On local smoothness classes of periodic functions

H. N. Mhaskar*
Department of Mathematics, California State University
Los Angeles, California, 90032, USA
email: hmhaska@calstatela.edu

J. Prestin
Institute of Mathematics, University of Lübeck
Wallstraße 40, 23560, Lübeck, Germany
email: prestin@math.uni-luebeck.de

Abstract

We obtain a characterization of local Besov spaces of periodic functions in terms of trigonometric polynomial operators. Several numerical examples are discussed, including applications to the problem of direction finding in phased array antennas and finding the location of jump discontinuities of derivatives of different order.

1 Introduction

If $f : \mathbb{R} \to \mathbb{R}$ is a 2π-periodic, (Lebesgue) measurable function, integrable on $[0, 2\pi]$, the Fourier coefficients of f are given by

$$\hat{f}(k) := \frac{1}{2\pi}\int_0^{2\pi} f(t)\exp(-ikt)dt, \qquad k = 0, \pm 1, \pm 2, \ldots. \tag{1.1}$$

It is well known that

$$\lim_{n\to\infty}\int_0^{2\pi}\left|f(x) - \sum_{|k|\le n}\left(1 - \frac{|k|}{n}\right)\hat{f}(k)\exp(ikx)\right|dx = 0,$$

and hence, the sequence of Fourier coefficients determines the function uniquely. Nevertheless, since the definition of the Fourier coefficients requires the knowledge of the

*The research of this author was supported, in part, by grant DMS-0204704 from the National Science Foundation and grant DAAD19-01-1-0001 from the U.S. Army Research Office.

function throughout the interval of periodicity, it is an interesting problem to obtain information about local behavior of the function, given the sequence of its Fourier coefficients; i.e., to construct trigonometric polynomial operators providing "time-frequency localization". There are several efforts in this direction during the last decade or so, for example, [3, 9, 16, 18, 19, 21].

One demonstration of the localization properties of such operators is their ability to detect the location of "singularities" (jump discontinuities of derivatives of different orders) of piecewise smooth functions. In its various forms, this problem has been investigated, and applied in different contexts, by many mathematicians, for example, [13, 5, 6, 7, 8, 11, 12]. In [15], we have studied a very general class of operators of the form $f \mapsto \sum g_{k,n}\hat{f}(k)\exp(ik\cdot)$. We have described a precise connection between the factors $g_{k,n}$ and the ability of the operator to detect the singularities of the function. Typically, the factors $g_{k,n}$ are of the form $g(k/n)$ for a suitable function g, and the time-frequency localization of the operators is determined by the smoothness and support of the function g.

The purpose of this paper is twofold. One objective is to report on some numerical experiments related to our paper [15], including applications to the problem of direction finding using phased array antennas. The second objective is to demonstrate further the time-frequency localizations of certain trigonometric polynomial operators by relating their growth on different subintervals of $[0, 2\pi]$ with the local Besov conditions satisfied by the function on these intervals.

Our main theoretical results are discussed in Section 2. In Section 3, we report on a number of numerical experiments to illustrate the results here (Section 3.1), as well as those in [15]. The proofs of the new results in Section 2 are given in Section 4.

2 Local smoothness spaces

If $A \subseteq \mathbb{R}$ is Lebesgue measurable, and $f : A \to \mathbb{R}$ is Lesbegue measurable, we write

$$\|f\|_{A,p} := \begin{cases} \left\{\int_A |f(t)|^p dt\right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \operatorname*{ess\,sup}_{t\in A} |f(t)|, & \text{if } p = \infty. \end{cases}$$

The class of measurable functions f for which $\|f\|_{A,p} < \infty$ is denoted by $L^p(A)$, with the standard convention that two functions are considered equal if they are equal almost everywhere on A. The symbol $X^p(A)$ denotes the space $L^p(A)$ if $1 \le p < \infty$, and the space of uniformly continuous and bounded functions on A (equipped with $\|\cdot\|_{A,\infty}$ as its norm) if $p = \infty$. In the sequel, we will simplify our notation by writing $\|f\|_p$ instead of $\|f\|_{[0,2\pi],p}$.

If $I = [a, b] \subseteq \mathbb{R}$, and $f \in L^p(I)$, we define the moduli of smoothness of f for $0 < \delta < (b-a)/r$ and integer $r \ge 1$ by

$$\omega_{I,r,p}(f,\delta) := \sup_{0<t\le\delta} \left\| \sum_{k=0}^{r} (-1)^{r-k}\binom{r}{k} f(\cdot + kt) \right\|_{[a,b-r\delta],p}. \tag{2.1}$$

If $\delta \geq (b-a)/r$, we find it convenient to define $\omega_{I,r,p}(f,\delta) := \min \|f-P\|_{I,p}$, where the minimum is taken over all *algebraic* polynomials of degree at most $r-1$. Let $1 \leq p \leq \infty$, $0 < \rho \leq \infty$, $\alpha > 0$, and r be the smallest integer $> \alpha$. For $I = [a,b] \subseteq \mathbb{R}$, and $f \in L^p(I)$, we write

$$\|f\|_{I,p,\rho,\alpha} := \begin{cases} \left\{ \sum_{n=0}^{\infty} \left(2^{n\alpha} \omega_{I,r,p}(f, 2^{-n}) \right)^{\rho} \right\}^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \sup_{n \geq 0, n \in \mathbb{Z}} 2^{n\alpha} \omega_{I,r,p}(f, 2^{-n}), & \text{if } \rho = \infty. \end{cases} \tag{2.2}$$

If $x_0 \in [0, 2\pi]$, the *local Besov space* $B_{p,\rho,\alpha}(x_0)$ consists of functions $f \in X^p$ satisfying the following property: there is a nondegenerate interval I, centered at x_0, such that $\|f\|_{I,p,\rho,\alpha} < \infty$. The interval may, of course, depend upon x_0 and f, as well as the other parameters. We note further that the modulus of smoothness, $\omega_{I,r,p}$ is the aperiodic modulus of smoothness, even though $B_{p,\rho,\alpha}(x_0)$ consists of periodic functions. For more information on Besov spaces we refer the reader to [4, 20].

Next, we discuss some of the trigonometric polynomial operators. Our starting point is to construct operators that provide a "good approximation". For $x \geq 0$, the class of all trigonometric polynomials of order at most x will be denoted by $\mathbb{H}_x$. For $f \in X^p$, and $x \geq 0$, the degree of approximation of f from $\mathbb{H}_x$ is defined by

$$E_{x,p}(f) := \min_{T \in \mathbb{H}_x} \|f - T\|_p. \tag{2.3}$$

A sequence of linear operators $U_n : X^p \to \mathbb{H}_{2^n}$, $n = 0, 1 \cdots$, will be called a *good approximation sequence (for X^p)* if $U_n(T) = T$ for each $T \in \mathbb{H}_{2^{n-1}}$, and $\sup_{n \geq 0} \|U_n(f)\|_p \leq c\|f\|_p$ for some constant $c > 0$, depending only on p and the sequence of operators. Let $\{U_n\}$ be a good approximation sequence, $n \geq 0$ be an integer, $f \in X^p$, and $T \in \mathbb{H}_{2^{n-1}}$ satisfy $\|f - T\|_p \leq 2E_{2^{n-1},p}(f)$. Then

$$\|f - U_n(f)\|_p \leq \|f - T - U_n(f - T)\|_p \leq (c+1)\|f - T\|_p \leq 2(c+1)E_{2^{n-1},p}(f).$$

Thus, we have

$$E_{2^n,p}(f) \leq \|f - U_n(f)\|_p \leq 2(c+1)E_{2^{n-1},p}(f), \quad f \in X^p, \ n = 0, 1, \cdots. \tag{2.4}$$

In order to construct good approximation sequences, we consider bi-infinite matrices of the form $H = (h_{k,2^n})_{k=0,\pm1,\pm2,\cdots,n=0,1,2,\cdots}$, where $h_{k,2^n} = 0$ if $|k| > 2^n$. Corresponding to any such matrix H, and $f \in X^1$, we define

$$\sigma_n(H, f; x) := \sum_{k \in \mathbb{Z}} h_{k,2^n} \hat{f}(k) \exp(ikx), \qquad x \in \mathbb{R}, \ n = 0, 1, 2, \cdots. \tag{2.5}$$

We find it convenient to define $\sigma_t(H, f) \equiv 0$ if $t < 0$. We note that for $f \in X^1$, $\sigma_n(H, f) \in \mathbb{H}_{2^n}$, and with

$$\Phi_n(H, x) := \sum_{k \in \mathbb{Z}} h_{k,2^n} \exp(ikx), \qquad x \in \mathbb{R}, \tag{2.6}$$

we have

$$\sigma_n(H, f; x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \Phi_n(H, x - t) dt, \qquad x \in \mathbb{R}. \tag{2.7}$$

A matrix H will be called a *dyadic summability matrix* if $h_{k,2^n} = 0$, for $|k| > 2^n$, $h_{k,2^n} = 1$, for $|k| \le 2^{n-1}$, and there exists a constant A_H such that

$$\sum_{k\in\mathbb{Z}} |k||h_{k+2,2^n} - 2h_{k+1,2^n} + h_{k,2^n}| \le A_H < \infty, \qquad n = 0, 1, 2, \cdots. \tag{2.8}$$

The following proposition summarizes some of the properties of the operators $\sigma_n(H, f)$. We adopt the following convention regarding constants. The letters $c, c_1, \cdots$ will denote positive constants that may depend upon H, p, ρ, α, and the parameter q after it is introduced below, and other indicated quantities. Their values may be different at different occurences, even within the same formula.

