LEBESGUE CONSTANTS FOR AN ORTHOGONAL POLYNOMIAL SCHAUDER BASIS

Roland Girgensohn[1] and Jürgen Prestin[1]

Dedicated to Professor P. L. Butzer on the occasion of his 70th birthday

Abstract. We examine a certain class of Schauder bases for the space $C[-1,1]$ consisting of algebraic polynomials orthogonal with respect to the Chebycheff weight of the first kind. We give an improved estimate for its Lebesgue constants.

Key words: Orthogonal polynomials, polynomial Schauder bases, Lebesgue constants, Chebycheff polynomials

AMS (1991) subject classification: Primary 41A10, 41A44, Secondary 42C15, 42C20

[1]GSF - National Research Center for Environment and Health, Institute of Biomathematics and Biometry, Ingolstädter Landstraße 1, D-85764 Neuherberg, Germany. e-mail: `girgen@gsf.de`, `prestin@gsf.de`

1. Introduction

Recently (see [3]), a class of Schauder bases for the space $(C[-1,1], \|\cdot\|_\infty)$, consisting of algebraic polynomials and orthogonal with respect to the Chebycheff weight of the first kind, has been constructed. These bases are of optimal degree in the sense that for given $\varepsilon > 0$ there is a basis $\{p_\mu\}$ with $\deg p_\mu \leq (1+\varepsilon)\,\mu$ for all $\mu \in I\!N_0$. A crucial step in the construction is the proof that its Lebesgue constants (i.e., the norms of its partial sum operators) are uniformly bounded. The purpose of the present paper is to give a much simplified proof of this fact, a proof which allows us to give explicit estimates for these Lebesgue constants.

Investigation of polynomial Schauder bases of optimal degree is quite recent. It was in 1987 that Privalov [6] showed that $\deg p_\mu = \mu$ is impossible for *any* polynomial basis $\{p_\mu\}$ of $C[-1,1]$ (orthogonal or not). In fact, he showed that there exists an $\varepsilon > 0$ such that $\max\limits_{\mu \leq n} \deg p_\mu \geq (1+\varepsilon)\, n$ if n is large enough. In 1990, Privalov constructed a basis for $C_{2\pi}$ of trigonometric polynomials of optimal degree [7]; in 1994, Lorentz and Sahakian, based on wavelet methods introduced by Offin and Oskolkov [5], gave an orthogonal trigonometric basis for $C_{2\pi}$ of optimal degree [4]; and in 1996, Kilgore, Prestin and Selig constructed an algebraic polynomial basis for $C[-1,1]$ of optimal degree, orthogonal with respect to the Chebycheff weight of the first kind [3, 8].

One might ask if such polynomial bases are possible with orthogonality regarding other weight functions w, for example Jacobi weights in general. This question is still open. An important step in any proof that a given orthogonal sequence of polynomials $\{p_\mu\}$ is a Schauder basis is the proof that its Lebesgue constants are uniformly bounded. The Lebesgue constants are given by $L_n := \sup\limits_{x\in[-1,1]} \| \sum\limits_{\mu=0}^{n} p_\mu(x) p_\mu(\,\cdot\,)\|_{L^1_w}$, where $\|f\|_{L^1_w} := \int_{-1}^{1} |f(t)|\, w(t)\, dt$. We note that the constant L_n is the norm of the orthogonal projection onto the subspace spanned by $\{p_0, \ldots, p_n\}$. In the present paper we give an easy method to estimate these quantities for the type of polynomial sequences introduced in [3]. This will lead us to explicit upper bounds for the L_n, and these can be regarded as the main result of this note. Similar but more refined methods are used in [2] to construct bases with Chebycheff orthogonality of the second, third and fourth kind; we are currently investigating possible generalizations.

In Section 2 we will (following the development in [3]) give a general construction for optimal degree Schauder bases for $C[-1,1]$, orthogonal with respect to the Chebycheff weight of the first kind. We will just give the construction, without many details or proofs; these can be found in [3] or, later, in [2]. Then, in Section 3, we give a new and short proof that the Lebesgue constants are bounded and compute the actual upper bounds which our method leads us to.

We will use the following notations. Let $f \in L^1(I\!R)$. We will denote by $\mathcal{F}(f) \equiv \hat{f}$

the Fourier transform of f, $\mathcal{F}(f)(\theta) \;:=\; \hat{f}(\theta) \;:=\; \int\limits_{-\infty}^{\infty} f(t)\, e^{-it\theta}\, dt$. The $L^1(I\!R)$-norm of f is $\|f\|_1 \;:=\; \int\limits_{-\infty}^{\infty} |f(\theta)|\, d\theta$.

2. Construction of Schauder bases

In what follows, the weight function w is the Chebycheff weight of the first kind on $[-1,1]$, $w(x) = \frac{2}{\pi}(1-x^2)^{-1/2}$. We will always have the transformation $x = \cos(t)$ in mind; for the norm that means

$$\|f\|_{L^1_w} \;=\; \int_{-1}^{1} |f(x)|\, w(x)\, dx \;=\; \frac{2}{\pi}\int_0^{\pi} |f(\cos(t))|\, dt.$$

The Chebycheff polynomials of the first kind, T_n, are defined by $T_n(\cos(t)) = \cos(nt)$ for $n \in I\!N_0$. Then the sequence $\{\frac{1}{\sqrt{2}}T_0, T_1, T_2, T_3, \ldots\}$ is orthonormal with respect to w, i.e., with respect to the scalar product

$$(f,g) \;:=\; \int_{-1}^{1} f(x)\, \overline{g(x)}\, w(x)\, dx \;=\; \frac{2}{\pi}\int_0^{\pi} f(\cos(t))\, \overline{g(\cos(t))}\, dt.$$

It follows from Privalov's result [6], or more directly from known results on uniform convergence of Fourier series, that this sequence of Chebycheff polynomials is not a Schauder basis for $C[-1,1]$.

