Sharp estimates of approximation of periodic functions in Hölder spaces

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Abstract. The main purpose of the paper is to study sharp estimates of approximation of periodic functions in the Hölder spaces $H_p^{r,\alpha}$ for all $0 < p \le \infty$ and $0 < \alpha \le r$. By using modifications of the classical moduli of smoothness, we give improvements of the direct and inverse theorems of approximation and prove the criteria for the precise order of decrease of the best approximation in these spaces. Moreover, we obtained strong converse inequalities for general methods of approximation of periodic functions in $H_p^{r,\alpha}$.

1. Introduction and notations

Let $\mathbb{T} \cong [0, 2\pi)$ be the torus. As usual, the space $L_p = L_p(\mathbb{T})$, $0 < p < \infty$, consists of measurable complex functions, which are 2π-periodic and

$$\|f\|_p = \left(\frac{1}{2\pi}\int_{\mathbb{T}} |f(x)|^p \mathrm{d}x\right)^{\frac{1}{p}} < \infty.$$

For simplicity, by $L_\infty = L_\infty(\mathbb{T})$ we denote the space of all 2π-periodic continuous functions on $\mathbb{T}$ which is equipped with the norm

$$\|f\|_\infty = \max_{x\in\mathbb{T}} |f(x)|.$$

Let us consider the linear Fourier means

$$\mathcal{L}_n(f, x) = \int_{\mathbb{T}} f(t)K_n(x-t)\mathrm{d}t, \quad x \in \mathbb{T}, \quad f \in L_p, \quad 1 \le p \le \infty, \tag{1.1}$$

which are generated by some kernel

$$K_n(x) = \sum_{\nu=-n}^{n} a_{\nu,n} e^{\mathrm{i}\nu x}, \quad a_{0,n} = 1, \quad n \in \mathbb{N}. \tag{1.2}$$

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To estimate the error of approximation of functions by Fourier means (1.1) one usually uses the classical moduli of smoothness:

$$\omega_k(f,t)_p = \sup_{0<\delta<t} \|\Delta_\delta^k f\|_p,$$

where

$$\Delta_\delta^k f(x) = \sum_{\nu=0}^{k} (-1)^\nu \binom{k}{\nu} f(x+\nu\delta).$$

For a long time, there has been some interest in investigating the approximation of functions by the Fourier means in the Hölder spaces. This interest originated from the study of a certain class of integro-differential equations and from applications in error estimations for singular integral equations (see, for example, [**11**] and [**22**]).

We will say that $f \in H_p^{r,\alpha}$, $0 < p \le \infty$, $0 < \alpha \le r$, $r \in \mathbb{N}$, if $f \in L_p$ and

$$\|f\|_{H_p^{r,\alpha}} = \|f\|_p + |f|_{H_p^{r,\alpha}} < \infty, \tag{1.3}$$

where

$$|f|_{H_p^{r,\alpha}} = \sup_{h>0} \frac{\|\Delta_h^r f\|_p}{h^\alpha} = \sup_{h>0} \frac{\omega_r(f,h)_p}{h^\alpha}.$$

Following the initial works of Kalandiya [**11**] and Prössdorf [**22**] the problems of approximation in Hölder spaces $H_p^{r,\alpha}$ were studied by Ioakimidis [**9**], Rempulska and Walczak [**25**], Bustamante and Roldan [**2**] and many others. One can find an interesting survey on this subject in [**4**], see also [**21**].

The standard results on approximation in $H_p^{r,\alpha}$, $1 \le p \le \infty$, have the following form. If the means $\{\mathcal{L}_n\}$ possess some good properties and $f \in H_p^{r,\alpha}$, then

$$\|f - \mathcal{L}_n(f)\|_{H_p^{r,\alpha}} \le C \sup_{0<h\le 1/n} \frac{\omega_r(f,h)_p}{h^\alpha}, \quad n \in \mathbb{N}, \tag{1.4}$$

(see, e.g., [**22**], [**21**], [**25**] or [**2**]). Note that, the estimates of type (1.4) in general are not sharp. One can verify this by using a function $f \in L_p$ with $f^{(s)} \in H_p^{r,\alpha}$ and sufficiently large s.