Proposition 2.1 *Let H be a bi-infinite matrix.*
(a) *If (2.8) is satisfied then*

$$\sup_{n\ge 0} \|\Phi_n(H)\|_1 < \infty. \tag{2.9}$$

(b) *If H is a dyadic summability matrix, then the sequence of operators $\{\sigma_n(H)\}$ is a good approximation sequence for every X^p, $1 \le p \le \infty$.*
(c) *Let $q \ge 1$ be an integer, and*

$$\sum_{k\in\mathbb{Z}} \left| \sum_{\ell=0}^{q+1} (-1)^\ell \binom{q+1}{\ell} h_{k+\ell,2^m} \right| \le c2^{-mq}, \qquad m = 1, 2, \cdots. \tag{2.10}$$

Then

$$|\Phi_n(H, x)| \le \frac{c_1}{2^{nq}|x \ mod \ 2\pi|^{q+1}}, \qquad x \ne 0 \ mod \ 2\pi, \ x \in \mathbb{R}. \tag{2.11}$$

We say that a dyadic summability matrix H is in the class $\mathcal{S}^q$ if the condition (2.10) is satisfied. A simple way to generate matrices in $\mathcal{S}^q$ is the following. We consider a function h that is a q-times iterated integral of a function of bounded variation, and that satisfies

$$h(x) = \begin{cases} 1, & \text{if } 0 \le x \le 1/2, \\ 0, & \text{if } x > 1, \\ h(-x), & \text{if } x < 0. \end{cases} \tag{2.12}$$

If $q \ge 1$ then the matrix $(h(k/2^n))$ is in $\mathcal{S}^q$.

Our objective in this section is to explain the relationship between local Besov spaces and the behavior of the operators

$$\tau_n(H, f) := \sigma_n(H, f) - \sigma_{n-1}(H, f) \tag{2.13}$$

near the point in question. This behavior will be described by the condition that certain norms of the operators belong to a sequential version of the Besov spaces, which we now define. For a sequence $\mathbf{a} = \{a_n\}_{n=0}^\infty$, and numbers $\alpha, \rho > 0$, we write

$$\|\mathbf{a}\|_{\rho,\alpha} := \begin{cases} \left\{ \displaystyle\sum_{n=0}^{\infty} (2^{n\alpha}|a_n|)^\rho \right\}^{1/\rho}, & \text{if } 0 < \rho < \infty, \\ \displaystyle\sup_{n\ge 0, n\in\mathbb{Z}} 2^{n\alpha}|a_n|, & \text{if } \rho = \infty. \end{cases} \tag{2.14}$$

The sequence $\mathbf{a}$ is said to be in the space $\mathsf{b}_{\rho,\alpha}$ if $\|\mathbf{a}\|_{\rho,\alpha} < \infty$. Thus, $f \in B_{p,\rho,\alpha}(x_0)$ if and only if the sequence $\{\omega_{I,r,p}(f, 2^{-n})\} \in \mathsf{b}_{\rho,\alpha}$ for some $r > \alpha$ and interval I centered at x_0.

Our first main theorem is the following.

Theorem 2.1 *Let $1 \le p \le \infty$, $f \in X^p$, $x_0 \in [0, 2\pi]$, $\alpha > 0$, $q > \max(1, \alpha)$ be an integer, $0 < \rho \le \infty$, and $H \in \mathcal{S}^q$. The following statements are equivalent.*
(a) $f \in B_{p,\rho,\alpha}(x_0)$.
(b) *There exists an interval I, centered at x_0, such that the sequence $\{\|\tau_n(H, f)\|_{I,p}\} \in \mathsf{b}_{\rho,\alpha}$.*
(c) *There exists an interval I, centered at x_0, such that for every infinitely often differentiable function ϕ, supported on I (and continued as a 2π-periodic function), the sequence $\{\|\tau_n(H, f\phi)\|_p\} \in \mathsf{b}_{\rho,\alpha}$.*

We observe that the functions $\tau_n(H, f) \in \mathbb{H}_{2^n}$ are trigonometric polynomials of order at most 2^n, and are frequency localized in the sense that the Fourier coefficients $\widehat{\tau_n(H, f)}(k)$ are equal to zero except when $2^{n-2} \le |k| \le 2^n$. Theorem 2.1 demonstrates that they are also time localized. The equivalence between parts (b) and (a) show that the behavior of $\tau_n(H, f)$ near a point depends only on the smoothness properties of f near that point. The equivalence between (c) and (a) shows that the global behavior of the operator, constructed only from values of f near a point is determined entirely by the smoothness of f near that point. Thus, the a priori assumption that $f \in X^p$ is not necessary in this context; the local Besov spaces are characterized by the global behavior of the operators based on the Fourier coefficients of $f\phi$ instead of those of f.

Next, we wish to discuss the discretized versions of Theorem 2.1, in the sense that the L^p norms in parts (b) and (c) above will be replaced by suitable discrete norms, and also in the sense that the operators will be evaluated using the values of the function rather than the Fourier coefficients.

Towards this end, we first introduce some further notations. If ν is a signed measure on $[0, 2\pi]$, we denote its total variation measure by $|\nu|$ (or $|d\nu|$ in the context of integration). If f is a ν-measurable function, and A is a ν-measurable set, we write

$$\|f\|_{A,\nu,p} := \begin{cases} \left\{ \int_A |f(t)|^p |d\nu(t)| \right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \max_{\xi \in \text{ supp } (\nu) \cap A} |f(\xi)|, & \text{if } p = \infty. \end{cases} \tag{2.15}$$

As before, we will write $\|f\|_{\nu,p} := \|f\|_{[0,2\pi],\nu,p}$. A signed measure ν will be called an *M-Z quadrature measure of order N* if each of the following conditions holds.

$$\|T\|_{\nu,p} \le c\|T\|_p, \qquad T \in \mathbb{H}_N,\ 1 \le p \le \infty, \tag{2.16}$$

and

$$\int T(t) d\nu(t) = \frac{1}{2\pi} \int_0^{2\pi} T(t) dt, \qquad T \in \mathbb{H}_N. \tag{2.17}$$

When the support of ν is a finite set, the inequality (2.16) is called a Marcinkiewicz-Zygmund inequality. A typical example is the measure that associates the mass $1/(3N)$ with each of the points $2\pi k/(3N)$, $k = 0, \cdots, 3N - 1$ (cf. [22, Chapter X, Formula (2.5), Theorems 7.5, 7.28]). It is proved in [14, Corollary 4.1 and Proposition 3.4] (with the estimate (4.9) in [14] as amended in the Corrigendum) that if $\mathcal{C}$ is any finite subset of $[0, 2\pi]$, and

$$\sup_{x\in[0,2\pi]} \min_{y\in\mathcal{C}} |e^{ix} - e^{iy}| \le c/N$$

for a sufficiently small absolute constant $c > 0$, then there exists a positive M-Z quadrature measure of order N, supported on $\mathcal{C}$.

The next theorem is an analogue of Theorem 2.1, when the norms of the operators are replaced by their discrete norms.

Theorem 2.2 *Let $1 \le p \le \infty$, $f \in X^p$, $x_0 \in [0, 2\pi]$, $\alpha > 0$, $q > \max(1, \alpha)$ be an integer, $0 < \rho \le \infty$, and $H \in \mathcal{S}^q$. For each integer $n \ge 0$, let ν_n be a (possibly signed) M-Z quadrature measure of order $6(2^n)$. Then*

$$f = \sum_{n=0}^{\infty} \tau_n(H, f) = \sum_{n=0}^{\infty} \int \tau_n(H, f, t)\{\Phi_{n+1}(H, \cdot - t) - \Phi_{n-2}(H, \cdot - t)\} d\nu_n(t), \qquad (2.18)$$

with the series converging in the sense of X^p. Moreover, the following statements are equivalent.
(a) *$f \in B_{p,\rho,\alpha}(x_0)$.*
(b) *There exists an interval I, centered at x_0, such that the sequence $\{\|\tau_n(H, f)\|_{I,\nu_n,p}\} \in \mathsf{b}_{\rho,\alpha}$.*
(c) *There exists an interval I, centered at x_0, such that for every infinitely often differentiable function ϕ, supported on I (and continued as a 2π-periodic function), the sequence $\{\|\tau_n(H, f\phi)\|_{\nu_n,p}\} \in \mathsf{b}_{\rho,\alpha}$.*

In the case when ν_n is supported on a finite set $\mathcal{C}_n$, the equation (2.18) shows that $\{\tau_n(H, f, \xi)\}_{\xi\in\mathcal{C}_n}$ is a sequence of coefficients in a series expansion of f. Theorem 2.2 thus gives the characterization of local Besov spaces in terms of the absolute values of these coefficients. We observe that the functions $\{\Phi_{n+1}(H, \cdot - \xi) - \Phi_{n-2}(H, \cdot - \xi)\}_{\xi\in\mathcal{C}_n}$ are not linearly independent. Therefore, it is possible for a function to have different expansions as a series in these functions. The following theorem shows that in some cases, the behavior of the coefficients of any such expansion implies the local Besov space condition.