Thus, for given $\varepsilon > 0$, we now construct a sequence $\{p_\mu\}_{\mu\in I\!N_0}$ of polynomials, orthogonal with respect to $(\cdot,\cdot)$ and such that $\deg p_\mu \le (1+\varepsilon)\,\mu$ for all μ, which will be a Schauder basis of $C[-1,1]$. The main ideas of the construction follow [3] and come from wavelet theory: The polynomials p_μ are constructed via a "multiresolution analysis" of $C[-1,1]$. Choose finite dimensional polynomial subspaces $V^{(j)}, W^{(j)} \subseteq C[-1,1]$ with

$$\begin{array}{ccccccccccc} V^{(0)} & \subseteq & V^{(1)} & \subseteq & \cdots & \subseteq & V^{(j-1)} & \subseteq & V^{(j)} & \subseteq & \cdots, \\ V^{(0)} & \oplus & W^{(1)} & \oplus & & \cdots & & \oplus & W^{(j)} & = & V^{(j)}, \end{array} \qquad (1)$$

that is, $W^{(j)}$ is the orthogonal complement of $V^{(j-1)}$ in $V^{(j)}$. Denote $\dim V^{(j)} = n_j + 1$, $\dim W^{(j)} = 2m_j = n_j - n_{j-1}$, and let $\{\varphi_k^{(j)}\}_{k=0}^{n_j}$ be an orthonormal basis of $V^{(j)}$ and $\{\psi_k^{(j)}\}_{k=0}^{2m_j-1}$ an orthonormal basis of $W^{(j)}$. Then the sequence of polynomials $\{p_\mu\}$ is defined by taking the basis elements of $V^{(0)}, W^{(1)}, W^{(2)}, \ldots$ in this order. That is,

$$p_k := \varphi_k^{(0)} \text{ for } k = 0,\ldots,n_0 \text{ and } p_{n_{j-1}+k+1} := \psi_k^{(j)} \text{ for } j \in I\!N, k = 0,\ldots,2m_j - 1. \quad (2)$$

This construction enables us to give upper bounds for the Lebesgue constants of the basis $\{p_\mu\}$ by estimating the Lebesgue constants of the individual spaces $V^{(j)}, W^{(j)}$:

$$\begin{aligned}\|\sum_{\mu=0}^{n} p_\mu(x)\, p_\mu(\cdot)\|_{L^1_w} &\le \|\sum_{\mu=0}^{n_{j-1}} p_\mu(x)\, p_\mu(\cdot)\|_{L^1_w} + \|\sum_{\mu=n_{j-1}+1}^{n} p_\mu(x)\, p_\mu(\cdot)\|_{L^1_w} \\ &= \|\sum_{k=0}^{n_{j-1}} \varphi_k^{(j)}(x)\, \varphi_k^{(j)}(\cdot)\|_{L^1_w} + \|\sum_{k=0}^{r} \psi_k^{(j)}(x)\, \psi_k^{(j)}(\cdot)\|_{L^1_w},\end{aligned}$$

for $n \in I\!N$, $n_{j-1} < n \le n_j$ and $n = n_{j-1} + r + 1$. Thus we have to estimate

$$\begin{aligned}\sup_{x\in[-1,1]} \|\sum_{k=0}^{r} \varphi_k^{(0)}(x)\, \varphi_k^{(0)}(\cdot)\|_{L^1_w} &\le A_0 \qquad \text{for } r = 0, \ldots, n_0, \\ \sup_{x\in[-1,1]} \|\sum_{k=0}^{n_j} \varphi_k^{(j)}(x)\, \varphi_k^{(j)}(\cdot)\|_{L^1_w} &\le A_1 \qquad \text{for } j \in I\!N_0 \quad \text{and} \\ \sup_{x\in[-1,1]} \|\sum_{k=0}^{r} \psi_k^{(j)}(x)\, \psi_k^{(j)}(\cdot)\|_{L^1_w} &\le A_2 \qquad \text{for } j \in I\!N,\ r = 0, \ldots, 2m_j - 1.\end{aligned} \tag{3}$$

The Lebesgue constants for the basis $\{p_\mu\}$ will then be bounded above by $\max\{A_0, A_1 + A_2\}$.

By (2), the degree of the poynomial p_μ depends on the values of the dimensions m_j and n_j. Computation shows that the conditions $\deg \varphi_k^{(0)} = k$ for $k = 0, \ldots, n_0$ and $3m_j/n_{j-1} \le \varepsilon$ for all $j \in I\!N$ are sufficient for the sequence $\{p_\mu\}$ to be of optimal degree, $\deg p_\mu \le (1+\varepsilon)\,\mu$ for all $\mu \in I\!N_0$. In Section 3, it will turn out that the estimates for the Lebesgue constants, A_1 and A_2, also depend on the dimensions m_j, n_j: the closer the quotient m_j/n_j is to 0, the larger will A_1 and A_2 be. Thus the dimensions must be chosen such that $\inf m_j/n_j > 0$, or, even better, such that the set $\{m_j/n_j : j \in I\!N\}$ is finite. That is always possible.

All of this means that definition of the sequence $\{p_\mu\}$ and computation of the Lebesgue constants can be reduced to considering the individual spaces $V^{(j)}, W^{(j)}$. We will now define these spaces, writing $V^{(M,N)}, W^{(M,N)}$ instead, where M and N stand for the relevant dimensions m_j, n_j.

Choose $M, N \in I\!N$ with $N > 3M$. The spaces $V^{(M,N)}, W^{(M,N)}$ will be defined by way of their orthogonal basis functions $\varphi_k^{(M,N)} \equiv \varphi_k$ and $\psi_k^{(M,N)} \equiv \psi_k$. For the spaces $V^{(M,N)}$, validity of Condition (3) does not depend on the choice of the orthogonal basis; therefore we start with $W^{(M,N)}$ and then just choose $V^{(M,N)}$ such that the orthogonality relations (1) are satisfied.