It turns out that one can obtain more general and sharper results by using the simple fact that any Fourier multiplier in L_p, $1 \le p \le \infty$, is also a Fourier multiplier in the Hölder spaces $H_p^{r,\alpha}$. Indeed, let $\mathcal{L}$ be any Fourier multiplier in L_p i.e., a bounded linear operator in L_p, which commutes with translates. Then

$$\|\Delta_h^r(\mathcal{L}f)\|_p = \|\mathcal{L}(\Delta_h^r f)\|_p \le \|\mathcal{L}\|_{L_p\to L_p} \|\Delta_h^r f\|_p. \tag{1.5}$$

Thus, some known results about approximation of functions in L_p spaces with $1 \le p \le \infty$ can be easily transferred to the Hölder spaces. For example, let means (1.1) be such that for all $f \in L_p$ and $n \in \mathbb{N}$:

$$\|f - \mathcal{L}_n(f)\|_p \asymp \omega_k\left(f, \frac{1}{n}\right)_p, \tag{1.6}$$

where, as usual, $A(f,n) \asymp B(f,n)$ means that there exist positive constants c and C such that $cA(f,n) \le B(f,n) \le CA(f,n)$ for all f and n. Then, by using (1.5) and (1.6), one can easily derive

$$\|f - \mathcal{L}_n(f)\|_{H_p^{r,\alpha}} \asymp \omega_k\left(f, \frac{1}{n}\right)_p + \sup_{h>0} h^{-\alpha} \omega_k\left(\Delta_h^r f, \frac{1}{n}\right)_p \tag{1.7}$$

(see Theorem 4.2).

Note that the inequalities of the form (1.6) were first obtained by Trigub in [**33**]. See also the results of Ditzian and Ivanov [**6**], in which such inequalities are called strong converse inequalities.

It is easy to see that (1.7) for arbitrary k independently of r and with a two-sided estimate is an essential improvement of an estimate like (1.4). In this sense the problem of approximation in the Hölder spaces $H_p^{r,\alpha}$ for $1 \le p \le \infty$ is an easy problem. But the analogous problem of approximation of functions in $H_p^{r,\alpha}$ with $0 < p < 1$ is much more difficult because the spaces L_p with $0 < p < 1$ are essentially different from the spaces L_p with $p \ge 1$: as it was mentioned by Peetre [**19**] these spaces are "pathological" spaces.

The main purpose of the paper is to study sharp estimates of approximation of periodic functions in the Hölder spaces $H_p^{r,\alpha}$ for all $0 < p \le \infty$ and $0 < \alpha \le r$. By using modifications of the classical moduli of smoothness, we give improvements of known direct and inverse theorems of approximation (see [**2**] and [**3**]) and prove the criteria of the precise order of decrease of the best approximation in $H_p^{r,\alpha}$. We also obtain estimates like (1.7) for the general Fourier means (1.1) and for the families of linear polynomial means

$$\mathcal{L}_{n,\lambda}(f,x) = \frac{1}{4n+1}\sum_{j=0}^{4n} f(t_j+\lambda)K_n(x-t_j-\lambda), \tag{1.8}$$

where $t_j = 2\pi j/(4n+1)$, $\lambda \in \mathbb{R}$, and the kernel K_n is as in (1.2). These means were intensively studied in several papers of Runovski and Schmeisser (see, e.g. [**26**], [**27**]).

The paper is organized as follows: The auxiliary results are formulated and proved in Section 2. In Section 3 we prove the analogs in $H_p^{r,\alpha}$ of some classical theorems of approximation theory. We also obtain some new properties of the best approximation in $H_p^{r,\alpha}$. In Section 4 we prove the two-sided inequalities for approximation of functions by some linear summation methods of Fourier series and by families of linear polynomial means. In Section 5 we consider some corollaries and make concluding remarks.

We denote by c and C positive constants depending on the indicated parameters. We also denote $p_1 = \min(p,1)$.

2. Preliminary remarks and auxiliary results

Let us recall several properties of the Hölder spaces $H_p^{r,\alpha}$, $0 < p \le \infty$.

LEMMA 2.1. *(See* [**5**, Ch. 2, Theorem 10.1 for the case $q=\infty$]*.) Let* $0 < p \le \infty$, $0 < \alpha < r < k$, *and* $k, r \in \mathbb{N}$. *Then the (quasi)-norms of a function in* $H_p^{r,\alpha}$ *and* $H_p^{k,\alpha}$ *are equivalent.*

Lemma 2.1 does not hold in the case $\alpha = r$. However, there exist the following descriptions of $H_p^{r,r}$ for $1 \le p < \infty$.