Theorem 2.3 *Let $1 \le p \le \infty$, $f \in X^p$, $x_0 \in [0, 2\pi]$, $\alpha > 0$, $q > \max(1, \alpha)$ be an integer, $0 < \rho \le \infty$, and h be a q-times iterated integral of a function of bounded variation, satisfying (2.12). Let $H = (h_{k,n})$, where $h_{k,n} = h(k/2^n)$. For each integer $n \ge 0$, let ν_n be a (possibly signed) M-Z quadrature measure of order $6(2^n)$, d_n be a ν_n-measurable function, $\|d_n\|_{\nu_n,p} \le c$, and*

$$f = \sum_{n=0}^{\infty} \int d_n(t)\{\Phi_{n+1}(H, \cdot - t) - \Phi_{n-2}(H, \cdot - t)\} d\nu_n(t), \qquad (2.19)$$

with the series converging in the sense of X^p. If there exists an interval I, centered at x_0, such that $\{\|d_n\|_{I,\nu_n,p}\} \in \mathsf{b}_{\rho,\alpha}$, then $f \in B_{p,\rho,\alpha}(x_0)$.

We note that the converse of the above theorem cannot be true, since the functions $\{\Phi_{n+1}(H, \cdot - \xi) - \Phi_{n-2}(H, \cdot - \xi)\}_{\xi \in \text{ supp } (\nu_n)}$ are not linearly independent.

Even though the main emphasis of this paper is to establish a connection between the local smoothness classes and the Fourier coefficients (via the operators τ_n), we will now turn our attention to the localization properties of the operators defined using the discrete measures ν_n. Of course, since the point evaluations are defined only for functions in X^∞, the following theorem will be stated only for the various supremum norms.

Let H be a dyadic summability matrix, for $n = 0, 1, \cdots$, ν_n be an M-Z quadrature measure of order $6(2^n)$, $\mathcal{C}_n := \text{supp}(\nu_n)$. For a ν_n-measurable function f, we define

$$\sigma_n^D(H, f, x) := \int f(t)\Phi_n(H, x - t)d\nu_n(t). \tag{2.20}$$

Clearly, $\sigma_n^D(H, f) \in \mathbb{H}_{2^n}$. These operators have properties analogous to those of $\sigma_n(H, f)$ (cf. Proposition 4.2). We note that the operator $\sigma_{n-1}^D(H)$ is defined using the measure ν_{n-1}. The following theorem gives a characterization of local Besov spaces in terms of the differences

$$\tau_n^D(H, f) := \sigma_n^D(H, f) - \sigma_{n-1}^D(H, f). \tag{2.21}$$

We note again that in the case when $\mathcal{C}_n$ is a finite set, the trigonometric polynomial $\tau_n^D(H, f) \in \mathbb{H}_{2^n}$ is defined using finitely many values of the function f.

Theorem 2.4 *Let $f \in X^\infty$, $x_0 \in [0, 2\pi]$, $\alpha > 0$, $q > \max(1, \alpha)$ be an integer, $0 < \rho \le \infty$, and $H \in \mathcal{S}^q$. For each integer $n \ge 0$, let ν_n be a (possibly signed) M-Z quadrature measure of order $6(2^n)$. The following statements are equivalent.*
(a) $f \in B_{\infty,\rho,\alpha}(x_0)$.
(b) *There exists an interval I, centered at x_0, such that the sequence $\{\|\tau_n^D(H, f)\|_{I,\infty}\} \in \mathsf{b}_{\rho,\alpha}$.*
(c) *There exists an interval I, centered at x_0, such that for every infinitely often differentiable function ϕ, supported on I (and continued as a 2π-periodic function), the sequence $\{\|\tau_n^D(H, f\phi)\|_\infty\} \in \mathsf{b}_{\rho,\alpha}$.*
(d) *There exists an interval I, centered at x_0, such that the sequence $\{\|\tau_n^D(H, f)\|_{I,\nu_n,\infty}\} \in \mathsf{b}_{\rho,\alpha}$.*
(e) *There exists an interval I, centered at x_0, such that for every infinitely often differentiable function ϕ, supported on I (and continued as a 2π-periodic function), the sequence $\{\|\tau_n^D(H, f\phi)\|_{\nu_n,\infty}\} \in \mathsf{b}_{\rho,\alpha}$.*

3 Numerical examples

In this section, we discuss some numerical examples to illustrate the theory presented in this paper, as well as in [15]. All the computations below were performed using MATLAB version 5.3 and/or Mathematica 4.0. We recall that for $q \ge 1$, the cardinal B-spline of order q is the function defined by (cf. [2, p. 131])

$$M_1(x) := \begin{cases} 1, & \text{if } 0 < x \le 1, \\ 0, & \text{otherwise,} \end{cases}$$
$$M_q(x) := \frac{1}{q-1}\{xM_{q-1}(x) + (q-x)M_{q-1}(x-1)\} \quad q \ge 2. \tag{3.1}$$

3.1 Local Lipschitz exponents

In this section, we will denote $B_{\infty,\infty,\alpha}(x_0)$ by Lip (α, x_0). We consider the function

$$f(x) := Q(x) + \begin{cases} |x-\pi|^{1/2}, & \text{if } \pi/2 \le x \le 3\pi/2, \\ (\pi/2)^{1/2} + |x-\pi/2|^{1/4}, & \text{if } 0 \le x < \pi/2, \\ (\pi/2)^{1/2} + |x-3\pi/2|^{3/4}, & \text{if } 3\pi/2 < x \le 2\pi, \end{cases} \tag{3.2}$$

where Q is a quadratic polynomial that makes the function continuously differentiable at 0, 2π, when extended as a 2π-periodic function. Clearly, $f \in$ Lip $(1/4, \pi/2)$, $f \in$ Lip $(1/2, \pi)$, and $f \in$ Lip $(3/4, 3\pi/2)$.

To analyse this function, we considered the function h defined by

$$h(x) = \sum_{j=-10}^{10} M_7(14x - j + 7/2). \qquad x \ge 0,$$

The function h is 5 times continuously differentiable on $\mathbb{R}$, $h^{(5)}$ is a piecewise constant function, $h(x) = 1$ if $|x| < 1/2$, and $h(x) = 0$ if $|x| > 1$. Therefore, $H = (h(k/2^n))$ is in $\mathcal{S}^6$. In this example, the Fourier coefficients of f were simplified and then approximated with sufficient exactness using Mathematica, and the supremum norms over different intervals are approximated with sufficient exactness (10^{-20}). Let $t = \pi/8$. To find the value of the Lipschitz exponent at a point x_0, we computed the quantity

$$\alpha_{\text{est}}(n, x_0) := \log_2\left(\frac{\|\tau_n(H,f)\|_{[x_0-t,x_0+t],\infty}}{\|\tau_{n+1}(H,f)\|_{[x_0-t,x_0+t],\infty}}\right)$$

and computed the following values:

n	$\alpha_{\text{est}}(n, \pi/2)$	$\alpha_{\text{est}}(n, \pi)$	$\alpha_{\text{est}}(n, 3\pi/2)$
5	0.27220	0.577207444	0.794
6	0.24239	0.502889989	0.709
7	0.24484	0.499999986	0.713
8	0.24693	0.500000048	0.72
9	0.24818	0.500000002	0.725

Similar, but less accurate, results were obtained using an approximation to the Fourier coefficients of f obtained by using a 2^{n+3} point `fft` command of Matlab.