Our definition of the space $W^{(M,N)}$ is governed by two ideas: 1) Let each basis function ψ_k be an element of span $\{T_{N-3M}, \ldots, T_{N+M}\}$. This makes it relatively simple to check orthogonality by Parseval's identity. 2) Define the ψ_k in such a way that they are localized around different points $y_k \in [-1, 1]$. The better the ψ_k are localized, the smaller the Lebesgue constants of $W^{(M,N)}$ will be, as we will see in Section 3.

This leads to the following definition. Choose a function

$$g : I\!R \to I\!R, \text{ continuous, with } \operatorname{supp} g \subseteq [-2,2] \text{ and } g(0) = 1. \tag{4}$$

Note that this implies $g(2) = g(-2) = 0$, which we will often use implicitly.

To achieve orthogonality, we also have to assume that g has the symmetry properties

$$g^2(1+x) + g^2(1-x) \;=\; 1 \quad \text{and} \quad g^2(-1+x) + g^2(-1-x) \;=\; 1 \tag{5}$$

for all $0 \le x \le 1$. Then define the basis function ψ_k as

$$\psi_k \;:=\; M^{-\frac{1}{2}} \sum_{s=-2M}^{2M} g\left(\frac{s}{M}\right) \cos((s-M)\,\theta_k)\, T_{N-M+s} \quad \text{for } k = 0, \ldots, 2M-1,$$

where $\theta_k := \frac{2k+1}{4M}\pi$. The cosine terms in this linear combination have the effect of localizing the function ψ_k around the point $y_k = \cos(\theta_k)$; they act as a generalized translation, as we will also see in Section 3. Now, after some computations involving the Parseval identity, we get that

$$(\psi_k, \psi_\ell) \;=\; \frac{2}{\pi} \int_0^\pi \psi_k(\cos(t))\, \psi_\ell(\cos(t))\, dt \;=\; \delta_{k,\ell}.$$

In fact, on using the identity $\cos((M+s)\theta_k) = -\cos((3M-s)\theta_k)$ together with the symmetry properties (5), we obtain for $k \neq \ell$,

$$\begin{aligned}
(\psi_k, \psi_\ell) \;&=\; \frac{1}{M} \sum_{s=-2M}^{2M} g^2\left(\tfrac{s}{M}\right) \cos((s-M)\theta_k) \cos(s-M)\theta_\ell) \\
&=\; \frac{1}{M}\left[\frac{1}{2} + \sum_{s=1}^{2M-1} \cos(s\theta_k)\cos(s\theta_\ell) + \frac{1}{2}\cos(2M\theta_k)\cos(2M\theta_\ell)\right] \\
&=\; \frac{1}{2M} \frac{\sin(\theta_k)\sin(2M\theta_k)\cos(2M\theta_\ell) - \sin(\theta_\ell)\cos(2M\theta_k)\sin(2M\theta_\ell)}{\cos(\theta_k) - \cos(\theta_\ell)} \\
&=\; 0,
\end{aligned}$$

and similarly $(\psi_k, \psi_k) = 1$.

In order to define the space $V^{(M,N)}$, let

$$g^*(\tfrac{M}{N}; x) \;:=\; g^*(x) \;:=\; \begin{cases} 1 & \text{if } 0 \le x \le 1 - \frac{M}{N}, \\ g(\frac{N}{M}(x-1)+1) & \text{if } x \ge 1 - \frac{M}{N}, \\ g^*(-x) & \text{if } x \le 0. \end{cases}$$

Note that g^* has the same properties as g, i.e., it satisfies (4) and (5) if g does. Moreover, it is even.

Now we can define the basis functions $\varphi_0 := \frac{1}{\sqrt{2}} T_0$, $\varphi_k := g^*(\frac{k}{N}) T_k + g^*(\frac{2N-k}{N}) T_{2N-k}$ for $k = 1, \ldots, N-1$ and $\varphi_N := T_N$. Then it is easy to see by (5) that these polynomials are an orthonormal system. Note that $\varphi_k = T_k$ for $k = 1, \ldots, N-M$.

The spaces $W^{(M,N)}$ and $V^{(M,N)}$ are now defined as the linear span of these orthonormal basis functions: $W^{(M,N)} := \mathrm{span}\{\psi_0, \ldots, \psi_{2M-1}\}$, $V^{(M,N)} := \mathrm{span}\{\varphi_0, \ldots, \varphi_N\}$.

Now how do we put together the sequence of spaces $W^{(j)} \equiv W^{(m_j,n_j)}$, $V^{(j)} \equiv V^{(m_j,n_j)}$? The spaces are determined by the dimensions m_j, n_j and by the function g used to define their orthogonal bases. For the dimensions, we use the following values. Choose $\eta \in I\!N$ such that $3/2^\eta \le \varepsilon$ and let $n_0 := 2^\eta$, $m_0 := 1$ and $m_j := 2^{[(j-1)/2^{\eta-1}]}$. That is, $(m_1, m_2, m_3, \ldots) = (1, \ldots, 1, 2, \ldots, 2, 4, \ldots, 4, 8, \ldots)$, where every power of 2 appears $2^{\eta-1}$ times. Then, if $j = q \cdot 2^{\eta-1} + r$ with $1 \le r \le 2^{\eta-1}$,

$$\begin{aligned} n_j &= n_0 + 2\sum_{i=1}^{j} m_i = (2^\eta + 2r)\, 2^q, \\ \left\{ \frac{m_j}{n_j} : j \in I\!N \right\} &= \left\{ \frac{1}{2^\eta + 2r} : r = 1, \ldots, 2^{\eta-1} \right\} \quad \text{and} \\ \frac{3m_j}{n_{j-1}} &= \frac{3}{2^\eta + 2r - 2} \le \frac{3}{2^\eta} \le \varepsilon. \end{aligned}$$

It now remains to specify the functions $g \equiv g^{(j)}$ which we use for defining the spaces $W^{(j)}, V^{(j)}$. We will use just two different functions, g and $\tilde{g}$, where g is even and $\tilde{g}$ is constructed out of g: If g with (4) and (5) is given, then $\tilde{g}$ is defined by

$$\tilde{g}(x) := \begin{cases} 0 & \text{if } x \le -3/2, \\ g(2x+1) & \text{if } -3/2 \le x \le -1/2, \\ 1 & \text{if } -1/2 \le x \le 0, \\ g(x) & \text{if } x \ge 0. \end{cases}$$

Note that $\tilde{g}$ also satisfies (4) and (5). We use the function $\tilde{g}$ for the spaces $W^{(j)}, V^{(j)}$ with $j = q \cdot 2^{\eta-1} + 1$, $q \ge 1$, and the function g for all the other spaces.