LEMMA 2.2. *(See* [**5**, Ch. 2, §9] *and* [**1**]*.) Let* $r \in \mathbb{N}$. *Then*

(1) *Suppose* $1 < p < \infty$. *Then* $f \in H_p^{r,r}$ *iff* $f^{(r-1)} \in AC$ *and* $f^{(r)} \in L_p$. *Moreover,*

$$\|f\|_{H_p^{r,r}} = \|f\|_p + \|f^{(r)}\|_p.$$

(2) $f \in H_1^{r,r}$ *iff* $f^{(r-2)} \in AC$ *(in the case* $r \ge 2$*) and* $f^{(r-1)}$ *is a function of bounded variation on* $[0,2\pi]$. *Moreover,*

$$\|f\|_{H_1^{r,r}} = \|f\|_1 + \mathrm{var}_{[0,2\pi)} f^{(r-1)}.$$

At the same time, if $1 \le p \le \infty$, $r \in \mathbb{N}$, and $f \in L_p$ is such that $\omega_r(f,h)_p = o(h^r)$, then $f \equiv \text{const}$. In the case $0 < p < 1$ the situation is totally different. In [**23**], it was shown that for any $r\in\mathbb{N}$ and $\alpha\in(0,r-1+1/p]$ there exists a function $f\in L_p$ such that $\omega_r(f,h)_p=\mathcal{O}(h^\alpha)$. In particular, for

$$f(x)=\sum_{\nu=0}^\infty \frac{\sin((2\nu+1)x-\pi(r-1)/2)}{(2\nu+1)^r} \tag{2.1}$$

it is known from [**23**] that

$$\omega_r(f,h)_p=\mathcal{O}(h^{r-1+\frac1p}). \tag{2.2}$$

For the complete description of functions satisfying condition (2.2) see [**17**] for the case $r=1$ and [**12**] for $r\ge 2$.

Note that, in spite of the fact that the Hölder spaces $H_p^{r,\alpha}$ make sense for all $\alpha\in(0,r-1+1/p]$ we will consider only the case $\alpha\le r$ (see Remark 5.1).

Let $\mathcal{T}_n$ be the set of all trigonometric polynomials of order at most n

$$\mathcal{T}_n=\left\{T(x)=\sum_{\nu=-n}^n c_\nu \mathrm{e}^{\mathrm{i}\nu x}\ :\ c_\nu\in\mathbb{C}\right\},$$

and let

$$E_n(f)_p=\inf_{T\in\mathcal{T}_n}\|f-T\|_p$$

be the error of the best approximation of a function f in L_p by trigonometric polynomials of order at most n. A trigonometric polynomial $T_n\in\mathcal{T}_n$ is called polynomial of best approximation in L_p, if

$$\|f-T_n\|_p=E_n(f)_p.$$

Note that for any $f\in L_p$ and $n\in\mathbb{N}\cup\{0\}$ such polynomials always exist (see [**5**, Ch. 3, §1]).

Recall the Jackson-type theorem in L_p-spaces, see [**28**] for $1\le p\le\infty$ and [**30**] for $0<p<1$ (see also [**29**] and [**10**]).

Theorem A. *Let $f\in L_p$, $0<p\le\infty$, $k\in\mathbb{N}$, and $n\in\mathbb{N}$. Then*

$$E_n(f)_p\le C\omega_k\left(f,\frac1n\right)_p, \tag{2.3}$$

where C is a constant independent of n and f.

We will use the following Stechkin–Nikolskii type inequality (see [**7**]).

Theorem B. *Let $0<p\le\infty$, $n\in\mathbb{N}$, $0<h\le\pi/n$, and $r\in\mathbb{N}$. Then the following two-sided inequality holds for any trigonometric polynomial $T_n\in\mathcal{T}_n$*

$$h^r\|T_n^{(r)}\|_p\asymp\|\Delta_h^r T_n\|_p, \tag{2.4}$$

where $\asymp$ is a two-sided inequality with absolute constants independent of T_n and h. Moreover, if T_n is a polynomial of the best approximation of a function $f\in L_p$, then we have

$$\|\Delta_h^r T_n\|_p\le C\omega_r\left(f,\frac1n\right), \tag{2.5}$$

where C is a constant independent of T_n, h, and f.