3.2 Detection of singularities

In this section, we find it convenient to treat $[-\pi, \pi]$ rather than $[0, 2\pi]$ as the basic domain of definition of 2π-periodic functions. We say that a 2π-periodic function $f : \mathbb{R} \to \mathbb{C}$ is *piecewise R-smooth,* if there exist points

$$-\pi = y_0 < y_1 < \cdots < y_m = \pi$$

such that the restriction of f to (y_i, y_{i+1}), $0 \le i \le m-1$, is R times continuously differentiable, and $f^{(r)}(y_i+)$ and $f^{(r)}(y_{i+1}-)$ exist as finite numbers for $0 \le r \le R$. Any such function can be expressed in a canonical form (3.3) as follows:

$$f(x) = \sum_{r=0}^{R} \sum_{j=1}^{m} z_{j,r} \Gamma_r(x - y_j) + F(x), \qquad x \in \mathbb{R}, \tag{3.3}$$

where $z_{j,r} \in \mathbb{C}$, F is an R times continuously differentiable 2π-periodic function on $\mathbb{R}$, and the functions Γ_r are defined by

$$\Gamma_r(x) = \sum_{k \in \mathbb{Z} \setminus \{0\}} \frac{e^{ikx}}{(ik)^{r+1}}, \qquad x \in \mathbb{R}. \tag{3.4}$$

In the representation (3.3), y_j will be called a *singularity of order* ℓ of f if $\ell = \min\{r : z_{j,r} \ne 0\}$. In [15], we have studied operators of the form

$$\tau_n^P(g, f, x) := \sum_{k=0}^{2^n - 1} g(k/2^n) \hat{f}(k) \exp(ikx), \tag{3.5}$$

where g is a suitably defined function, and f is a real valued, piecewise R-smooth function. We note that $\tau_n^P(g, f)$ utilizes only the coefficients $\hat{f}(k)$ for positive values of k. Since f is real valued, these already contain all the needed information. Moreover, the results in [15] indicate that these "one-sided" transforms are better suited to reveal the singularities of different orders, odd or even. Although some numerical examples are discussed in [15] on the use of these operators in the detection of the points y_j, we find it worthwhile to discuss a few additional issues that have been raised in connection with this problem in the last few years.

To illustrate these issues, we consider the function

$$f(x) = \begin{cases} -\cos x - (1/\pi - 1/2)(x + \pi) - 1/\pi, & \text{if } -\pi \le x \le -\pi/2, \\ \sin x - (1/\pi + 1/2)x - 1/\pi, & \text{if } |x| < \pi/2, \\ -\cos x - (1/\pi - 1/2)(x - \pi) - 1/\pi, & \text{if } \pi/2 \le x \le \pi. \end{cases} \tag{3.6}$$

It is easy to verify that f is piecewise 2-smooth, with a singularity of order 2 at $-\pi/2$, of order 1 at $\pi/2$, and no other singularities of order ≤ 2.

The Fourier coefficients of this function are given by

$$\hat{f}(k) = \begin{cases} 0, & \text{if } k = 0, \\ -(1+i)/4 + i/\pi, & \text{if } k = 1, \\ \dfrac{i}{k^2\pi} \left(\sin(k\pi/2) + \dfrac{k(ik+1)\cos(k\pi/2)}{k^2 - 1} \right), & \text{if } k = 2, 3, \cdots, \\ \overline{\hat{f}(-k)}, & \text{if } k < 0. \end{cases} \tag{3.7}$$

To analyse the function f, we considered the function g, given by

$$g(x) := M_5(10x - 5), \tag{3.8}$$

which is a 3 times continuously differentiable function, supported on $[1/2, 1]$. Therefore, it should be kept in mind that the operator $\tau_n^P(g, f)$ uses only 2^{n-1} Fourier coefficients of f.

To find the singularities, we performed the following steps repeatedly.

1. Find
$$x_0 = \arg\max|\tau_n^P(g, f, x)|$$

2. Find
$$m_1 := |\tau_n^P(g, f, x_0)|,\ m_2 := |\tau_{n-1}^P(g, f, x_0)|.$$

3. Let $r = \text{round}\ (\log_2(m_2/m_1))$, and $d = \tau_n^P(g, f, x_0)/\tau_n^P(g, \Gamma_r, 0)$.

4. Reset $\hat{f}(k) = \hat{f}(k) - d\exp(-ikx_0)/(ik)^{r+1}$.

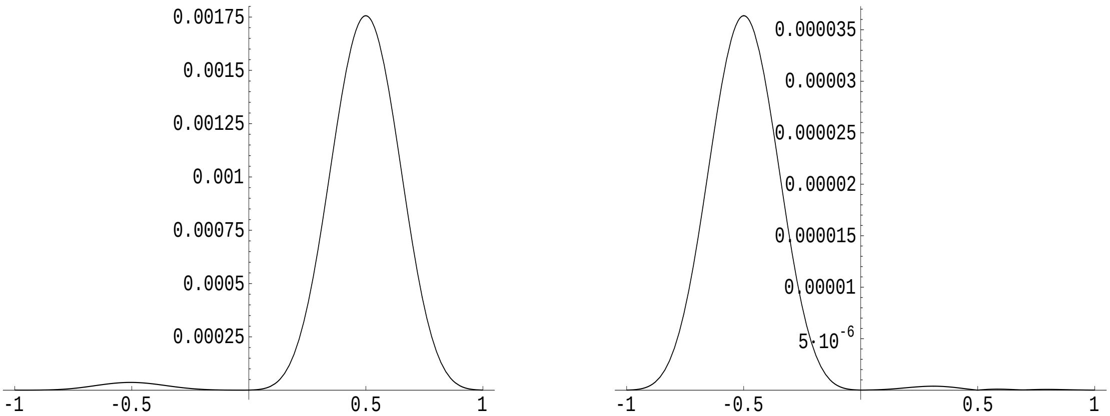

Figure 1: The graph of $|\tau_n^P(g, f, x)|$ on the left, and the graph at the start of the next iteration, after deleting the singularity at $\pi/2$ on the right. The x-axis is labeled with multiples of π.

Figure 1 shows the graphs of $|\tau_n^P(g, f, x)|$ before and after the first singularity at $\pi/2$ is deleted. Using $n = 5$, we found a singularity of order 1 at 0.4997π (with $m_1 = 0.0018$), a singularity of order 2 at -0.4995π (with $m_1 = (3.636)(10)^{-5}$), and an extra singularity of order 5 at -0.347π (with $m_1 = (2.0365)(10)^{-6}$). The results were similar when the Fourier coefficients were estimated using MATLAB and `fft` with 256 samples of f.

Next, we proceeded to investigate the singularity at $\pi/2$ a bit further. Thus, we observe that in addition to the jump $f'(\pi/2+) - f'(\pi/2-) = 2$ in f' at $\pi/2$, we also have $f''(\pi/2+) - f''(\pi/2-) = 1$, $f'''(\pi/2+) - f'''(\pi/2-) = -1$, and $f^{(4)}(\pi/2+) - f^{(4)}(\pi/2-) = -1$. We wish to find all of these jumps. To do this, in place of the step 3 above, we tried to solve the system of equations

$$\tau_n^P(g, f, x_0) = \sum_{j=1}^{4} d_j \tau_n^P(g, \Gamma_j, 0), \qquad n = 5, 6, 7, 8. \tag{3.9}$$

This procedure is similar to Richardson extrapolation. The jump in f' was found to be exactly 2. However, the jumps in the second, third, and fourth order derivatives were found to be $-65.70, 400.60, 33955.87$ respectively. On the other hand, if we solve the system (3.9) with $\pi/2$ in place of x_0, then the results are exact (up to 3 or more places of decimal accuracy). This effect is related to the condition number of the matrix of this system which is $(5.2)(10)^6$. Therefore, it is clear that one needs to improve upon the location of x_0.

It turns out that a very impressive improvement is obtained by a sort of "information fusion", where, after locating the singularity approximately using the Fourier coefficients, one uses adaptively samples of the function. We recall that the Chebyshev polynomials of degree m (adapted to $[a, b]$) are defined by

$$T_m([a,b], \frac{a+b}{2} + \frac{b-a}{2}\cos\theta) = \cos m\theta. \tag{3.10}$$

We used an analogue of τ_n^P based on Chebyshev polynomials:

$$\tau_n^T(a,b,g;f,x) := \sum_{k=0}^{2^n-1} g(k/2^n) C_k(a,b,f) T_k([a,b],x), \tag{3.11}$$

where

$$C_k(a,b,f) = \frac{1}{\pi}\int_a^b f(t)T_k([a,b],t)((b-t)(t-a))^{-1/2}dt.$$

We improved upon the location of x_0 by considering the location of the highest peak of $|\tau_{16}^T(x_0-\epsilon, x_0+\epsilon, g; f, x)|$ over two intervals of the form $[x_0-\epsilon, x_0+\epsilon]$ for a decreasing value of ϵ, updating x_0 in the second step.

Computing the integrals with sufficient exactness, we found x_0 to be within $(-5.29)(10)^{-10}$ of $\pi/2$ and the jump of f'' was improved to 0.99995. The jumps of height -1 of the third and fourth order derivative were detected as -0.998 and -0.977, respectively.