This completely determines the sequence of spaces $W^{(j)}, V^{(j)}$. It can be checked directly that these spaces satisfy the orthogonality and inclusion relations (1); this also follows from the arguments in [3]. The sequence of polynomials $\{p_\mu\}$ is now defined via (2), where we use the basis elements φ_k and ψ_k as they are given above. In order to prove that $\{p_\mu\}$ indeed is a Schauder basis of $(C[-1,1], \|\cdot\|_\infty)$, all that remains is to prove the relations (3), that is to show that the Lebesgue constants of the spaces $W^{(M,N)}, V^{(M,N)}$ are uniformly bounded. We will do that in Section 3. It turns out that we will need a further condition on the function g in order to bound the Lebesgue constants: we will require g to satisfy

$$\hat{g}, \hat{\tilde{g}}, \mathcal{F}(g^*(\tfrac{1}{2^\eta+2r}; \cdot)) \in L^1(I\!R) \quad \text{for all } 1 \le r \le 2^{\eta-1}. \tag{6}$$

This is satisfied if the compactly supported g is absolutely continuous and g' is of bounded variation, as follows from Lemma 3. All of our examples will be of this type. The simplest example of a function g with (4), (5) and (6) is the function

$$g_c(x) = \begin{cases} \cos(\pi x/4) & \text{if } |x| \le 2, \\ 0 & \text{otherwise.} \end{cases} \tag{7}$$

In [3], there is a small mistake: the basis given there in Definition 2.1 is not orthogonal. (It would be orthogonal if the sine were to be replaced by a cosine; making this change would even simplify the proofs.) Thus we will now give, for the first time correctly, a class of Schauder bases $\{p_\mu\}$ of optimal degree, consisting of orthogonal polynomials. For $\mu \le 2^\eta$, we will have $p_\mu = \varphi_\mu^{(0)}$. For $\mu > 2^\eta$, there are unique numbers $q \in \mathbb{N}$ and $r \in \{1, \ldots, 2^{\eta-1}\}$ with

$$(2^\eta + 2r - 2)\, 2^q \;<\; \mu \;\le\; (2^\eta + 2r)\, 2^q.$$

Then the basis element p_μ will come from the space $W^{(j)}$, where j is given by $j = q \cdot 2^{\eta-1} + r$. More precisely, we will have $p_\mu = \psi_k^{(j)}$ if $\mu = (2^\eta + 2r - 2)\, 2^q + 1 + k$ with $0 \le k \le 2m_j - 1 = 2^{q+1} - 1$.

Theorem 1. *Given $\varepsilon > 0$. Choose $\eta \in \mathbb{N}$ such that $3/2^\eta \le \varepsilon$. Let the sequence of polynomials $\{p_\mu\}_{\mu \in \mathbb{N}_0}$ be defined as follows. For $\mu \le 2^\eta$, set*

$$p_0 \;:=\; \frac{1}{\sqrt{2}} T_0 \quad \textit{and} \quad p_\mu \;:=\; T_\mu \quad \textit{for } \mu > 0.$$

For $\mu > 2^\eta$, find the uniquely determined numbers $q \in \mathbb{N}$, $r \in \{1, \ldots, 2^{\eta-1}\}$ and $k \in \{0, \ldots, 2^{q+1} - 1\}$ with $\mu = (2^\eta + 2r - 2)\, 2^q + 1 + k$, set $M := 2^q$, $N := (2^\eta + 2r)\, 2^q$, $\theta_k := \frac{2k+1}{4M}\pi$, and define

$$p_\mu \;:=\; M^{-\frac{1}{2}} \sum_{s=-2M}^{2M} \tilde{g}\left(\frac{s}{M}\right) \cos((s - M)\,\theta_k)\, T_{N-M+s} \qquad \textit{if } r = 1,$$

$$p_\mu \;:=\; M^{-\frac{1}{2}} \sum_{s=-2M}^{2M} g\left(\frac{s}{M}\right) \cos((s - M)\,\theta_k)\, T_{N-M+s} \qquad \textit{if } r > 1,$$

with an even function g satisfying (4), (5) and (6). Then the sequence $\{p_\mu\}$ is a Schauder basis for $(C[-1,1], \|\cdot\|_\infty)$, consisting of polynomials with $\deg p_\mu \le (1+\varepsilon)\,\mu$, and orthonormal with respect to the Chebycheff weight of the first kind.

3. The Lebesgue constants

It now remains to prove Condition (3) and to find explicit bounds for the constants A_0, A_1, A_2 appearing there.

Lemma 1. *Suppose that g with (4) is such that $\hat{g} \in L^1(\mathbb{R})$. Then*

$$\|\psi_k\|_{L^1_w} \;\le\; \frac{1}{\pi}\,\|\hat{g}\|_1 \quad \text{and} \quad \sup_{x\in[-1,1]} \sum_{k=0}^{2M-1} |\psi_k(x)| \;\le\; \Big(\frac{1}{\pi}+1\Big)\,\|\hat{g}\|_1 .$$

Proof. With the transformation $x = \cos(t)$, and using addition theorems for the cosine, we get

$$\begin{aligned}
\psi_k(x) &= \sum_{s=-2M}^{2M} g(\tfrac{s}{M}) \cos((s-M)\,\theta_k)\, \cos((N-M+s)\,t) \\
&= \frac{1}{2}\sum_{s=-2M}^{2M} g(\tfrac{s}{M}) \cos\Big(s(t+\theta_k) + (N-M)t - M\theta_k\Big) \\
&\qquad + \frac{1}{2}\sum_{s=-2M}^{2M} g(\tfrac{s}{M}) \cos\Big(s(t-\theta_k) + (N-M)t + M\theta_k\Big) \\
&= \frac{1}{2}\,\Re\Big[e^{i((N-M)t-M\theta_k)}\cdot K_M(t+\theta_k) + e^{i((N-M)t+M\theta_k)}\cdot K_M(t-\theta_k)\Big] ,
\end{aligned}$$

where $K_M(t) := \sum_{s=-2M}^{2M} g(\frac{s}{M})\, e^{ist}$. (Recall that g is real-valued.)