In our paper we will often use the following two properties of moduli of smoothness (see [**5**, Ch. 2, § 7 and Ch. 12, § 5] or [**32**, Ch. 4]). Let $f\in L_p$, $0<p\le\infty$, $r\le k$, $k,r\in\mathbb{N}$. Then

$$\omega_k(f,h)_p\le 2^{\frac{k-r}{p_1}}\omega_r(f,h)_p\le 2^{\frac{k}{p_1}}\|f\|_p,\quad h>0, \tag{2.6}$$

$$\omega_r(f,\lambda h)_p\le r^{\frac1{p_1}-1}(1+\lambda)^{\frac1{p_1}+r-1}\omega_r(f,h)_p,\quad h>0,\quad \lambda>0. \tag{2.7}$$

In this paper, we will deal with functions in $L_p(\mathbb{T}^2)$ which depend additionally on a parameter $\lambda \in \mathbb{T}$. We denote by $\|\cdot\|_{\overline{p}}$ the p-(quasi-)norm with respect to both the main variable $x \in \mathbb{T}$ and the parameter $\lambda \in \mathbb{T}$, i.e.

$$\|\cdot\|_{\overline{p}} = \|\|\cdot\|_{p;\,x}\|_{p;\,\lambda}\,,$$

where $\|\cdot\|_{p;\,x}$ and $\|\cdot\|_{p;\,\lambda}$ are the p-norms (quasi-norms, if $0<p<1$) with respect to x and λ, respectively.

We will understand the error of approximation of $f \in L_p$ by a family of linear operators $\{\mathcal{L}_{n,\,\lambda}\}_{n\in\mathbb{N},\,\lambda\in\mathbb{T}}$ in the Hölder spaces $H_p^{r,\alpha}$ in the following sense:

$$\|f-\mathcal{L}_{n,\,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}} = \|f-\mathcal{L}_{n,\,\lambda}(f)\|_{\overline{p}} + |f-\mathcal{L}_{n,\,\lambda}(f)|_{H_{\overline{p}}^{r,\alpha}}, \tag{2.8}$$

where

$$|f-\mathcal{L}_{n,\,\lambda}(f)|_{H_{\overline{p}}^{r,\alpha}} = \sup_{h>0}\frac{\|\Delta_h^r(f-\mathcal{L}_{n,\,\lambda}(f))\|_{\overline{p}}}{h^\alpha}. \tag{2.9}$$

As usual, the norm of a linear and bounded operator $\mathcal{L}_n$ in L_p is given by

$$\|\mathcal{L}_n\|_{(p)} = \sup_{\|f\|_p\le 1}\|\mathcal{L}_n(f)\|_p.$$

A sequence $\{\mathcal{L}_n\}_{n\in\mathbb{N}}$ is said to be bounded if the sequence of its norms is bounded by some positive constant independent of n. Now let us consider the family of operators $\{\mathcal{L}_{n,\,\lambda}\}$ in $L_p(\mathbb{T}^2)$. We define the (averaged) norm of such a family by

$$\|\{\mathcal{L}_{n,\lambda}\}\|_{(p)} = \sup_{\|f\|_p\le 1}\|\mathcal{L}_{n,\lambda}(f,x)\|_{\overline{p}}.$$

By the analogy, a family $\{\mathcal{L}_{n,\lambda}\}$ is said to be bounded if the sequence of its norms is bounded. Everywhere below $\{\mathcal{L}_{n,\,\lambda}\}$ stands for families of linear polynomial means of type (1.8).

Let us consider the inequalities of the type (1.5). It is well-known that there are no non-trivial Fourier multipliers in L_p with $0<p<1$, but even if we replace in (1.5) the operator $\mathcal{L}$ by some family $\{\mathcal{L}_{n,\lambda}\}$, this inequality does not hold in L_p, $0<p<1$. Indeed, suppose the following inequality holds for some non-trivial family $\{\mathcal{L}_{n,\lambda}\}$

$$\left(\int_{\mathbb{T}}\|\Delta_h^r\mathcal{L}_{n,\lambda}(f)\|_p^p\mathrm{d}\lambda\right)^{\frac1p} \le C_{r,p}\|\Delta_h^r f\|_p,\quad f\in L_p,\quad 0<p<1.$$

Note that, for the function f from (2.1) we have $\|\Delta_h^r f\|_p = \mathcal{O}(h^{r-1+1/p})$, but by Theorem B one only has that $\|\Delta_h^r\mathcal{L}_{n,\lambda}(f)\|_{\overline{p}} \asymp h^r$ as $h\to 0$. Thus, we have a contradiction.