In the Matlab experiments, we estimated the coefficients C_k using a 64 point Chebyshev quadrature rule. After two iterations, we obtained x_0 within $(3.7289)(10)^{-9}$ of $\pi/2$ and the jump of f'' was improved to 1.009. The remaining two jumps were not accurate. Here, we used 16 Fourier coefficients and 16 Chebyshev coefficients (obtained from 128 function values).

3.3 Phased array antennas

Our discussion in this paragraph is based on [10, 17]. A (linear) phased array antenna is a device consisting of a number of *antenna elements*, which are arranged in a linear pattern. Each element is an electronic device that produces an alternating current, upon receiving an electromagnetic signal. The current from the k-th element is given by $z\exp(iku)$, where $z \in \mathbb{C}$ gives the strength of the signal, and u is related in a one-to-one manner to the direction of arrival of the signal, relative to the position of the antenna. With m signals arriving simultaneously, the total current from the k-th element assumes the form

$$I(k) = \sum_{j=1}^{m} z_j \exp(-iku_j).$$

The *direction finding problem* in this context consists of finding the quantities u_j, given finitely many values of $I(k)$. Usually, the data $I(k)$ is mixed with noise. Most of the methods known in this connection are based on the statistical properties of this noise, and require several observations on $I(k)$. One popular method is to use another device, called a *phase shifter*, that produces a pattern of the form

$$\left|\sum_{k=1}^{N} g(k)I(k)\exp(ikx)\right|.$$

With proper choices of the factor $g(k)$, the peaks of this pattern give the locations of u_j.

Clearly, for $k \neq 0$, $I(k)/(ik)^2$ are the Fourier coefficients of $\sum_{j=1}^{m} z_j\Gamma_1(\cdot - u_j)$. Thus, we may apply the algorithm described in Section 3.2 to find the locations of u_j.

We considered the case

$$I(k) = \exp(ik\pi/2) + 34 + 300\exp(-ik\pi/4), \qquad k = 1, \cdots, 64,$$

and added a uniform noise in the range $[0, 3]$ to each $I(k)$, where we observe that the noise exceeds the lowest strength of the signal from $-\pi/2$. Rather than taking several observations on $I(k)$, we applied the algorithm of Section 3.2 to each observation of the contaminated $I(k)$'s. This experiment was repeated 500 times. The average detected values of the u_j's were 0.249975π (with a strength of 300.006), followed by $(-3.45)10^{-6}$ (with a strength of 35.5), followed by -0.501π (with a strength of 0.4). We also had one more repetition of the algorithm, giving a spurious average direction of 0.246π (with a strength of -0.01). The standard deviations in these experiments were respectively $(9.5)10^{-5}$, $(1.6)10^{-5}$, 0.028 for the first three directions (and 1.4, 0.2, 0.96 for their respective strengths). Since the standard deviation for the spurious direction was 1.9, it is clear that the algorithm found all the three legitimate directions correctly, and no more. It is interesting to note that we do not need to assume an a priori knowledge about the number of signals present.

4 Proofs

In this section, we provide the proofs of the results in Section 2. We begin with the proof of Proposition 2.1.

Proof of Proposition 2.1. Part (a) is proved in [15, Proposition 2.1(b)] and part (c) is proved in [15, Proposition 2.2]. Next, let H be a dyadic summability matrix. It is clear from the definition that $\sigma_n(H, f) \in \mathbb{H}_{2^n}$ for $f \in X^p$, and $\sigma_n(H, T) = T$ for $T \in \mathbb{H}_{2^{n-1}}$. Consequently, using [22, Theorem 1.15, Chapter II] (with $\Phi_n(H)$ in place of g and 1 in place of q there), and part (a) of this proposition, we obtain for any integer $n \geq 0$,

$$\|\sigma_n(H, f)\|_p = \left\|\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\Phi_n(H, \cdot - t)dt\right\|_p \leq \|f\|_p \sup_{n\geq 0}\|\Phi_n(H)\|_1 = c\|f\|_p.$$

This proves part (b). □

In order to prove Theorem 2.1, we first prove a similar characterization for global smoothness classes. Since we need the characterization in terms of the sequence $\{E_{2^n,p}(f)\}$ rather than in terms of spline functions, we have to define these global smoothness classes using a periodic modulus of smoothness rather than $\omega_{[0,2\pi],r,p}$. For $1 \le p \le \infty$, $f \in X^p$, $\delta > 0$, this modulus is defined by

$$\omega^*_{r,p}(f,\delta) := \sup_{0<t\le\delta} \left\| \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(\cdot + kt) \right\|_p , \tag{4.1}$$

where, in contrast to the aperiodic modulus of smoothness, the norm is taken over the entire period of f, using the periodicity of f in the case when the translates go outside of $[0, 2\pi]$. If $1 \le p \le \infty$, $\alpha > 0$, $0 < \rho \le \infty$, the *periodic Besov space* $B^*_{p,\rho,\alpha}$ consists of $f \in X^p$, for which the sequence $\{\omega^*_{r,p}(f, 2^{-n})\} \in \mathsf{b}_{\rho,\alpha}$ for some integer $r > \alpha$. The spaces are known to be independent of the choice of r, as long as $r > \alpha$ [4, Theorem 10.1, Chapter 2].

Theorem 4.1 *Let $1 \le p \le \infty$, $f \in X^p$, $\alpha > 0$, $0 < \rho \le \infty$, and $\{U_n\}$ be a good approximation sequence for X^p. The following statements are equivalent.*
(a) $f \in B^*_{p,\rho,\alpha}$.
(b) $\{E_{2^n,p}(f)\} \in \mathsf{b}_{\rho,\alpha}$.
(c) *The sequence* $\{\|U_n(f) - U_{n-1}(f)\|_p\} \in \mathsf{b}_{\rho,\alpha}$.

In order to prove this theorem, we recall a consequence of the discrete Hardy inequality.

Lemma 4.1 *Let $\alpha > 0$, $0 < \rho \le \infty$, $\{a_n\} \in \mathsf{b}_{\rho,\alpha}$. Then the sequence* $\{\sum_{m=n+1}^{\infty} |a_m|\} \in \mathsf{b}_{\rho,\alpha}$.

Proof. This follows from the discrete Hardy inequality [4, Lemma 3.4, p. 27]. We note that the notation in [4] is different from ours. □

Proof of Theorem 4.1. The equivalence between (a) and (b) follows from [4, Theorem 9.2, Chapter 7]. Now, suppose that (b) holds. The estimate (2.4) implies that for $n \ge 2$,

$$\|U_n(f) - U_{n-1}(f)\|_p \le \|U_n(f) - f\|_p + \|f - U_{n-1}(f)\|_p \le cE_{2^{n-2},p}(f).$$

Since $\{E_{2^{n-2},p}(f)\} \in \mathsf{b}_{\rho,\alpha}$, it follows that $\{\|U_n(f) - U_{n-1}(f)\|_p\} \in \mathsf{b}_{\rho,\alpha}$; i.e., (c) holds.

Next, let (c) hold. From (2.4), we deduce that $f = \sum_{m=0}^{\infty}(U_m(f) - U_{m-1}(f))$, with convergence in the sense of X^p, and hence, that

$$E_{2^n,p}(f) \le \sum_{m=n+1}^{\infty} \|U_m(f) - U_{m-1}(f)\|_p. \tag{4.2}$$

Since $\{\|U_m(f) - U_{m-1}(f)\|_p\} \in \mathsf{b}_{\rho,\alpha}$, Lemma 4.1 implies that the sequence on the right hand side above is in $\mathsf{b}_{\rho,\alpha}$. Therefore, the sequence $\{E_{2^n,p}(f)\} \in \mathsf{b}_{\rho,\alpha}$ as well. Thus, (c) implies (b). □

Next, we establish a connection between the local and global Besov spaces. Even though the local Besov spaces are defined using the aperiodic moduli of smoothness, since the functions are periodic, there is no loss of generality in assuming that the point in question, x_0, is π.