Since the support of g is contained in $[-2,2]$, $\hat{g}$ is an entire function of finite degree, and for the total variation of $\hat{g}$ on $\mathbb{R}$, $V(\hat{g})$, we have by the Bernstein inequality that

$$V(\hat{g}) \;=\; \int_{-\infty}^{\infty} |\hat{g}'(t)|\,dt \;\le\; 2\|\hat{g}\|_1$$

(see [9], Chapter 4.8.61.(28)). Thus, $\hat{g} \in L^1(\mathbb{R}) \cap BV(\mathbb{R})$, and we can apply the Poisson summation formula in the following form: The identity

$$K_M(t) \;=\; \sum_{s=-2M}^{2M} g(\tfrac{s}{M})\, e^{ist} \;=\; M \sum_{n=-\infty}^{\infty} \overline{\hat{g}}(M(t+2\pi n)) \tag{8}$$

holds for each $t \in [-\pi,\pi]$. This follows, e.g., from [1], Proposition 5.1.29(ii), by exchanging the roles of f and $\hat{f}$ and applying the inversion formula, Proposition 5.1.10.

This implies, using the theorem of B. Levi in the third step, that

$$\begin{aligned}
\|K_M\|_{L^1[-\pi,\pi]} &:= \int_{-\pi}^{\pi} |K_M(t)|\,dt \;\le\; M\int_{-\pi}^{\pi} \sum_{n=-\infty}^{\infty} |\hat{g}(M(t+2\pi n))|\,dt \\
&= M \sum_{n=-\infty}^{\infty} \int_{-\pi}^{\pi} |\hat{g}(M(t+2\pi n))|\,dt \\
&= M\int_{-\infty}^{\infty} |\hat{g}(Mt)|\,dt \;=\; \int_{-\infty}^{\infty} |\hat{g}(t)|\,dt \;=\; \|\hat{g}\|_1 .
\end{aligned}$$

Now, using in the second step that $\psi_k(\cos(t))$ is an even function, and in the fourth step that K_M is 2π-periodic, we get

$$\begin{aligned}
\|\psi_k\|_{L^1_w} &= \frac{2}{\pi}\int_0^\pi |\psi_k(\cos(t))|\,dt \;=\; \frac{1}{\pi}\int_{-\pi}^{\pi} |\psi_k(\cos(t))|\,dt \\
&\le \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_M(t-\theta_k)|\,dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} |K_M(t+\theta_k)|\,dt \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} |K_M(t)|\,dt \;\le\; \frac{1}{\pi}\|\hat{g}\|_1.
\end{aligned}$$

To estimate the sum $\sum |\psi_k(x)|/\|\psi_k\|^2_{L^2_w}$, we need the so-called Timan inequality (see [9], Chapter 4.9.1.(3)). If $\tau(t,\theta)$ is a trigonometric polynomial of degree n in the second variable, then for all $m \in I\!N$ and t,

$$\sum_{k=0}^{m-1} \left|\tau(t, \tfrac{2k}{m}\,\pi)\right| \;\le\; \left(\frac{m}{2\pi} + n\right)\int_{-\pi}^{\pi} |\tau(t,\theta)|\,d\theta. \tag{9}$$

We apply this, with $m = 4M$, to K_M, which has degree at most $2M$:

$$\begin{aligned}
\sum_{k=0}^{2M-1} \frac{|\psi_k(x)|}{\|\psi_k\|^2_{L^2_w}} &= \frac{1}{M}\sum_{k=0}^{2M-1} |\psi_k(x)| \\
&\le \frac{1}{2M}\left(\sum_{k=0}^{2M-1} |K_M(t + \tfrac{2k+1}{4M}\,\pi)| + \sum_{k=0}^{2M-1} |K_M(t - \tfrac{2k+1}{4M}\,\pi)|\right) \\
&= \frac{1}{2M}\sum_{k=0}^{4M-1} |K_M(t + \tfrac{2k+1}{4M}\pi)| \\
&\le \frac{1}{2M}\left(\frac{4M}{2\pi} + 2M\right)\int_{-\pi}^{\pi} |K_M(t + \tfrac{\pi}{4M} + \theta)|\,d\theta \;\le\; \left(\frac{1}{\pi} + 1\right)\|\hat{g}\|_1.
\end{aligned}$$

□

An upper bound for the constant A_2 in (3) is therefore $\frac{1}{\pi}\left(\frac{1}{\pi}+1\right)\cdot \max\{\|\hat{g}\|_1^2, \|\widehat{g}\|_1^2\}$.

