 1$ implies

$$|\hat{f}(\xi)| \leq \frac{V(f')}{\xi^2} \quad \text{for all} \quad \xi \in \mathbb{R}.$$

This means that additionally $(\cdot\hat{f}) \in L^2(\mathbb{R})$, so that (2.1) holds. We apply Lemma 7 for $p = 2$ to $(\cdot f)$ to obtain

$$|\mathcal{F}(\cdot f)(\xi)| \leq \frac{V((\cdot f)')}{\xi^2} \quad \text{a.e.}$$

Lemma 1 yields that $\hat{f}$ is admissible, which completes the proof. $\square$

As the sampling of compactly supported functions produces trigonometric polynomials, which are of special interest in many applications, we conclude this section with a result concerning such functions.

Corollary 4.
If $f \in AC(\mathbb{R})$ is compactly supported and $f' \in BV(\mathbb{R})$, then $\hat{f}$ is admissible and Theorem 6 is applicable.

Proof. It is straightforward to see that the conditions of Corollary 3 are satisfied. $\square$

7. Examples

Some examples, related to several previous investigations and combined with new computational tests, should help to illustrate the results of the previous sections. We apply Theorem 6 or one of its corollaries to sampling different generating functions, starting with a Gaussian and a characteristic function, yielding best and worst uncertainty products, respectively. In a sense, these are the two extremal cases of sampling B-splines, which we will consider in general at the end of this section. We start to investigate specific continuous piecewise linear splines, related to certain scaling functions and wavelets in a trigonometric multiresolution analysis. At first we consider a class of such linear splines with just one support interval, and then another class, where the supports consist of two closed intervals that are not connected.

We begin by considering the generating function f to be a Gaussian function of the form given in Theorem 1. The Fourier transform of such a Gaussian is again a Gaussian. Therefore all conditions of Theorem 6 are satisfied and we obtain, together with Theorem 1, the following.

Corollary 5.

If f is a Gaussian function, then

$$\lim_{a\to\infty} U_{2\pi}(f_a^{sam}) = U_{\mathbb{R}}(f) = \frac{1}{2},$$

hence, the constant $1/2$ is optimal for the periodic uncertainty principle of Theorem 2.

Thus any Gaussian gives rise to an asymptotically optimal family f_a^{sam} for the periodic uncertainty principle. Specifically for the simple Gaussian with $f(x) = e^{-x^2}$, we obtain for any $a > 0$

$$f_a^{sam}(t) = \sum_{k=-\infty}^{\infty} e^{-(k/a)^2} e^{ikt},$$

which is an even, positive and smooth function. It can be derived from the Theta functions

$$\theta(z,q) = \sum_{k=-\infty}^{\infty} q^{k^2} z^k \quad (z \in \mathbb{C}, |q| < 1)$$

by setting $z = e^{it}$ and $q = e^{-1/a^2}$. Originally the optimality of the constant $1/2$ in Theorem 2 was proven in [12] by directly estimating the angular and frequency variances of the family of functions f_a^{sam} (though specified with the parameter $h = 1/a^2$ going to zero instead of a going to infinity).

Now we turn to trigonometric polynomials by considering $a = n \in \mathbb{N}$. In order to generate trigonometric polynomials as f_n^{sam}, we need symmetric generating functions f of compact support. As the simplest example, we start with a characteristic function.

Example 1. **The Dirichlet kernel:** For the generating function

$$f(x) := \chi_{[-\frac{1}{2},\frac{1}{2}]}(x) = \begin{cases} 1 & \text{for } x \in [-\frac{1}{2},\frac{1}{2}], \\ 0 & \text{otherwise,} \end{cases}$$

we obtain the Dirichlet kernel as

$$f_{2n}^{sam}(t) = \sum_{k=-n}^{n} f\left(\frac{k}{2n}\right) e^{ikt} = 1 + 2\sum_{k=1}^{n} \cos kt = D_n(t).$$

The Fourier transform of f does not satisfy the set of conditions in Theorem 6, as

$$\hat{f}(\xi) = \frac{\sin(\xi/2)}{\xi/2}.$$

Knowing f and $\hat{f}$, we find directly that $U_{\mathbb{R}}(f) = \infty$. In [12] it was computed explicitly for the Dirichlet kernel that $U_{2\pi}(D_n) \sim \sqrt{n}$. Thus we can conclude that

$$\lim_{n\to\infty} U_{2\pi}(f_{2n}^{sam}) = U_{\mathbb{R}}(f) = \infty,$$

even though this case is not covered by our results in this paper.