However, the following lemma holds.

Lemma 2.3. *Let $f\in L_p$, $0<p\le\infty$, $0<\alpha\le r$, $r,n\in\mathbb{N}$, and let the family $\{\mathcal{L}_{n,\lambda}\}$ be bounded in L_p. Then*

$$\left(\int_{\mathbb{T}}\sup_{h>0}\|h^{-\alpha}\Delta_h^r\mathcal{L}_{n,\lambda}(f)\|_p^p\mathrm{d}\lambda\right)^{\frac1p} \le C\sup_{h\ge 1/n}\frac{\omega_r(f,h)_p}{h^\alpha}, \tag{2.10}$$

with the usual modification in the case $p=\infty$, where C is a constant independent of α, f and n.

Proof. We treat only the case $0<p<\infty$; similar arguments apply when $p=\infty$.

Let us first prove that for any $n\in\mathbb{N}$

$$\left(\int_{\mathbb{T}}\sup_{h\ge 1/n}\|h^{-\alpha}\Delta_h^r\mathcal{L}_{n,\lambda}(f)\|_p^p\mathrm{d}\lambda\right)^{\frac1p} \le C\sup_{h\ge 1/n}\frac{\omega_r(f,h)_p}{h^\alpha}. \tag{2.11}$$

By [**27**, p. 28], we have

$$\mathcal{L}_{n,\lambda}(T_n) = \mathcal{L}_n(T_n) \tag{2.12}$$

for any $T_n \in \mathcal{T}_n$, $n \in \mathbb{N}$, and $\lambda \in \mathbb{R}$. Therefore, we have also

$$\Delta_h^r \mathcal{L}_{n,\lambda}(f) = \Delta_h^r \mathcal{L}_{n,\lambda}(f - T_n) + \Delta_h^r \mathcal{L}_{n,\lambda}(T_n) = \Delta_h^r \mathcal{L}_{n,\lambda}(f - T_n) + \mathcal{L}_{n,\lambda}(\Delta_h^r T_n) \tag{2.13}$$

for any $h > 0$ and $\lambda \in \mathbb{R}$. By Theorem A, we choose polynomials $T_n \in \mathcal{T}_n$, $n \in \mathbb{N}$, such that

$$\|f - T_n\|_p \le C\omega_r\left(f, \frac{1}{n}\right)_p . \tag{2.14}$$

Thus, using (2.12)–(2.14), and the boundedness of $\{\mathcal{L}_{n,\lambda}\}$, we obtain (2.11) by

$$\begin{aligned}
&\int_{\mathbb{T}} \sup_{h \ge 1/n} \frac{\|\Delta_h^r \mathcal{L}_{n,\lambda}(f)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda \\
&\quad \le C \left(\int_{\mathbb{T}} \sup_{h \ge 1/n} \frac{\|\Delta_h^r \mathcal{L}_{n,\lambda}(f - T_n)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda + \int_{\mathbb{T}} \sup_{h \ge 1/n} \frac{\|\mathcal{L}_n(\Delta_h^r T_n)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda \right) \\
&\quad \le C \left(n^{\alpha p} \int_{\mathbb{T}} \|\mathcal{L}_{n,\lambda}(f - T_n)\|_p^p \mathrm{d}\lambda + \sup_{h \ge 1/n} \int_{\mathbb{T}} \frac{\|\mathcal{L}_{n,\lambda}(\Delta_h^r T_n)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda \right) \\
&\quad \le C \|\{\mathcal{L}_{n,\lambda}\}\|_{(p)}^p \left(n^{\alpha p} \|f - T_n\|_p^p + \sup_{h \ge 1/n} \frac{\|\Delta_h^r T_n\|_p^p}{h^{\alpha p}} \right) \\
&\quad \le C \left(n^{\alpha p} \|f - T_n\|_p^p + \sup_{h \ge 1/n} \frac{\|\Delta_h^r f\|_p^p}{h^{\alpha p}} \right) \\
&\quad \le C \left(n^{\alpha p} \omega_r(f, 1/n)_p^p + \sup_{h \ge 1/n} \frac{\|\Delta_h^r f\|_p^p}{h^{\alpha p}} \right) \le C \sup_{h \ge 1/n} \frac{\omega_r(f, h)_p^p}{h^{\alpha p}}.
\end{aligned}$$

