# Polynomial Wavelets and Wavelet Packet Bases

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**Abstract**

We discuss wavelet-oriented ideas to construct bases of algebraic polynomials. In particular, the splitting in the frequency domain is extended in order to define wavelet packets.

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## 1 Introduction

We show here how algebraic polynomials on the interval $[-1,1]$ can be treated as wavelets and can be handled by wavelet techniques. Benefits include the potential for computational efficiency and accuracy in applications, for example to approximation problems. Theoretical developments also follow from systematic development and exploitation of orthogonality and from a generalization of the concept of shift invariance, which allow the application of the wavelet techniques on the interval. We will give here some of the basic ideas and techniques used in the wavelet approach to polynomials, which is also related to an important application, the construction of a series of mutually orthogonal polynomials of "optimal degree."

As the idea of wavelets originated in connection with signal analysis, let us look first at the original setting. Signal analysis naturally involves a "time domain" and a "frequency domain". One splits the frequency domain dyadically into *wavelet spaces*, with dilations and translations of a single function (mother wavelet) employed systematically to construct bases for these spaces. *Wavelet packet spaces* are subspaces which in turn further split the wavelet spaces, using smaller frequency ranges. In signal analysis, a function (signal) is "time-localized" if it is relatively large in magnitude at a certain "time" and relatively small otherwise. A "frequency-localized" function on the other hand is more or less of a single frequency. In a manifestation of Heisenberg's uncertainty principle, perfect time localization and perfect frequency localization are mutually incompatible. Thus, one goal in signal analysis is the construction of "time-frequency-localized" bases, involving a balanced consideration of both domains. In a wavelet treatment of polynomials on $[-1,1]$, the time domain clearly should correspond to the underlying interval $[-1,1]$, while the frequency domain should correspond more or less to the degree of involved monomials. More precise statements and adaptations of this and of other concepts require more systematic treatment.

Wavelet techniques for polynomials on the interval $[-1,1]$ with respect to the Chebyshev weight have been developed in Kilgore and Prestin [4], in Tasche [12] and in Plonka, Selig, and Tasche [7], where the generalized Chebyshev shift was discussed and applied to the development of wavelets on the interval. An adaptation of the uncertainty principle can be found in Rösler and Voit [10], which in turn could be applied to wavelets on $[-1,1]$ analogous to Narcowich and Ward [6] and to Selig [11].

More recently, in Kilgore, Prestin, and Selig [5], wavelet techniques have been used to show the existence of and to perform the construction of an orthogonal Schauder basis of polynomials of optimal degree for the space $C[-1,1]$, where optimal degree signifies that the $n$th polynomial in the basis is always of degree less than $n(1+\epsilon)$, for previously given $\epsilon > 0$. Here, the use of wavelet packets is precisely what is needed to construct a polynomial basis in which the degree of the polynomials grows within the prescribed limitations; as $\epsilon$ decreases, the dimension of the packet spaces decreases, and the number of packet spaces into which a given wavelet space must be split increases. This basis problem has a long history which is discussed in further detail in the paper [5].

Here, we will construct different wavelet bases and wavelet packet bases on the interval. At first, we will define polynomial subspaces by means of bases with the most frequency localization. Then, the idea of time-frequency-localized bases will be realized by building finite linear combinations in order to obtain wavelets and wavelet packets as generalized translates within each subspace.

The wavelet spaces as well as the wavelet packet spaces will be *orthogonal*; their orthogonality is given with respect to the weighted inner product

$$\langle f, g\rangle = \frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}}\ .$$

Hence, we use the orthogonal Chebyshev polynomials $T_n(x) = \cos n \arccos x$ $(n \in \mathbb{N}_0)$ for which

$$\langle T_n, T_m\rangle = \begin{cases} 2 & \text{for } n = m = 0, \\ 1 & \text{for } n = m > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

We will directly and explicitly describe the algebraic polynomials used in our wavelet and wavelet packet bases by giving their Chebyshev expansions. Our examples are related to the trigonometric Dirichlet kernel and the de la Vallée Poussin kernels and corresponding shift-invariant spaces (see e.g. Privalov [9], and Prestin and Selig [8]).