The piecewise linear generating functions of the following two examples, however, satisfy the admissibility conditions in Definition 3.

Example 2. **De la Vallée Poussin means:** For any fixed $r \in \mathbb{R},\ r \geq 1$, the generating function

$$\begin{aligned} f_r(x) &:= \frac{1}{2}\left(\chi_{[-r,r]} * \chi_{[-1,1]}\right)(x) \\ &= \begin{cases} 1 & \text{for } |x| \in [0, r-1], \\ (r+1-|x|)/2 & \text{for } |x| \in (r-1, r+1), \\ 0 & \text{otherwise,} \end{cases} \end{aligned}$$

yields the de la Vallée Poussin means φ_N^M for $N, M \in \mathbb{N},\ N \geq M$, as

$$\left(f_{\frac{N}{M}}\right)_M^{sam}(t) = \sum_{k=-N-M}^{N+M} f_{\frac{N}{M}}\left(\frac{k}{M}\right) e^{ikt} = \frac{1}{2M}\sum_{n=N-M}^{N+M-1} D_n(t) =: \varphi_N^M(t).$$

Since f_r is absolutely continuous with compact support $[-r-1, r+1]$ and since

$$f_r'(x) = \frac{1}{2}\,\mathrm{sign}(x)\ \chi_{[r-1,r+1]}(|x|)$$

has total variation 2 for any r, we can apply Corollary 4. Both f_r and its Fourier transform

$$\hat{f}_r(\xi) = \frac{2\sin(r\xi)\sin(\xi)}{\xi^2}$$

are symmetric, hence we only need to compute

$$\begin{aligned} \|\hat{f}_r\|^2 &= 2\pi\|f_r\|^2 = 4\pi\left(r - 1/3\right), \\ \|\cdot\hat{f}_r\|^2 &= 2\pi\|f_r'\|^2 = 2\pi, \\ \|\cdot f_r\|^2 &= 2(r-1)(r^2+1)/3 + 8/15, \end{aligned}$$

and obtain

$$U^2_{\mathbb{R}}(f_r) \;=\; \frac{\|\cdot f_r\|^2 \|\cdot \hat{f}_r\|^2}{\|f_r\|^2 \|\hat{f}_r\|^2} \;=\; \frac{3(5r^3 - 5r^2 + 5r - 1)}{10(3r-1)^2}. \tag{7.1}$$

De la Vallée Poussin means can be used to generate multiresolution analyses (see [17]). For fixed $r \in \mathbb{N}_0$, $r \geq 3$, let $N_j = 2^j r$ and $M_j = 2^j$ for all $j \in \mathbb{N}_0$. For the scaling functions $\phi_j = (2N_j)^{-1/2}(f_{N_j/M_j})^{sam}_{M_j}$ we have shown in [16] that the uncertainty product is uniformly bounded, namely $U^2_{2\pi}(\phi_j) < r/5$ for all j. The expressions for $U^2_{2\pi}(\phi_j)$, however, were quite complicated. Using Theorem 6, we now obtain easily the exact limit $\lim_{j\to\infty} U_{2\pi}(\phi_j) = U_{\mathbb{R}}(f_r)$ from (7.1). In particular, for $r = 3$, which is the minimal parameter that allows to generate a multiresolution analysis, we have $\lim_{j\to\infty} U_{2\pi}(\phi_j) = U_{\mathbb{R}}(f_3) = \sqrt{39/80} \approx 0.7$. Compared to the limit $1/2$ for best time-frequency localized functions, our polynomial scaling functions ϕ_j for $j \in \mathbb{N}_0$ are very well localized.