Lemma 2. *Suppose that g with (4) and (5) is such that $\hat{g}, \widehat{g^*} \in L^1(I\!R)$. Then*

$$\sup_{x\in[-1,1]} \left\| \sum_{k=0}^{N} \varphi_k(x)\cdot\varphi_k(\cdot) \right\|_{L^1_w} \;\le\; \frac{1}{2\pi}\left(\|\mathcal{F}(g^{*2})\|_1 + \|\mathcal{F}(2g(1+\cdot)g(1-\cdot))\|_1\right).$$

Proof. We can give an explicit expression for the sum to be estimated. Writing $x = \cos(t)$ and $\xi = \cos(\theta)$, we get

$$\sum_{k=0}^{N} \varphi_k(x)\varphi_k(\xi)$$

$$
\begin{aligned}
&= \frac{1}{2} + \sum_{k=1}^{N-M-1} T_k(x)\,T_k(\xi) + T_N(x)\,T_N(\xi) \\
&\quad + \sum_{k=N-M}^{N-1} \Big(g^*(\tfrac{k}{N})\,T_k(x) + g^*(\tfrac{2N-k}{N})\,T_{2N-k}(x)\Big)\Big(g^*(\tfrac{k}{N})\,T_k(\xi) + g^*(\tfrac{2N-k}{N})\,T_{2N-k}(\xi)\Big) \\
&= \frac{1}{2} + \sum_{k=1}^{2N} g^{*2}(\tfrac{k}{N})\,T_k(x)\,T_k(\xi) + \frac{1}{2}T_N(x)\,T_N(\xi) \\
&\qquad\qquad + \sum_{k=1}^{M} g^*(\tfrac{N-k}{N})g^*(\tfrac{N+k}{N})\,(T_{N-k}(x)\,T_{N+k}(\xi) + T_{N+k}(x)\,T_{N-k}(\xi)) \\
&= \frac{1}{2} + \frac{1}{2}\sum_{k=1}^{2N} g^{*2}(\tfrac{k}{N})(\cos(k(\theta+t)) + \cos(k(\theta-t))) + \frac{1}{2}\cos(N\theta)\cos(Nt) \\
&\quad + \sum_{k=1}^{M} g(\tfrac{M-k}{M})g(\tfrac{M+k}{M})(\cos(N(\theta+t))\,\cos(k(\theta-t)) + \cos(N(\theta-t))\,\cos(k(\theta+t))) \\
&= \frac{1}{4}K_N^*(\theta+t) + \frac{1}{4}K_N^*(\theta-t) + \frac{1}{4}\cos(N(\theta+t))\,K_M^{*,1}(\theta-t) \\
&\qquad\qquad + \frac{1}{4}\cos(N(\theta-t))\,K_M^{*,1}(\theta+t),
\end{aligned}
$$

where for the second sum the addition theorems for the cosine have to be used twice, and where we denote $K_N^*(t) := 1 + 2\sum_{k=1}^{2N} g^{*2}(\frac{k}{N})\cos(kt) = \sum_{k=-2N}^{2N} g^{*2}(\frac{k}{N})e^{ikt}$ and $K_M^{*,1}(t) := 1 + 2\sum_{k=1}^{M} 2g(1-\frac{k}{M})g(1+\frac{k}{M})\cos(kt) = \sum_{k=-M}^{M} 2g(1-\frac{k}{M})g(1+\frac{k}{M})e^{ikt}$.

Thus we get, for every $x \in [-1,1]$,

$$
\begin{aligned}
\Big\|\sum_{k=0}^{N}\varphi_k(x)\varphi_k(\cdot)\Big\|_{L_w^1} &= \frac{2}{\pi}\int_0^\pi \Big|\sum_{k=0}^{N}\varphi_k(x)\varphi_k(\cos(\theta))\Big|\,d\theta \\
&= \frac{1}{\pi}\int_{-\pi}^\pi \Big|\sum_{k=0}^{N}\varphi_k(x)\varphi_k(\cos(\theta))\Big|\,d\theta \\
&\le \frac{1}{4\pi}\Big(\int_{-\pi}^\pi |K_N^*(\theta+t)|\,d\theta + \int_{-\pi}^\pi |K_N^*(\theta-t)|\,d\theta \\
&\qquad\qquad + \int_{-\pi}^\pi |K_M^{*,1}(\theta+t)|\,d\theta + \int_{-\pi}^\pi |K_M^{*,1}(\theta-t)|\,d\theta\Big) \\
&= \frac{1}{2\pi}\Big(\int_{-\pi}^\pi |K_N^*(\theta)|\,d\theta + \int_{-\pi}^\pi |K_M^{*,1}(\theta)|\,d\theta\Big) \\
&\le \frac{1}{2\pi}\left(\|\mathcal{F}(g^{*2})\|_1 + \|\mathcal{F}(2g(1-\cdot)g(1+\cdot))\|_1\right),
\end{aligned}
$$

by the same arguments as before. □

Lemma 3. *Let f be an absolutely continuous function on the real line with compact support and let f' be of bounded total variation on $\mathbb{R}$, i.e., $V(f') < \infty$. Then $\hat{f} \in L^1(\mathbb{R})$ and*

$$\|\hat{f}\|_1 \le 4\sqrt{V(f')\,\|f\|_1}.$$

Proof. Choose $a, b \in \mathbb{R}$ such that $\operatorname{supp} f \subseteq (a,b)$. For all $\theta \ne 0$ we obtain by partial integration in Stieltjes integrals

$$\begin{aligned} |\hat{f}(\theta)| &= \left|\int_a^b f(t)e^{-it\theta}\,dt\right| = \left|\frac{1}{\theta}\int_a^b f'(t)e^{-it\theta}\,dt\right| = \frac{1}{\theta^2}\left|\int_a^b f'(t)\,de^{-it\theta}\right| \\ &= \frac{1}{\theta^2}\left|\int_a^b e^{-it\theta}\,df'(t)\right| \le \frac{1}{\theta^2}V(f'). \end{aligned}$$

If $\|f\|_1 = 0$ there is nothing to prove. Otherwise let $\kappa := \sqrt{\frac{V(f')}{\|f\|_1}} > 0$. Then

$$\begin{aligned} \|\hat{f}\|_1 &= \left(\int\limits_{|\theta|\ge\kappa} + \int\limits_{|\theta|\le\kappa}\right)|\hat{f}(\theta)|\,d\theta \\ &\le V(f')\int\limits_{|\theta|\ge\kappa}\frac{1}{\theta^2}\,d\theta + \int\limits_{|\theta|\le\kappa}\int\limits_{\mathbb{R}}|f(t)|\,dt\,d\theta \\ &\le V(f')\frac{2}{\kappa} + \|f\|_1 2\kappa = 4\sqrt{V(f')\,\|f\|_1}. \end{aligned}$$

□

Lemma 3 gives an easy way to find functions f with Fourier transform in $L^1(\mathbb{R})$, and to estimate $\|\hat{f}\|_1$. This estimate is good enough for most of our purposes, but in order to get good enough bounds for the Lebesgue constants, we have to estimate these norms for one class of functions directly.