Note that in our construction the Chebyshev polynomials can be replaced by other polynomial systems orthonormal with respect to an arbitrary weight function $w$ which yield corresponding bases for $L_w^2$. Having similar frequency localization in terms of the involved orthonormal polynomials, the resulting polynomials will differ in their properties of localization on $[-1,1]$ according to the weight $w$ which will be related to different approximation properties of the bases. For the construction of the wavelets orthogonal with respect to an arbitrary weighted inner product we refer to Depczynski and Jetter [1, 2] and Fischer and Prestin [3]. However, results for the wavelet packets and the uncertainty principles are still in progress. Therefore, we restrict ourselves here to the Chebyshev weight and Chebyshev polynomials.

## 2 Wavelets and Wavelet Packets on $[-1, 1]$

Let $N, M \in \mathbb{N}$ be fixed, with $N = 2^{\eta+1}M$ for some $\eta \in \mathbb{N}$, $\eta \geq 2$. Furthermore, let us introduce, for any $\ell = 0, \ldots, 2^\eta - 2$, real coefficients

$$a_M^\ell(k), \qquad (k = -M, \ldots, M) \qquad \text{and} \qquad a_{2M}^0(k), \qquad (k = -2M, \ldots, 2M).$$

With any fixed set of such coefficients, we define the following spaces of polynomials

$$\begin{aligned} V_N^M \;:=\; & \operatorname{span}\,(\{T_k :\; k = 0, \ldots, N-M\} \cup \\ & \cup \{a_M^0(k-M)T_{N-M+k} + a_M^0(M-k)T_{N+M-k} :\; k = 1, \ldots, M\}), \end{aligned}$$

$$\begin{aligned} W_N^M \;:=\; & \operatorname{span}\,(\{a_M^0(-k)T_{N+k} - a_M^0(k)T_{N-k} :\; k = 1, \ldots, M-1\} \cup \\ & \cup \{T_k :\; k = N+M, \ldots, 2N-2M\} \cup \\ & \cup \{a_{2M}^0(k-2M)T_{2N-2M+k} + a_{2M}^0(2M-k)T_{2N+2M-k} :\; k = 1, \ldots, 2M\}), \end{aligned}$$

$$\begin{aligned} W_{N,\ell}^M \;:=\; & \operatorname{span}\,(\{a_M^{\ell-1}(-k)T_{N+2M(\ell-1)+k} - a_M^{\ell-1}(k)T_{N+2M(\ell-1)-k} :\; k = 1, \ldots, M-1\} \cup \\ & \cup \{a_M^\ell(k-M)T_{N+M(2\ell-1)+k} + a_M^\ell(M-k)T_{N+M(2\ell+1)-k} :\; k = 0, \ldots, M\}), \end{aligned}$$

for $\ell = 1, \ldots, 2^\eta - 2$, and

$$\begin{aligned} W_{N,2^\eta-1}^M \;:=\; & \operatorname{span}\,(\{a_M^{2^\eta-2}(-k)T_{2N-4M+k} - a_M^{2^\eta-2}(k)T_{2N-4M-k} :\; k = 1, \ldots, M-1\} \cup \\ & \cup \{T_k :\; k = 2N-3M, \ldots, 2N-2M\} \cup \\ & \cup \{a_{2M}^0(k-2M)T_{2N-2M+k} + a_{2M}^0(2M-k)T_{2N+2M-k} :\; k = 1, \ldots, 2M\}). \end{aligned}$$

Given a general scheme for constructing the coefficients $a_M^\ell$, it is then possible to double repeatedly the values of $M$ and $N$ together. This successive doubling gives a nested sequence of spaces $V_N^M$, a corresponding sequence of spaces $W_N^M$, and inside of each space $W_N^M$ a set of subspaces $W_{N,1}^M, \ldots, W_{N,2^\eta-1}^M$.

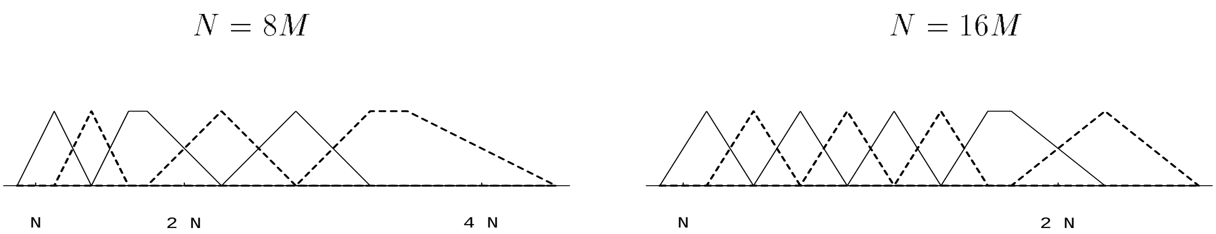
Three relevant examples for the choice of the coefficients are the following, where for all $\ell = 1, \ldots, 2^\eta - 2$