Corresponding to band-pass functions or wavelets, we can use generating functions that vanish around 0. Related to the previous example let us consider a piecewise linear version with two unconnected support components.

Example 3. **Band-pass polynomials:** Fix $r \in \mathbb{R}$, $r \geq 3$, and let the generating function be

$$\begin{aligned} g_r(x) \;&:=\; f_r\left(\frac{x}{2}\right) - f_r(x) \\ &=\; \begin{cases} (|x| - (r-1))/2 & \text{for } |x| \in (r-1, r+1), \\ 1 & \text{for } |x| \in [r+1, 2r-2], \\ (2r+2-|x|)/4 & \text{for } |x| \in (2r-2, 2r+2), \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Then, for $N, M \in \mathbb{N}$, $N \geq M$, sampling of the form

$$\left(g_{\frac{N}{M}}\right)^{sam}_M(t) \;=\; 2\sum_{k=N-M}^{2(N+M)} g_{\frac{N}{M}}\left(\frac{k}{M}\right)\cos kt \;=\; \varphi^{2M}_{2N}(t) - \varphi^{M}_{N}(t)$$

yields trigonometric polynomials in the frequency band $\pm(N-M, 2N+2M)$.

The function g_r is even and absolutely continuous with compact support. Its derivative

$$g_r'(x) \;=\; \frac{1}{2}\,\mathrm{sign}(x)\;\chi_{[r-1,r+1]}(|x|) - \frac{1}{4}\,\mathrm{sign}(x)\;\chi_{[2r-2,2r+2]}(|x|)$$

is of course odd and of total variation 3 for any r, so that we have $\langle \cdot g_r, g_r\rangle = 0$ and $\langle g_r', g_r\rangle = 0$, and hence by Lemma 2,

$$U^2_{\mathbb{R}}(g_r) \;=\; \frac{\|\cdot g_r\|^2 \|g_r'\|^2}{\|g_r\|^4} \;=\; \frac{35r^3 - 45r^2 + 35r - 9}{20(r-1)^2}.$$

Wavelets $\psi_j = (2N_j)^{-1/2}(g_{N_j/M_j})^{sam}_{M_j}$ with the same parameters as in the previous example were studied in [16], where a uniform bound $U^2_{2\pi}(\psi_j) < 2r + 2.78$ was obtained for the uncertainty product for all j. With the help of Theorem 6 we find the exact limit $\lim_{j\to\infty} U_{2\pi}(\psi_j) = U_{\mathbb{R}}(g_r)$. For the smallest possible r this yields $\lim_{j\to\infty} U_{2\pi}(\psi_j) = U_{\mathbb{R}}(g_3) = \sqrt{159/20} \approx 2.82$. This value is significantly bigger than the optimum $1/2$, since the variance increases as the mass is divided into two parts moving away from each other. Hence the uncertainty product for band-pass functions cannot reach the limit $1/2$. Still by choosing an absolutely continuous generating function and by increasing the band width proportional to its location, we can guarantee the uniform boundedness from above for the periodic uncertainty products.

Now, let us look at smoother compactly supported generating functions.

Definition 4. (Centralized m^{th} order cardinal B-spline)
Let

$$B_1(x) := \begin{cases} 1 & \text{for } x \in (-1/2, 1/2), \\ 1/2 & \text{for } x = \pm 1/2, \\ 0 & \text{otherwise,} \end{cases}$$

and for $m \in \mathbb{N}$, $m \geq 2$, recursively

$$B_m(x) := (B_{m-1} * \chi_{[-\frac{1}{2},\frac{1}{2}]})(x) = \int_{-\frac{1}{2}}^{\frac{1}{2}} B_{m-1}(x-u)du.$$

The Fourier transform of B_m is given by

$$\hat{B}_m(\xi) = \left(\frac{\sin(\xi/2)}{\xi/2}\right)^m.$$

Since the function B_m has compact support $[-\frac{m}{2}, \frac{m}{2}]$, is $m-2$ times continuously differentiable and symmetric with respect to zero, the conditions of Corollary 4 are satisfied for $m \geq 2$ and we obtain