Proposition 4.1 *Let $1 \le p \le \infty$, $f \in X^p$, $\alpha > 0$, $0 < \rho \le \infty$. Then $f \in B_{p,\rho,\alpha}(\pi)$ if and only if there exists an interval I, centered at π, such that for every infinitely often differentiable function ϕ, supported on I, the function $f\phi$ (extended to $\mathbb{R}$ as a 2π-periodic function) is in $B^*_{p,\rho,\alpha}$.*

Proof. Let $f \in B_{p,\rho,\alpha}(\pi)$, and I be an interval, centered at π, such that $|I| < \pi/2$, and $\omega_{I,r,p}(f, 2^{-n}) \in \mathsf{b}_{\rho,\alpha}$ for some $r > \alpha$. In view of [4, Theorem 2.4, Chapter 6], there exists a function f_1, which is an r times iterated integral of a function ${f_1}^{(r)} \in X^p(I)$, such that

$$\|f - f_1\|_{I,p} + 2^{-nr}\|{f_1}^{(r)}\|_{I,p} \le c(I)\omega_{I,r,p}(f, 2^{-n}).$$

It follows that

$$\|f - f_1\|_{I,p} + 2^{-nr}\{\|f_1\|_{I,p} + \|{f_1}^{(r)}\|_{I,p}\} \le c(I, f)\{\omega_{I,r,p}(f, 2^{-n}) + 2^{-nr}\}. \tag{4.3}$$

Now, let ϕ be an infinitely often differentiable function, supported on I, and let

$$f_2(x) := \begin{cases} f_1(x)\phi(x), & \text{if } x \in I, \\ 0, & \text{if } x \in [0, 2\pi] \setminus I. \end{cases}$$

Then f_2 may be extended as a 2π-periodic function on $\mathbb{R}$, and this extension is an r times iterated periodic integral of a function ${f_2}^{(r)} \in X^p$. Moreover, using Leibniz formula, we deduce that

$$\|{f_2}^{(r)}\|_p \le c(\phi) \sum_{k=0}^{r} \|{f_1}^{(k)}\|_{I,p}.$$

In view of [4, Theorem 5.6, Chapter 2] (with $u = |I|/2$) we have for integer k, $0 \le k \le r$,

$$\|{f_1}^{(k)}\|_{I,p} \le c(I)\{\|f_1\|_{I,p} + \|{f_1}^{(r)}\|_{I,p}\}.$$

Therefore, (4.3) implies that

$$\begin{aligned} \|f\phi - f_2\|_p + 2^{-nr}\|{f_2}^{(r)}\|_p &\le c(I, \phi)\left\{\|f - f_1\|_{I,p} + 2^{-nr}\left(\|f_1\|_{I,p} + \|{f_1}^{(r)}\|_{I,p}\right)\right\} \\ &\le c(f, I, \phi)\{\omega_{I,r,p}(f, 2^{-n}) + 2^{-nr}\}. \end{aligned}$$

Using [4, Theorem 2.4, Chapter 6] again, this time with the periodic modulus of smoothness of $f\phi$ (extended to $\mathbb{R}$ as a 2π-periodic function), we deduce that

$$\omega^*_{r,p}(f\phi, 2^{-n}) \le c(f, I, \phi)\{\omega_{I,r,p}(f, 2^{-n}) + 2^{-nr}\}.$$

Since both the sequences $\{\omega_{I,r,p}(f, 2^{-n})\}$ and $\{2^{-nr}\}$ are in $\mathsf{b}_{\rho,\alpha}$, we conclude that $f\phi \in B^*_{p,\rho,\alpha}$.

To prove the converse, let I be an interval, centered at π, such that $|I| < \pi/2$, and for every infinitely often differentiable function ϕ, supported on I, the function $f\phi$ (extended to $\mathbb{R}$ as a 2π-periodic function) is in $B^*_{p,\rho,\alpha}$. Let J be the interval centered at π with $|J| = |I|/2$, and ψ be an infinitely often differentiable function, supported on I, and equal to 1 on J. Since $f\psi = f$ on J, it follows that

$$\omega_{J,r,p}(f, 2^{-n}) = \omega_{J,r,p}(f\psi, 2^{-n}) \le \omega^*_{r,p}(f\psi, 2^{-n}).$$

Consequently, $f \in B_{p,\rho,\alpha}(\pi)$. □

We find it convenient to summarize certain calculations needed in the proof of Theorem 2.1 in the following two lemmata.

Lemma 4.2 *Let $q \ge 1$ be an integer, $H \in \mathcal{S}^q$. Let I be an interval centered at π, $|I| < \pi/2$, J_1 (respectively J) be the interval, centered at π, with length equal to $|I|/2$ (respectively, $|I|/4$), and ψ be a C^∞ function, supported on I, such that $\psi(x) = 1$ for all $x \in J_1$. Let $1 \le p \le \infty$, ν be a signed measure, and f be a function defined on the support of ν with $N_f := \int |f(t)||d\nu(t)| < \infty$. Then for the operator*

$$\sigma_{n,\nu}(H, f, x) := \int f(t)\Phi_n(H, x - t)d\nu(t)$$

we have

$$\|\sigma_{n,\nu}(H, (1-\psi)f)\|_{J,\infty} \le \frac{c(I, H, \psi)N_f}{2^{nq}}. \tag{4.4}$$

Proof. For $x \in J$, we have

$$\begin{aligned}
|\sigma_{n,\nu}(H, (1-\psi)f, x)| &\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |\Phi_{2^n}(H, x-t)f(t)(1-\psi(t))|\,|d\nu(t)| \\
&= \frac{1}{2\pi}\int_{[-\pi,\pi]\setminus J_1} |\Phi_{2^n}(H, x-t)f(t)(1-\psi(t))|\,|d\nu(t)| \\
&\le \frac{1}{2\pi}\int_{\{t\,:\,|x-t|\ge|I|/8\}} |\Phi_{2^n}(H, x-t)f(t)(1-\psi(t))|\,|d\nu(t)|.
\end{aligned}$$

In view of the estimate (2.11), we conclude that

$$\|\sigma_{n,\nu}(H, (1-\psi)f)\|_{J,\infty} \le \frac{c(I, H, \psi)N_f}{2^{nq}}. \tag{4.5}$$

□

Lemma 4.3 *Let $1 \le p \le \infty$, and $\{U_n\}$ be a good approximation sequence for X^p. If, for some interval I, centered at π, $\{\|U_n(f) - U_{n-1}(f)\|_{I,p}\} \in \mathsf{b}_{\rho,\alpha}$, then for every infinitely often differentiable function ϕ supported on I (and extended to $\mathbb{R}$ as a 2π-periodic function), $\{E_{2^n,p}(f\phi) \in \mathsf{b}_{\rho,\alpha}\}$; i.e., $f\phi \in B^*_{p,\rho,\alpha}$.*

Proof. Using the direct theorem of approximation theory [4, Corollary 2.4, Chapter 7], we may choose $T \in \mathbb{H}_{2^n}$ such that

$$\|\phi - T\|_\infty \le c2^{-nq}. \tag{4.6}$$

Further, since $f \in X^p$, (2.4) implies that

$$f = \sum_{m=0}^{\infty}(U_m(f) - U_{m-1}(f)) = U_n(f) + \sum_{m=n+1}^{\infty}(U_m(f) - U_{m-1}(f)),$$

where the convergence is in the sense of X^p. Then $TU_n(f) \in \mathbb{H}_{2^{n+1}}$, and using the fact that $\|U_n(f)\|_p \le c\|f\|_p$, we deduce that

$$\begin{aligned}
E_{2^{n+1},p}(f\phi) &\le \|f\phi - TU_n(f)\|_p \\
&\le \|(\phi - T)U_n(f)\|_p + \|(f - U_n(f))\phi\|_p \\
&\le c\|f\|_p\|\phi - T\|_\infty + c\|\phi\|_\infty\|f - U_n(f)\|_{I,p} \\
&\le c(f, \phi)\left\{2^{-nq} + \sum_{m=n+1}^{\infty}\|U_m(f) - U_{m-1}(f)\|_{I,p}\right\}.
\end{aligned}$$

Lemma 4.1 implies that

$$\{\sum_{m=n+1}^{\infty}\|U_m(f)-U_{m-1}(f)\|_{I,p}\}\in \mathsf{b}_{\rho,\alpha}.$$

Since $\{2^{-nq}\}\in \mathsf{b}_{\rho,\alpha}$, we conclude that $\{E_{2^{n+1},p}(f\phi)\}\in \mathsf{b}_{\rho,\alpha}$. In view of Theorem 4.1, this completes the proof. □

We are now in a position to prove Theorem 2.1.

PROOF OF THEOREM 2.1. In view of Proposition 4.1, part (a) holds (with $x_0=\pi$) if and only if there is an interval, centered at π, such that for every infinitely often differentiable function ϕ, supported on I, the function $f\phi$ (extended to $\mathbb{R}$ as a 2π-periodic function) is in $B^*_{p,\rho,\alpha}$. According to Theorem 4.1 (applied with $\{\sigma_n(H)\}$), this is equivalent to part (c). Thus, parts (a) and (c) are equivalent. Now, suppose (c) holds, and I be the interval chosen as in that part. Without loss of generality, we may assume that $|I|<\pi/2$. Let J_1 (respectively J) be the interval, centered at π with length equal to $|I|/2$ (respectively, $|I|/4$). We take ψ to be a C^∞ function, supported on I, such that $\psi(x)=1$ for all $x\in J_1$. Then Lemma 4.2, applied with the normalized Lebesgue measure in place of ν, implies that

$$\|\sigma_n(H,(1-\psi)f)\|_{J,\infty}\le \frac{c(I,f,H,\psi)}{2^{nq}}.$$

Therefore,

$$\begin{aligned}&\|\tau_n(H,f)\|_{J,p}\\ &\le \|\tau_n(H,f\psi)\|_{J,p}+\|\sigma_n(H,(1-\psi)f)\|_{J,\infty}+\|\sigma_{n-1}(H,(1-\psi)f)\|_{J,\infty}\\ &\le \|\tau_n(H,f\psi)\|_{p}+\frac{c(I,f,H,\psi)}{2^{nq}}.\end{aligned}$$

Since $q>\alpha$, both $\{\|\tau_n(H,\psi f)\|_p\}$ and $\{2^{-nq}\}$ are in $\mathsf{b}_{\rho,\alpha}$. Therefore, $\{\|\tau_n(H,f)\|_{J,p}\}\in \mathsf{b}_{\rho,\alpha}$, and part (b) is proved.