$$\begin{array}{lll}
\text{(a)} & a_M^\ell(k) = \begin{cases} 1\,, & -M \le k \le 0\,, \\ 0\,, & 0 < k \le M\,, \end{cases} & \\
\text{(b)} & a_M^\ell(k) = \frac{M-k}{2M}\,, & -M \le k \le M\,, \\
\text{(c)} & a_M^\ell(k) = \frac{M-k}{\sqrt{2M^2+2k^2}}\,, & -M \le k \le M\,,
\end{array}$$

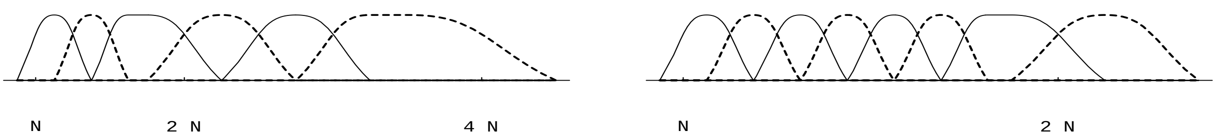
and $a_{2M}^0(k) = a_M^0(k/2)$ for all $-2M \le k \le 2M$. Arisen from their trigonometric analogs, example (a) yields functions related to the Dirichlet kernel whereas examples (b) and (c) come from de la Vallée Poussin means and from an orthogonalization procedure applied to translates thereof, respectively.

For the sake of good time localization for the wavelet and wavelet packet bases to be constructed we generally suggest that the coefficients $a_M^\ell(k)$ should decrease monotonically with increasing $k$ and should be normalized such that $a_M^\ell(-M) = 1$.

Based on the examples (b) and (c), the following graphs represent the size of the coefficients $a_M^\ell$ with respect to their distribution in the frequency domain (see also the definition of the wavelet packet functions on page 6) and thus illustrate some of the many possibilities for constructing the spaces $W_{N,\ell}^M$, for $\ell = 1, \ldots, 2^\eta - 1$ and $\eta = 2$ (left) and $\eta = 3$ (right). The graphs for $\eta = 2$ also depict one doubling of $N$ and $M$.




Example (b)




Example (c)

Now we study the spaces defined above and show that under certain conditions on the coefficients they span

$$\begin{aligned} V_{2N}^{2M} \;:=\; & \text{span}\,(\{T_k :\; k = 0, \ldots, 2N-2M\} \cup \\ & \cup \{a_{2M}^0(k-2M)T_{2N-2M+k} + a_{2M}^0(2M-k)T_{2N+2M-k} :\; k = 1, \ldots, 2M\})\,. \end{aligned}$$

**Theorem 2.1** *For any real coefficients $a^\ell_M(k)$ and $a^0_{2M}(k)$, it holds that*

$$V^M_N \cup W^M_N \subset V^{2M}_{2N}, \qquad V^M_N \perp W^M_N,$$

*and*

$$W^M_{N,\ell} \subset W^M_N, \qquad \textit{for } \ell = 1, ..., 2^\eta - 1.$$

*If*

$$a^\ell_M(M) = 0, \qquad \textit{for } \ell = 1, ..., 2^\eta - 2, \tag{2}$$

*then*

$$W^M_{N,\ell_1} \perp W^M_{N,\ell_2}, \qquad \textit{for } 1 \le \ell_1 < \ell_2 \le 2^\eta - 1,$$

*and if moreover the coefficients satisfy for all $\ell = 0, \dots, 2^\eta - 2$*

$$\begin{aligned} (a^\ell_M(k))^2 + (a^\ell_M(-k))^2 &> 0, \qquad \textit{for all } \; k = 0, \dots, M, \\ (a^0_{2M}(k))^2 + (a^0_{2M}(-k))^2 &> 0, \qquad \textit{for all } \; k = 0, \dots, 2M-1, \end{aligned} \tag{3}$$

*then we have*

$$V^{2M}_{2N} = V^M_N \oplus W^M_N \tag{4}$$

*and*

$$W^M_N = \bigoplus_{\ell=1}^{2^\eta-1} W^M_{N,\ell}. \tag{5}$$

Proof. The inclusions $V^M_N \cup W^M_N \subset V^{2M}_{2N}$ and $W^M_{N,\ell} \subset W^M_N$ follow directly from the definition of the spaces.