Lemma 4. *Let $0 < \lambda \le \frac{1}{2}$ and*

$$v(\lambda;x) := v(x) := \begin{cases} 1, & \text{if } |x| \le 1-\lambda, \\ \cos^2\!\left(\frac{\pi}{4}\left(1+\frac{|x|-1}{\lambda}\right)\right), & \text{if } 1-\lambda < |x| \le 1+\lambda, \\ 0, & \text{otherwise.} \end{cases}$$

Then

$$\|\hat{v}\|_1 \le \frac{8}{\pi}\ln\frac{1}{\lambda} + 14.33.$$

Proof. One computes

$$\hat{v}(\theta) \;=\; \frac{2\pi^2\cos(\lambda\theta)\sin(\theta)}{\theta(\pi^2-4\lambda^2\theta^2)}.$$

Hence

$$\begin{aligned}
\|\hat{v}\|_1 &= 2\pi^2\left(\int\limits_{|\theta|\le\frac{\pi}{2}} + \int\limits_{|\theta|\ge\frac{\pi}{2}}\right)\left|\frac{\cos(\theta)}{\pi^2-4\theta^2}\cdot\frac{\sin(\frac{\theta}{\lambda})}{\theta}\right| d\theta \\
&\le 2\pi^2\left(\max_{|t|\le\frac{\pi}{2}}\left|\frac{\cos(t)}{\pi^2-4t^2}\right|\int\limits_{|\theta|\le\frac{\pi}{2}}\left|\frac{\sin(\frac{\theta}{\lambda})}{\theta}\right| d\theta + \max_{|t|\ge\frac{\pi}{2}}\left|\frac{\sin(\frac{t}{\lambda})}{t}\right|\int\limits_{|\theta|\ge\frac{\pi}{2}}\left|\frac{\cos(\theta)}{\pi^2-4\theta^2}\right| d\theta\right) \\
&\le 4\int_0^{\frac{\pi}{2}}\frac{|\sin(\frac{\theta}{\lambda})|}{\theta}\,d\theta + 4\int_{\frac{\pi}{2}}^{\infty}\left|\frac{\cos(\theta)}{\pi-2\theta}+\frac{\cos(\theta)}{\pi+2\theta}\right| d\theta \\
&\le 4\sum_{k=0}^{\frac{\pi}{2\lambda}}(-1)^k\int_{k\pi}^{(k+1)\pi}\frac{\sin(\theta)}{\theta}\,d\theta + 4\sum_{k=1}^{\infty}(-1)^{k+1}\int_{\frac{(2k-1)\pi}{2}}^{\frac{(2k+1)\pi}{2}}\left(\frac{\cos(\theta)}{\pi-2\theta}+\frac{\cos(\theta)}{\pi+2\theta}\right) d\theta \\
&\le 4\int_0^{\pi}\frac{\sin(\theta)}{\theta}\,d\theta + 4\sum_{k=1}^{\frac{\pi}{2\lambda}}\frac{2}{k\pi} + 2\sum_{k=1}^{\infty}(-1)^{k+1}\int_{(k-1)\pi}^{(k+1)\pi}\frac{\sin(\theta)}{\theta}\,d\theta \\
&\le 6\int_0^{\pi}\frac{\sin(\theta)}{\theta}\,d\theta + \frac{8}{\pi}\left(0.8107+\ln(\frac{\pi}{2\lambda})\right),
\end{aligned}$$

since $\sum_{k=1}^{y}\frac{1}{k} \;\le\; \ln(y)+\gamma+\frac{1}{2(y-1)}$, where γ denotes the Euler-Mascheroni constant. □

Theorem 2. *Assume that the basis* $\{p_\mu\}_{\mu\in I\!N_0}$ *is constructed via Theorem 1, where the constant* $\eta \in I\!N$ *additionally satisfies* $\varepsilon < 3/2^{\eta-1}$.
a) If the function g *is absolutely continuous with a derivative of bounded variation, then*

$$\sup_{x\in[-1,1]}\|\sum_{\mu=0}^{n}p_\mu(x)\,p_\mu(\cdot)\|_{L^1_w} \;\le\; \frac{4}{\pi^2}\ln\frac{1}{\varepsilon}+C,$$

where C *is an absolute constant depending only on* g *and bounded above by*

$$\begin{aligned}
C \;\le\; & 3.29+\frac{2}{\pi}\sqrt{V((g^2-g_c^2)')\|g^2-g_c^2\|_1} \\
& +\frac{4}{\pi}\sqrt{V((g(1+\cdot)g(1-\cdot))')\|g(1+\cdot)g(1-\cdot)\|_1} \qquad (10)\\
& +\frac{16(\pi+1)}{\pi^2}\max\{V(g')\|g\|_1, V(\tilde{g}')\|\tilde{g}\|_1\},
\end{aligned}$$

where g_c is the function given by (7).
b) If we choose $g \equiv g_c$, then the Lebesgue constants satisfy

$$\sup_{x\in[-1,1]} \| \sum_{\mu=0}^{n} p_\mu(x)\, p_\mu(\cdot)\|_{L^1_w} \ \le \ \frac{4}{\pi^2} \ln \frac{1}{\varepsilon} + 80.72.$$

Proof. The proof is based on the splitting (3) and the lemmas above. The first estimate in (3) remains to be shown, but it is well-known that for $n \le 2^\eta < 6/\varepsilon$,

$$\sup_{x\in[-1,1]} \left\| \frac{1}{2} T_0(x) T_0 + \sum_{\mu=1}^{n} T_\mu(x) T_\mu \right\|_{L^1_w}$$
$$= \ \frac{1}{\pi} \int_0^\pi \frac{|\sin(\frac{2n+1}{2}t)|}{\sin(\frac{t}{2})}\, dt \ < \ 1.51 + \frac{4}{\pi^2} \ln n \ \le \ 1.51 + \frac{4}{\pi^2} \ln 2^\eta \ < \ 2.24 + \frac{4}{\pi^2} \ln \frac{1}{\varepsilon}.$$

The standard proof for this estimate follows the same lines as the estimate for $\|\hat{v}\|_1$ in the proof of Lemma 4 and can also be found, e.g., in [1], Proposition 1.2.3.