Finally, let part (b) hold, and I be the interval as in that part. Without loss of generality, we may assume that $|I|<\pi/2$. Let ϕ be any infinitely often differentiable function supported on I, and extended to $\mathbb{R}$ as a 2π-periodic function. In view of Lemma 4.3 (applied with $\sigma_n(H)$ in place of U_n), we see that $f\phi\in B^*_{p,\rho,\alpha}$. This proves part (c) of Theorem 2.1. □

Our proof of Theorem 2.2 depends upon the following lemmata.

Lemma 4.4 *Let ν be any signed, Borel measure on $[0,2\pi]$, F be $|\nu|$–integrable, and G be a bounded, Borel measurable function on $[0,2\pi]$. Let*

$$F*_\nu G(x):=\int F(t)G(x-t)d\nu(t),\qquad x\in[0,2\pi]. \tag{4.7}$$

Then for $1\le p\le\infty$,

$$\|F*_\nu G\|_p\le \|F\|_{\nu,p}\|G\|_{\nu,1}^{1-1/p}\|G\|_1^{1/p}. \tag{4.8}$$

In particular, if $N\ge 0$ is an integer, and ν is an M–Z quadrature measure of order N, then

$$\|F*_\nu G\|_p\le c\|F\|_{\nu,p}\|G\|_1,\qquad G\in\mathbb{H}_N. \tag{4.9}$$

PROOF. The estimate (4.8) is obvious for $p = \infty$, and is easy to verify in the case when $p = 1$. The intermediate cases follow from these cases, and the Riesz-Thorin interpolation theorem [1, Theorem 1.1.1]. The estimate (4.9) follows from (4.8) and (2.16). □

Lemma 4.5 *Let $\{\nu_n\}$ be as in Theorem 2.2, $1 \le p \le \infty$. Then for integer $n \ge 0$ and $T \in \mathbb{H}_{2^{n+1}}$,*

$$c_1\|T\|_p \le \|T\|_{\nu_n,p} \le c_2\|T\|_p. \tag{4.10}$$

Moreover, let I, J be intervals centered at π, with $J \subset I$, and $Q \ge 1$ be an integer. Then for $T \in \mathbb{H}_{2^n}$,

$$\|T\|_{J,p} \le c(Q,I,J)\{\|T\|_{I,\nu_n,p} + 2^{-nQ}\|T\|_p\}, \quad \|T\|_{J,\nu_n,p} \le c(Q,I,J)\{\|T\|_{I,p} + 2^{-nQ}\|T\|_p\}. \tag{4.11}$$

PROOF. Let H be a dyadic summability matrix. We observe that $T\Phi_{n+2}(H) \in \mathbb{H}_{6(2^n)}$. Therefore, (2.17) may be applied to obtain (cf. Proposition 2.1(b))

$$T(x) = \frac{1}{2\pi}\int_0^{2\pi} T(t)\Phi_{n+2}(H, x-t)dt = \int T(t)\Phi_{n+2}(H, x-t)d\nu_n(t).$$

The first inequality in (4.10) now follows from (4.9) and (2.9). The second inequality there is (2.16).

Let ψ be an infinitely often differentiable, 2π periodic function, which takes the value 1 on J and is zero on $[0, 2\pi] \setminus I$, and further satisfies $\|\psi\|_\infty = 1$. Then we may find $R \in \mathbb{H}_{2^n}$ such that

$$\|\psi - R\|_\infty \le c(Q,I,J)2^{-nQ}. \tag{4.12}$$

Then $TR \in \mathbb{H}_{2^{n+1}}$ and (4.10) holds with TR in place of T. Therefore,

$$\begin{aligned}
\|T\|_{J,p} &= \|T\psi\|_{J,p} \le \|T\psi\|_p \le \|TR\|_p + \|(\psi - R)T\|_p \\
&\le c(Q,I,J)\{\|TR\|_{\nu_n,p} + 2^{-nQ}\|T\|_p\} \\
&\le c(Q,I,J)\{\|T\psi\|_{\nu_n,p} + \|(\psi - R)T\|_{\nu_n,p} + 2^{-nQ}\|T\|_p\} \\
&\le c(Q,I,J)\{\|T\|_{I,\nu_n,p} + 2^{-nQ}\|T\|_{\nu_n,p} + 2^{-nQ}\|T\|_p\} \\
&\le c(Q,I,J)\{\|T\|_{I,\nu_n,p} + 2^{-nQ}\|T\|_p\}.
\end{aligned}$$

This proves the first inequality in (4.11). The second inequality is proved similarly, and we omit the proof. □

PROOF OF THEOREM 2.2. We observe that $\widehat{\Phi_{n+1}(H)}(k) - \widehat{\Phi_{n-2}(H)}(k) = 1$ if $2^{n-1} \le |k| \le 2^n$. Using (2.17), we deduce that for $x \in \mathbb{R}$,

$$\tau_n(H,f,x) = \int \tau_n(H,f,t)\{\Phi_{n+1}(H,x-t) - \Phi_{n-2}(H,x-t)\}d\nu(t).$$

The representation (2.18) follows from the fact that $f = \sum_{n=0}^\infty \tau_n(H,f)$, with convergence in the sense of X^p. We apply Lemma 4.5 with q in place of Q. Since $\{2^{-nq}\} \in \mathsf{b}_{\rho,\alpha}$ and $\|\sigma_n(H,f)\|_p \le c\|f\|_p$, the estimates (4.11) show that the statement (b) in Theorem 2.2 is equivalent to the statement (b) of Theorem 2.1, and hence, to the fact that $f \in B_{\rho,\alpha,p}(\pi)$.

The equivalence between the statements (c) and (a) is proved similarly, using (4.10) and the corresponding equivalence between the statements (c) and (a) of Theorem 2.1. □

PROOF OF THEOREM 2.3. In this proof only, let $g_{k,m} = h_{k,m} - h_{k,m-1}$, $y_{k,n} = h_{k,n+1} - h_{k,n-2}$, and ψ_j be defined for $j \in \mathbb{Z}$ by

$$\psi_j(x) = \left(h(2^{-j-1}x) - h(2^{-j+2}x)\right)(h(x) - h(2x)).$$

Then $g_{k,m} = 0$ if $|k| < 2^{m-2}$ or $|k| \geq 2^m$, and $y_{k,n} = 0$ if $|k| < 2^{n-3}$ or $|k| \geq 2^{n+1}$. Therefore, for $x \in [0, 2\pi]$, (2.19) implies that

$$\begin{aligned}
\tau_m(H, f, x) &= \sum_{n=0}^{\infty} \int d_n(t) \int_0^{2\pi} \sum_{\ell \in \mathbb{Z}} y_{\ell,n} e^{i\ell(y-t)} \sum_{k \in \mathbb{Z}} g_{k,m} e^{ik(x-y)} dy d\nu(t) \\
&= \sum_{n=m-3}^{m+3} \int d_n(t) \sum_{k \in \mathbb{Z}} y_{k,n} g_{k,m} e^{ik(x-t)} d\nu_n(t) \\
&= \sum_{j=-3}^{3} \int d_{m+j}(t) \sum_{k \in \mathbb{Z}} y_{k,m+j} g_{k,m} e^{ik(x-t)} d\nu_{m+j}(t) \\
&= \sum_{j=-3}^{3} \int d_{m+j}(t) \sum_{k \in \mathbb{Z}} \psi_j(k/2^m) e^{ik(x-t)} d\nu_{m+j}(t). \qquad (4.13)
\end{aligned}$$

Now, we observe that each of the functions ψ_j is a q times iterated integral of a function of bounded variation. Therefore, each of the matrix $(\psi_j(k/2^m))$ satisfies (2.9) and (2.10), and Proposition 2.1(c) implies that for $j = 0, \pm 1, \pm 2, \pm 3$,

$$\left| \sum_{k \in \mathbb{Z}} \psi_j(k/2^m) e^{ik(x-t)} \right| \leq c 2^{-mq} |x - t|^{-q-1}, \qquad x \neq t \bmod 2\pi.$$