Using (1) and (2) the orthogonality $V^M_N \perp W^M_N$ and $W^M_{N,\ell_1} \perp W^M_{N,\ell_2}$ can be easily checked. In particular, for $\ell = \ell_1 = \ell_2 - 1$ we obtain for any $k = 0, \dots, M-1$

$$\begin{aligned} &\langle a^\ell_M(-k) T_{N+2M\ell-k} + a^\ell_M(k) T_{N+2M\ell+k}\,,\; a^\ell_M(k) T_{N+2M\ell-k} - a^\ell_M(-k) T_{N+2M\ell+k} \rangle \\ &\quad = \; a^\ell_M(-k)\, a^\ell_M(k) \left( \langle T_{N+2M\ell-k}, T_{N+2M\ell-k} \rangle - \langle T_{N+2M\ell+k}, T_{N+2M\ell+k} \rangle \right) \\ &\quad = \; 0. \end{aligned}$$

For $|\ell_1 - \ell_2| > 1$ the orthogonality $W^M_{N,\ell_1} \perp W^M_{N,\ell_2}$ is evident.

Let us now prove (4) and (5). From (3) it follows that not both $a^\ell_M(-k)$ and $a^\ell_M(k)$ can vanish. Hence,

$$\begin{aligned} \dim \operatorname{span} \{ a^\ell_M(-k) T_{N+2M\ell+k} - a^\ell_M(k) T_{N+2M\ell-k} : \\ k = 1, \dots, M-1 \} &= M - 1, \\ \dim \operatorname{span} \{ a^\ell_M(k-M) T_{N+M(2\ell-1)+k} + a^\ell_M(M-k) T_{N+M(2\ell+1)-k} : \\ k = 0, \dots, M \} &= M + 1, \\ \dim \operatorname{span} \{ a^0_{2M}(k-2M) T_{2N-2M+k} + a^0_{2M}(2M-k) T_{2N+2M-k} : \\ k = 1, \dots, 2M \} &= 2M. \end{aligned}$$

Then, for the dimensions of the spaces we obtain

$$\dim V_N^M \;=\; N+1\,, \qquad \dim W_N^M \;=\; N\,, \qquad \dim W_{N,\ell}^M \;=\; 2M\,,$$

for $\ell = 1, ..., 2^\eta - 2$, and

$$\dim W_{N,2^\eta-1}^M \;=\; 4M\,.$$

Hence

$$\dim V_{2N}^{2M} \;=\; \dim V_N^M + \dim W_N^M$$

and

$$\dim W_N^M \;=\; \sum_{\ell=1}^{2^\eta-1} \dim W_{N,\ell}^M\,.$$

Together with the imbedding and orthogonality relations this proves the assertion. □

Following [7] one can define scaling functions and wavelets in terms of Chebyshev polynomials as generalized Chebyshev shifts of one function.

We define scaling functions for $s = 0, \ldots, N$, by

$$\phi_{N,s}^M \;:=\; \frac{1}{2}\,T_0 + \sum_{k=1}^{N-M} \cos\tfrac{ks\pi}{N}\;T_k + \sum_{k=N-M+1}^{N+M-1} a_M^0(k-N)\;\cos\tfrac{ks\pi}{N}\;T_k\,,$$

and wavelets, for $s = 1, \ldots, N$, by

$$\begin{aligned}\psi_{N,s}^M \;:=\;& \sum_{k=N-M+1}^{N+M-1} a_M^0(N-k)\;\cos\tfrac{k(2s-1)\pi}{2N}\;T_k + \sum_{k=N+M}^{2N-2M} \cos\tfrac{k(2s-1)\pi}{2N}\;T_k + \\ &+ \sum_{k=2N-2M+1}^{2N+2M-1} a_{2M}^0(k-2N)\;\cos\tfrac{k(2s-1)\pi}{2N}\;T_k\,.\end{aligned}$$