Hence if we set $A_0 := 2.24 + \frac{4}{\pi^2} \ln \frac{1}{\varepsilon}$, $A_1 := \frac{1}{2\pi} \left(\|\mathcal{F}(g^{*2})\|_1 + \|\mathcal{F}(2g(1+\cdot)g(1-\cdot))\|_1 \right)$ and $A_2 := \frac{\pi+1}{\pi^2} \max\{\|\hat{g}\|_1^2, \|\widehat{\widetilde{g}}\|_1^2\}$, then the estimates in (3) are satisfied, and $\max\{A_0, A_1 + A_2\}$ is an upper bound of the Lebesgue constants for $\{p_\mu\}$. In order to give explicit numerical bounds, we estimate $\|\mathcal{F}(2g(1+\cdot)g(1-\cdot))\|_1$, $\|\hat{g}\|_1$ and $\|\widehat{\widetilde{g}}\|_1$ by Lemma 3, and for $\|\mathcal{F}(g^{*2})\|_1$ we use Lemma 4 with $\lambda = m_j/n_j \ge 1/2^{\eta+1} > \varepsilon/12$:

$$\begin{aligned}
\|\mathcal{F}(g^{*2})\|_1 \ &\le \ \|\mathcal{F}(v)\|_1 + \|\mathcal{F}(g^{*2} - v)\|_1 \\
&\le \ \frac{8}{\pi} \ln \frac{1}{\lambda} + 14.33 + 4\sqrt{V((g^{*2} - v)')\|g^{*2} - v\|_1} \\
&\le \ \frac{8}{\pi} \ln \frac{1}{\varepsilon} + 20.66 + 4\sqrt{V((g^2 - g_c^2)')\|g^2 - g_c^2\|_1},
\end{aligned}$$

since rescaling yields

$$V((g^{*2} - v)') = \frac{1}{\lambda} V((g^2 - g_c^2)') \qquad \text{and} \qquad \|g^{*2} - v\|_1 = \lambda \|g^2 - g_c^2\|_1.$$

If we choose $g \equiv g_c$, then we can use Lemma 4 for $\|\mathcal{F}(g^{*2})||_1$, and we can explicitly compute

$$\mathcal{F}(2g(1+\cdot)g(1-\cdot))(\theta) \ = \ 4\pi \frac{\cos(\theta)}{\pi^2 - 4\theta^2}, \qquad \hat{g}(\theta) \ = \ 8\pi \frac{\cos(2\theta)}{\pi^2 - 16\theta^2},$$

and then, along the lines of the proof of Lemma 4,

$$\|\mathcal{F}(2g(1+\cdot)g(1-\cdot))\|_1 \ = \ \|\hat{g}\|_1 \ = \ 4 \int_0^\pi \frac{\sin t}{t}\, dt.$$

For $\|\hat{\tilde{g}}\|_1$ we have to use Lemma 3, however, which gives us

$$\|\hat{\tilde{g}}\|_1 \ \le\ 4\sqrt{V(\tilde{g}')\|\tilde{g}\|_1} \ =\ 4\sqrt{9+\frac{3\pi}{4}}.$$

Putting all of this together, we obtain

$$\sup_{x\in[-1,1]} \|\sum_{\mu=0}^{n} p_\mu(x)\, p_\mu(\cdot)\|_{L^1_w} \ \le\ \frac{4}{\pi^2}\ln\frac{1}{\varepsilon} + 80.72.$$

□

In [3], the function

$$g(x) \ :=\ \begin{cases} \frac{2-|x|}{\sqrt{2+2(|x|-1)^2}} & \text{if } |x|\le 2, \\ 0 & \text{otherwise,} \end{cases}$$

was used to construct the bases. In this case we obtain directly from Theorem 2,

$$C \ \le\ 82.3.$$

Here we have numerically estimated $V((g^2-g_c^2)') \approx 1.875$; the other quantities can be computed analytically.

References

[1] P.L. Butzer, R.J. Nessel, *Fourier Analysis and Approximation*, Birkhäuser, Basel 1971.

[2] R. Girgensohn, Polynomial Schauder bases for $C[-1,1]$ with Chebycheff orthogonality, *submitted to Appl. Comput. Harmon. Anal.*

[3] T. Kilgore, J. Prestin, K. Selig, Orthogonal algebraic polynomial Schauder bases of optimal degree, *J. Fourier Anal. Appl.* **2** (1996), 597–610.

[4] R.A. Lorentz, A.A. Sahakian, Orthogonal trigonometric Schauder bases of optimal degree for $C(0,2\pi)$, *J. Fourier Anal. Appl.* **1** (1994), 103–112.

[5] D. Offin, K. Oskolkov, A note on orthonormal polynomial bases and wavelets, *Constr. Approx.* **9** (1993), 319–325.

[6] A.A. Privalov, On the growth of degrees of polynomial bases and approximation of trigonometric projectors, *Mat. Zametki* **42** (1987), 207–214. English translation in *Math. notes* **42** (1987), 619–623.

[7] A.A. Privalov, Growth of degrees of polynomial bases, *Mat. Zametki* **48** (1990), 69–78. English translation in *Math. notes* **48** (1990), 1017–1024.

[8] K. Selig, *Periodische Wavelets und eine gradoptimale Schauderbasis*, Dissertation, Universität Rostock 1997, and Shaker Verlag, Aachen 1998.

[9] A.F. Timan, *Theory of Approximation of Functions of a Real Variable*, Pergamon Press, Oxford 1963.