Example 4. **B-spline sampling:** For the centralized B-spline of order $m \geq 2$ we have

$$\begin{aligned}(B_m)^{sam}_n(t) &= \sum_{k=-[nm/2]}^{[nm/2]} B_m\left(\frac{k}{n}\right) e^{ikt} \\ &= B_m(0) + 2\sum_{k=1}^{[nm/2]} B_m\left(\frac{k}{n}\right) \cos kt.\end{aligned}$$

The limit behaviour is then

$$\lim_{n\to\infty} U_{2\pi}((B_m)^{sam}_n) = U_{\mathbb{R}}(B_m).$$

Note that for the case $m = 2$ sampling of the piecewise linear hat function yields the Féjer kernel

$$(B_2)_n^{sam}(t) = \sum_{k=-n}^{n} B_2\left(\frac{k}{n}\right) e^{ikt} = 1 + 2\sum_{k=1}^{n-1} \frac{n-k}{n} \cos kt = F_{n-1}(t),$$

for which

$$\lim_{n\to\infty} U_{2\pi}(F_n) = U_{\mathbb{R}}(B_2) = \sqrt{\frac{3}{10}}$$

was already computed explicitly in [13], and which also can be obtained from (7.1) for $r = 1$.

The trigonometric kernels obtained by sampling cardinal B-splines in the integers date back to Schoenberg [14]. Polya frequency functions (see [15] and the references therein) as generalizations of B-splines could also serve as generating functions in this context, although this idea is not pursued any further in this paper. More recently, B-spline kernels have been used in preconditioning Toeplitz matrices, see [3] and [11], and in scattered data interpolation on the sphere [10].

Finally we investigate how to find a family of asymptotically optimal trigonometric polynomials. Both the B-splines and their Fourier transforms converge to Gaussian functions as the order m tends to infinity (see e.g. [19]), so that

$$\lim_{m\to\infty} U_{\mathbb{R}}(B_m) = \frac{1}{2}.$$

Thus we can formulate

Corollary 6.

For the centralized B-splines from Definition 4 we have

$$\lim_{m\to\infty}\lim_{n\to\infty} U_{2\pi}((B_m)_n^{sam}) = \lim_{m\to\infty} U_{\mathbb{R}}(B_m) = \frac{1}{2}.$$

Thus, B-spline sampling can be used to define sequences $(p_\ell)_{\ell\in\mathbb{N}}$ of asymptotically optimal trigonometric polynomials for the uncertainty principle of Theorem 2, i.e., for which $U_{2\pi}(p_\ell) \to 1/2$ as $\ell \to \infty$.

Such a sequence $\left((B_{m_\ell})_{n_\ell}^{sam}\right)_\ell$ can be generated in the following manner. For a given $\ell \in \mathbb{N}$, first choose a number m_ℓ such that $U_{\mathbb{R}}(B_{m_\ell}) - 1/2 < 1/(2\ell)$, and then for this m_ℓ select an index n_ℓ for which $|U_{2\pi}((B_{m_\ell})_{n_\ell}^{sam}) - U_{\mathbb{R}}(B_{m_\ell})| < 1/(2\ell)$.

We want to illustrate Corollary 6 by providing some explicitly computed numbers $U^2_{2\pi}((B_m)^{sam}_n)$ for different values of m and n in the following table.

	$n=4$	$n=16$	$n=64$	$n=256$	$n\to\infty$
$m=1$	0.293333	1.309552	5.311829	21.312336	∞
$m=2$	0.324545	0.301469	0.300092	0.300006	0.3
$m=3$	0.268207	0.254696	0.253891	0.253840	0.253837
$m=4$	0.262740	0.251930	0.251275	0.251234	0.251232
$m=10$	0.254816	0.250449	0.250180	0.250163	0.250162
$m=20$	0.252374	0.250184	0.250048	0.250040	0.250039

These numerical results raise the issue of making the approach truly constructive in the sense of computing specific values m_ℓ and n_ℓ for a given ℓ. This leads to the question of finding convergence rates for the limit processes we have studied which are dependent on smoothness properties of the generating functions. Such quantitative aspects are the subject of currently ongoing research by the authors.

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