In this paper we introduce corresponding wavelet packet functions, for $p = 1, ..., 2^{\eta-1}-1$ and $s = 1, \ldots, 2M$, by

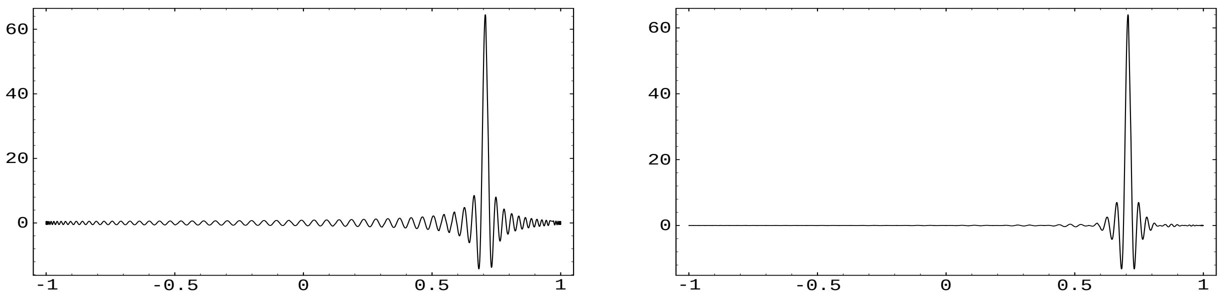
$$\begin{aligned}\psi_{N,2p-1,s}^M \;:=\;& \sum_{k=N+(4p-5)M+1}^{N+(4p-3)M-1} a_M^{2p-2}(N+(4p-4)M-k)\;\sin\tfrac{k(2s-1)\pi}{4M}\;T_k + \\ &+ \sum_{k=N+(4p-3)M}^{N+(4p-1)M-1} a_M^{2p-1}(k-N-(4p-2)M)\;\sin\tfrac{k(2s-1)\pi}{4M}\;T_k\,, \\ \psi_{N,2p,s}^M \;:=\;& \sum_{k=N+(4p-3)M+1}^{N+(4p-1)M-1} a_M^{2p-1}(N+(4p-2)M-k)\;\cos\tfrac{k(2s-1)\pi}{4M}\;T_k + \\ &+ \sum_{k=N+(4p-1)M}^{N+(4p+1)M-1} a_M^{2p}(k-N-4pM)\;\cos\tfrac{k(2s-1)\pi}{4M}\;T_k\,,\end{aligned}$$

and for $s = 1, \ldots, 4M$, by

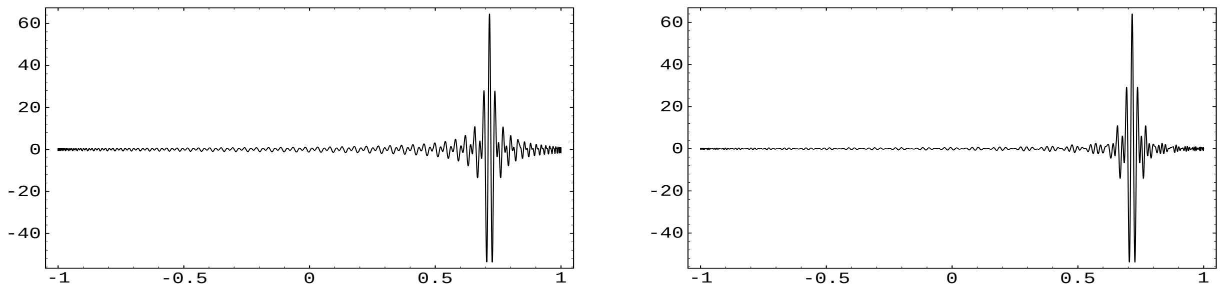
$$\begin{aligned}\psi^M_{N,2^\eta-1,s} \quad := \quad & \sum_{k=2N-5M+1}^{2N-3M-1} a_M^{2^\eta-2}(k-2N-4M)\ \cos\tfrac{k(2s-1)\pi}{8M}\ T_k + \\ & + \sum_{k=2N-3M}^{2N-2M} \cos\tfrac{k(2s-1)\pi}{8M}\ T_k + \\ & + \sum_{k=2N-2M+1}^{2N+2M-1} a^0_{2M}(k-2N)\ \cos\tfrac{k(2s-1)\pi}{8M}\ T_{2N-k}\ .\end{aligned}$$

In order to illustrate their time localization, we have drawn corresponding functions for the coefficients from the examples (a) and (b) for $N = 128$ and $M = 16$. The corresponding functions for the example (c) are not shown; they would be quite similar to those for (b).