Let J be the interval, centered at π, and having length $|I|/2$. Then for $x \in J$ and $j = 0, \pm 1, \pm 2, \pm 3$,

$$\left| \int_{t \in [0,2\pi] \setminus I} d_{m+j}(t) \sum_{k \in \mathbb{Z}} \psi_j(k/2^m) e^{ik(x-t)} d\nu_n(t) \right| \leq c(I) 2^{-mq} \|d_{m+j}\|_{p,\nu_{m+j}} \leq c(I) 2^{-mq}. \qquad (4.14)$$

Let $\chi(t)$ the characteristic function of I. Using Lemma 4.4, (2.16), and (2.9), we obtain that for $x \in J$ and $j = 0, \pm 1, \pm 2, \pm 3$,

$$\begin{aligned}
&\left| \int d_{m+j}(t) \sum_{k \in \mathbb{Z}} \psi_j(k/2^m) e^{ik(x-t)} d\nu_{m+j}(t) \right| \\
&\leq \left| \int d_{m+j}(t) \chi(t) \sum_{k \in \mathbb{Z}} \psi_j(k/2^m) e^{ik(x-t)} d\nu_{m+j}(t) \right| + \frac{c(I)}{2^{mq}} \\
&\leq c(I) \{ \|d_{m+j}\|_{I,\nu_{m+j}} + 2^{-mq} \}. \qquad (4.15)
\end{aligned}$$

Along with (4.13), this implies that

$$\|\tau_m(H,f)\|_{J,p} \le c(I)\left\{\sum_{j=-3}^{3}\|d_{m+j}\|_{I,\nu_{m+j}} + 2^{-mq}\right\}.$$

Therefore, $\{\|\tau_m(H,f)\|_{J,p}\} \in \mathsf{b}_{\rho,\alpha}$, and Theorem 2.1 implies that $f \in B_{p,\rho,\alpha}(\pi)$. □

In order to prove Theorem 2.4, we first prove the analogue of Proposition 2.1.

Proposition 4.2 *If H be a dyadic summability matrix, then the sequence of operators $\{\sigma_n^D(H)\}$ is a good approximation sequence for X^∞.*

Proof. It is obvious from the definition that $\sigma_n^D(H,f) \in \mathbb{H}_{2^n}$ for $f \in X^\infty$. In view of (2.17), we have for $T \in \mathbb{H}_{2^{n-1}}$, $\sigma_n^D(H,T) = \sigma_n(H,T) = T$. The estimates (2.9) and (4.9) lead to the fact that $\|\sigma_n^D(H,f)\|_\infty \le c\|f\|_\infty$. □

Proof of Theorem 2.4. The proof of this theorem is almost a summary of all the arguments we have given so far in this paper. As usual, we assume $x_0 = \pi$. In view of Proposition 4.1, part (a) holds if and only if there is an interval, centered at π, such that for every infinitely often differentiable function ϕ, supported on I, the function $f\phi$ (extended to $\mathbb{R}$ as a 2π-periodic function) is in $B^*_{\infty,\rho,\alpha}$. Since $\{\sigma_n^D(H)\}$ is a good approximation sequence for X^∞, Theorem 4.1 implies that this is equivalent to part (c).

Now, suppose (c) holds, and I be the interval chosen as in that part. Without loss of generality, we may assume that $|I| < \pi/2$. Let J_1 (respectively J) be the interval, centered at π with length equal to $|I|/2$ (respectively, $|I|/4$). We take ψ to be a C^∞ function, supported on I, such that $\psi(x) = 1$ for all $x \in J_1$. Using Lemma 4.2 with ν_n in place of ν, we see that $\|\sigma_n^D(H,(1-\psi)f)\|_{J,\infty} \le c(I,f,H,\psi)2^{-nq}$, and similarly, using Lemma 4.2 with ν_{n-1} in place of ν, $\|\sigma_{n-1}^D(H,(1-\psi)f)\|_{J,\infty} \le c(I,f,H,\psi)2^{-nq}$. Therefore,

$$\begin{aligned}
&\|\tau_n^D(H,f)\|_{J,\infty}\\
&\quad\le \|\tau_n^D(H,f\psi)\|_{J,\infty} + \|\sigma_n^D(H,(1-\psi)f)\|_{J,\infty} + \|\sigma_{n-1}^D(H,(1-\psi)f)\|_{J,\infty}\\
&\quad\le \|\tau_n^D(H,f\psi)\|_\infty + \frac{c(I,f,H,\psi)}{2^{nq}}.
\end{aligned}$$

Since $q > \alpha$, both $\{\|\tau_n^D(H,\psi f)\|_\infty\}$ and $\{2^{-nq}\}$ are in $\mathsf{b}_{\rho,\alpha}$. Therefore, $\{\|\tau_n^D(H,f)\|_{J,\infty}\} \in \mathsf{b}_{\rho,\alpha}$, and part (b) is proved.

Next, let part (b) hold, and I be the interval as in that part. Without loss of generality, we may assume that $|I| < \pi/2$. Let ϕ be any infinitely often differentiable function supported on I, and extended to $\mathbb{R}$ as a 2π-periodic function. The part (c) follows from Lemma 4.3 applied with $p = \infty$ and $\sigma_n^D(H)$ in place of U_n.

The equivalence of part (d) with part (b) and that of part (e) with part (c) follows immediately from Lemma 4.5 as in the proof of Theorem 2.2. □

References

[1] J. Bergh and J. Löfström, "Interpolation spaces, an introduction", Springer Verlag, Berlin, 1976.

[2] C. de Boor, "A practical guide to splines", Springer Verlag, New York, 1978.

[3] C. K. Chui and H. N. Mhaskar, *On trigonometric wavelets,* Constr. Approx. **9** (1993), 167-190.

[4] R. A. DeVore and G. G. Lorentz, "Constructive approximation", Springer Verlag, Berlin, 1993.

[5] K. S. Eckhoff, *Accurate reconstructions of functions of finite regularity from truncated Fourier series expansions,* Math. Comp., **64** (1995), 671–690.

[6] K. S. Eckhoff, *On a high order numerical method for functions with singularities,* Math. Comp., **67** (1998), 1063–1087.

[7] A. Gelb and E. Tadmor, *Detection of edges in spectral data,* Appl. Comput. Harmon. Anal. 7 (1999), no. 1, 101–135.

[8] A. Gelb and E. Tadmor, *Enhanced spectral viscosity approximations for conservation laws,* Proceedings of the Fourth International Conference on Spectral and High Order Methods (ICOSAHOM 1998) (Herzliya). Appl. Numer. Math. 33 (2000), no. 1-4, 3–21.

[9] Y. W. Koh, S. L. Lee and H. H. Tan, *Periodic orthogonal splines and wavelets,* Appl. Comput. Harmonic Anal. **2** (1995), 201-218.

[10] H. Krim and M. Vidberg, *Sensor array signal processing: two decades later,* Laboratory for Information and Decision Systems, Massachusetts Institute of Technology, LIDS P-2282, 1995.

[11] G. Kvernadze, *Approximation of the singularities of a bounded function by the partial sums of its differentiated Fourier series,* Appl. Comput. Harmon. Anal. 11 (2001), no. 3, 439–454.

[12] G. Kvernadze, *Determination of the jumps of a bounded function by its Fourier series,* J. Approx. Theory 92 (1998), no. 2, 167–190.

[13] C. Lanczos, "Applied analysis", Dover, New York, 1988.

[14] H. N. Mhaskar, F. J. Narcowich and J. D. Ward, *Spherical Marcinkiewicz-Zygmund inequalities and positive quadrature,* Math. Comp. **70** (2001), no. 235, 1113–1130. (Corrigendum: Math. Comp. **71** (2001), 453–454.)

[15] H. N. Mhaskar and J. Prestin, *On the detection of singularities of a periodic function,* Advances in Computational Mathematics, **12** (2000), 95–131.

[16] F. J. Narcowich and J. D. Ward, *Wavelets associated with periodic basis functions,* Appl. Comput. Harmonic Anal. **3** (1996), 40-56.

[17] S. U. Pillai, "Array signal processing", Springer Verlag, New York, 1989.

[18] G. Plonka and M. Tasche, *A unified approach to periodic wavelets,* in Wavelets: theory, algorithms, and applications, (Eds. C. K. Chui, L. Montefusco, and L. Puccio), Academic Press, New York, 1994, 137–151.

[19] J. Prestin and E. Quak, *Trigonometric interpolation and wavelet decompositions,* Numerical Algorithms, **9** (1995), 293–318.

[20] H.-J. Schmeisser and H. Triebel, "Topics in Fourier Analysis and Function Spaces", Akademische Verlagsgesellschaft Geest& Portig, Leipzig, 1987.

[21] J. Prestin and K. Selig, *Interpolatory and orthonormal trigonometric wavelets,* in: Signal and Image Representation in Combined Spaces (Eds. J. Zeevi and R. Coifman), Academic Press, New York, 1998, 201–255.

[22] A. Zygmund, "Trigonometric Series", Cambridge University Press, Cambridge, 1977.