Example (a) Example (b)

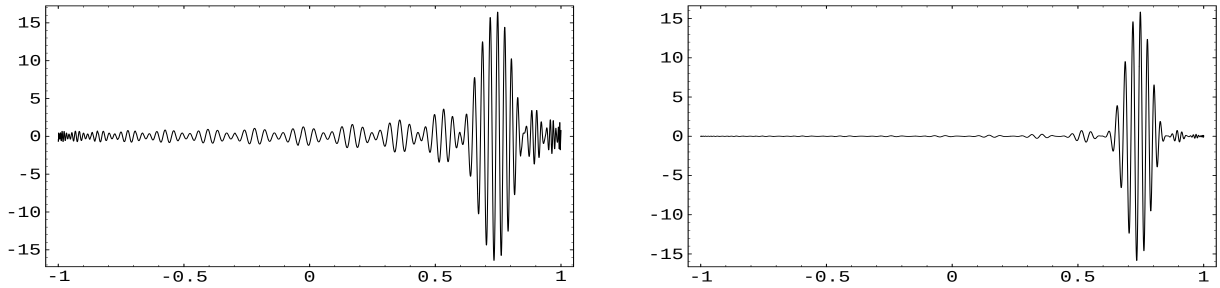



Scaling functions $\phi^{16}_{128,32}$




Wavelets $\psi^{16}_{128,32}$

Example (a) Example (b)




Wavelet packet functions $\psi_{128,1,8}^{16}$

We can show that the functions defined above build bases of the previously defined spaces.

**Theorem 2.2** *For $N, M \in I\!N$, with $N = 2^{\eta+1}M$, we have*

$$\begin{aligned} V_N^M &= \text{span}\,\{\phi_{N,s}^M :\; s = 0, \dots, N\}, \\ W_N^M &= \text{span}\,\{\psi_{N,s}^M :\; s = 1, \dots, N\}, \\ W_{N,\ell}^M &= \text{span}\,\{\psi_{N,\ell,s}^M :\; s = 1, \dots, 2M\}, \end{aligned}$$

*for $\ell = 1, \dots, 2^\eta - 2$, and*

$$W_{N,2^\eta-1}^M = \text{span}\,\{\psi_{N,2^\eta-1,s}^M :\; s = 1, \dots, 4M\}.$$

Proof. Let us write the basis used in the definition of $V_N^M$ in the order given there as a column vector $\underline{\boldsymbol{v}}_N^M$, that is,

$$\underline{\boldsymbol{v}}_N^M = \Big(T_0\,,\; \dots\,,\; T_{N-M},\; a_M^0(1-M)T_{N-M+1} + a_M^0(M-1)T_{N+M-1}\,, \\ \dots\,,\; a_M^0(-1)T_{N-1} + a_M^0(1)T_{N+1}\,,\; 2a_M^0(0)T_N\Big)^T \quad .$$

Let us similarly represent the given basis of $W_N^M$ as a column vector $\underline{\boldsymbol{w}}_N^M$ and the given basis of $W_{N,\ell}^M$ as a column vector $\underline{\boldsymbol{w}}_{N,\ell}^M$.

Now we can write

$$\left(\phi_{N,s}^M\right)_{s=0}^N = \boldsymbol{A}_N\,\underline{\boldsymbol{v}}_N^M\,, \qquad \left(\psi_{N,s}^M\right)_{s=1}^N = \boldsymbol{B}_N\,\underline{\boldsymbol{w}}_N^M$$

where

$$\boldsymbol{A}_N = \left(\frac{2-\delta_{k,0}-\delta_{k,N}}{2}\cos\frac{ks\pi}{N}\right)_{s,k=0}^N\,, \qquad \boldsymbol{B}_N = \left(\frac{2-\delta_{k,2N}}{2}\cos\frac{k(2s-1)\pi}{2N}\right)_{s=1,k=N+1}^{N,2N}\,.$$

Also, for $\ell = 1, \dots, 2^\eta - 2$, we have

$$\left(\psi_{N,\ell,s}^M\right)_{s=1}^{2M} = \boldsymbol{C}_{N,\ell}^M\,\underline{\boldsymbol{w}}_{N,\ell}^M\,, \qquad \left(\psi_{N,2^\eta-1,s}^M\right)_{s=1}^{4M} = \boldsymbol{C}_{N,2^\eta-1}^M\,\underline{\boldsymbol{w}}_{N,2^\eta-1}^M\,,$$

where, for $p = 1, \dots, 2^{\eta-1} - 1$,

$$\boldsymbol{C}^M_{N,2p} = \left( \frac{2 - \delta_{k,N+4pM}}{2} \cos \frac{k(2s-1)\pi}{4M} \right)^{2M,N+4pM}_{s=1,k=N+(4p-2)M+1} ,$$

$$\boldsymbol{C}^M_{N,2p-1} = \left( \frac{2 - \delta_{k,N+(4p-2)M}}{2} \sin \frac{k(2s-1)\pi}{4M} \right)^{2M,N+(4p-2)M}_{s=1,k=N+(4p-4)M+1} ,$$

and

$$\boldsymbol{C}^M_{N,2^\eta-1} = \left( \frac{2 - \delta_{k,2N}}{2} \cos \frac{k(2s-1)\pi}{8M} \right)^{4M,2N}_{s=1,k=2N-4M+1} .$$

The proof of the theorem is now completed by noting that the regularity of these matrices is well-known and follows directly from (see Tasche [12])

$$\boldsymbol{A}_N \boldsymbol{A}_N^T = \left( \frac{N}{2} \delta_{s,k} \right)^N_{s,k=0} ,$$

$$\boldsymbol{B}_N^T \boldsymbol{B}_N = \left( \frac{N}{2} \delta_{s,k} (2 - \delta_{k,N}) \right)^N_{s,k=1}$$

and

$$(\boldsymbol{C}^M_{N,2p})^T \boldsymbol{C}^M_{N,2p} = (\boldsymbol{C}^M_{N,2p-1})^T \boldsymbol{C}^M_{N,2p-1} = (M \delta_{s,k} (2 - \delta_{k,2M}))^{2M}_{s,k=1} .$$

□

Note that in the above proof the transformation matrices between corresponding bases of the scaling function spaces, wavelet spaces, and wavelet packet spaces are given. The transformation from one basis to another can also be carried out by use of fast algorithms (cf. [12, 7]).

# 3 Orthogonal bases

Here we further impose orthogonality of the bases given in Theorem 2.2 . It turns out to be guaranteed by a certain condition on the coefficients $a^\ell_M$.

**Theorem 3.1** *If*

$$a^\ell_M(M) = 0 , \qquad \text{for } \ell = 1, ..., 2^\eta - 2 , \tag{6}$$

*and*

$$\begin{aligned} (a^\ell_M(k))^2 + (a^\ell_M(-k))^2 &= 1 \qquad \text{for } k = 0, \dots, M , \quad \ell = 0, \dots, 2^\eta - 2 , \\ (a^0_{2M}(k))^2 + (a^0_{2M}(-k))^2 &= 1 \qquad \text{for } k = 0, \dots, 2M-1 , \end{aligned} \tag{7}$$

*then we have the orthogonality properties*

$$\langle \phi_{N,r}^M \ , \ \phi_{N,s}^M \rangle = N\delta_{r,s} \frac{1+\delta_{s,0}+\delta_{s,N}}{2} \, , \qquad \textit{for } r,s = 0, \ldots, N \, , \tag{8}$$

$$\langle \psi_{N,r}^M \ , \ \psi_{N,s}^M \rangle = N\delta_{r,s} \, , \qquad \textit{for } r,s = 1, \ldots, N \, , \tag{9}$$

$$\langle \psi_{N,\ell,r}^M \ , \ \psi_{N,\ell,s}^M \rangle = M\delta_{r,s} \qquad \textit{for all } \ell = 1, \ldots, 2^\eta - 1 \textit{ and } r,s = 1, \ldots, 2M \, , \tag{10}$$

$$\langle \psi_{N,2^\eta-1,r}^M \ , \ \psi_{N,2^\eta-1,s}^M \rangle = 2M\delta_{r,s} \qquad \textit{for } r,s = 1, \ldots, 4M \, . \tag{11}$$

Notice the connection between the conditions (3) giving linear independence and (7) giving orthogonality.

Proof. For the proof, we will use the orthogonality properties (1) of the Chebyshev polynomials $T_k$. In order to show (8), we note that

$$\begin{aligned}
\langle \phi_{N,r}^M \ , \ \phi_{N,s}^M \rangle &= \frac{1}{2} + \sum_{k=1}^{N-M} \cos\frac{kr\pi}{N} \cos\frac{ks\pi}{N} + \\
&\quad + \sum_{k=-M+1}^{M-1} \frac{(2M-k)^2}{2M^2+2(M-k)^2} \cos\frac{(N-k)r\pi}{N} \cos\frac{(N-k)s\pi}{N} \\
&= \frac{1+(-1)^{r-s}}{2} + \sum_{k=1}^{N-1} \cos\frac{kr\pi}{N} \cos\frac{ks\pi}{N} \\
&= \frac{2+(-1)^{r-s}+(-1)^{r+s}}{4} + \frac{1}{2}\sum_{k=1}^{N-1} \left( \cos\frac{k(r-s)\pi}{N} + \cos\frac{k(r+s)\pi}{N} \right) \\
&= N\delta_{r,s} \frac{1+\delta_{s,0}+\delta_{s,N}}{2} \, ,
\end{aligned}$$

where we used that

$$\frac{1}{2} + \frac{(-1)^r}{2} + \sum_{k=1}^{N-1} \cos\frac{kr\pi}{N} \;=\; N\delta_{r,0\mathrm{mod}2N} \, .$$

The proof of (9) - (11) follows the same ideas. □

The conditions (6) - (7) hold for our example (c). For this special case, the functions are

$$\phi_{N,s}^M \;=\; \frac{1}{2}\, T_0 + \sum_{k=1}^{N-M} \cos\tfrac{ks\pi}{N}\, T_k + \sum_{k=1}^{2M-1} \tfrac{2M-k}{\sqrt{2M^2+2(M-k)^2}} \cos\tfrac{(N-M+k)s\pi}{N}\, T_{N-M+k} \, ,$$

$$\psi^M_{N,s} = \sum_{k=1}^{2M-1} \frac{k}{\sqrt{2M^2+2(M-k)^2}} \cos\frac{(N-M+k)(2s-1)\pi}{2N}\, T_{N-M+k} + \sum_{k=N+M}^{2N-2M} \cos\frac{k(2s-1)\pi}{2N}\, T_k + \\ + \sum_{k=1}^{4M-1} \frac{4M-k}{\sqrt{8M^2+2(k-2M)^2}} \cos\frac{(2N-2M+k)(2s-1)\pi}{2N}\, T_{2N-2M+k}\,,$$

$$\psi^M_{N,2p,s} = \sum_{k=-2M+1}^{2M-1} \frac{2M-|k|}{\sqrt{2M^2+2(M-|k|)^2}} \cos\left((k-M)\frac{(2s-1)\pi}{4M}\right)\, T_{N+(4p-1)M+k}\,,$$

$$\psi^M_{N,2p-1,s} = \sum_{k=-2M+1}^{2M-1} \frac{2M-|k|}{\sqrt{2M^2+2(M-|k|)^2}} \sin\left((k-3M)\frac{(2s-1)\pi}{4M}\right)\, T_{N+(4p-3)M+k}\,,$$

and

$$\psi^M_{N,2^\eta-1,s} = \sum_{k=1}^{2M-1} \frac{k}{\sqrt{2M^2+2(M-k)^2}} \cos\frac{k(2s-1)\pi}{8M}\, T_k + \\ + \sum_{k=2N-3M}^{2N-2M} \cos\frac{k(2s-1)\pi}{8M}\, T_k + \\ + \sum_{k=1}^{4M-1} \frac{4M-k}{\sqrt{8M^2+2(k-2M)^2}} \cos\frac{(2N-2M+k)(2s-1)\pi}{8M}\, T_{2N-2M+k}\,.$$

With appropriate choices of $N$ and $M$ given by successive doubling of certain initial $N$ and $M$, the pairwise orthogonal wavelet packet functions just described can be used to define an orthogonal Schauder basis for $C[-1,1]$ consisting of polynomials of optimal degree at most $n(1+\epsilon)$. For this construction, the initial values of $N$ and $M$ are determined by the given value of $\epsilon$. We have mentioned this problem already in the introduction; the details are given in Kilgore, Prestin, and Selig [5].

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