Now, by using Theorem B and (2.11), we derive

$$\begin{aligned}
\int_{\mathbb{T}} \sup_{h > 0} \frac{\|\Delta_h^r \mathcal{L}_{n,\lambda}(f)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda &\le \int_{\mathbb{T}} \left\{ \sup_{0 < h < 1/n} + \sup_{h \ge 1/n} \right\} \frac{\|\Delta_h^r \mathcal{L}_{n,\lambda}(f)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda \\
&\le C \left(n^{\alpha p} \int_{\mathbb{T}} \|\Delta_{1/n}^r \mathcal{L}_{n,\lambda}(f)\|_p^p \mathrm{d}\lambda + \sup_{h \ge 1/n} \frac{\omega_r(f, h)_p^p}{h^{\alpha p}} \right) \\
&\le C \sup_{h \ge 1/n} \frac{\omega_r(f, h)_p^p}{h^{\alpha p}}.
\end{aligned}$$

The last inequality implies (2.10). □

We will also use the following result, which can be obtained by repeating the proofs of formulas (6.1) and (6.3) from [**27**].

LEMMA 2.4. *Let $f \in L_p$, $1 \le p \le \infty$, $n \in \mathbb{N}$, and let $\{\mathcal{L}_{n,\lambda}\}$ be bounded operators in L_p. Then the following equivalence holds*

$$\|f - \mathcal{L}_n(f)\|_p \asymp \|f - \mathcal{L}_{n,\lambda}(f)\|_{\overline{p}}.$$

The lemma below gives an equivalent definition of (2.8)–(2.9).

LEMMA 2.5. *Let $f \in L_p$, $0 < p \le \infty$, $0 < \alpha \le r$, $r, n \in \mathbb{N}$, and let the family $\{\mathcal{L}_{n,\lambda}\}$ be bounded in L_p. Then*

$$|f - \mathcal{L}_{n,\lambda}(f)|_{H_{\overline{p}}^{r,\alpha}} \asymp \left(\int_{\mathbb{T}} |f - \mathcal{L}_{n,\lambda}(f)|^p_{H_p^{r,\alpha}} \mathrm{d}\lambda\right)^{\frac{1}{p}}$$

with the usual modification in the case $p = \infty$.

PROOF. The estimate from above is evident. Let us prove the estimate from below. As in the proof of Lemma 2.3, for the convenience, we treat only the case $0 < p < \infty$.

Let $T_n \in \mathcal{T}_n$, $n \in \mathbb{N}$, be such that $|f - T_n|_{H_p^{r,\alpha}} = \inf_{T \in \mathcal{T}_n} |f - T|_{H_p^{r,\alpha}}$. By using two times Lemma 2.3 and (2.12), we obtain

$$\begin{aligned}
&\int_{\mathbb{T}} |f - \mathcal{L}_{n,\lambda}(f)|^p_{H_p^{r,\alpha}} \mathrm{d}\lambda \\
&\le C\left(|f - T_n|^p_{H_p^{r,\alpha}} + \int_{\mathbb{T}} |T_n - \mathcal{L}_{n,\lambda}(T_n)|^p_{H_p^{r,\alpha}} \mathrm{d}\lambda + \int_{\mathbb{T}} |\mathcal{L}_{n,\lambda}(T_n - f)|^p_{H_p^{r,\alpha}} \mathrm{d}\lambda\right) \\
&\le C\left(|f - T_n|^p_{H_p^{r,\alpha}} + |T_n - \mathcal{L}_n(T_n)|^p_{H_p^{r,\alpha}} + \sup_{h \ge 1/n} \frac{\omega_r(f - T_n, h)_p^p}{h^{\alpha p}}\right) \\
&\le C\left(|f - T_n|^p_{H_p^{r,\alpha}} + \sup_{h>0} \int_{\mathbb{T}} \frac{\|\Delta_h^r(T_n - \mathcal{L}_{n,\lambda}(T_n))\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda\right) \\
&\le C\left(|f - T_n|^p_{H_p^{r,\alpha}} + |f - \mathcal{L}_{n,\lambda}(f)|^p_{H_{\overline{p}}^{r,\alpha}} + \sup_{h>0} \int_{\mathbb{T}} \frac{\|\Delta_h^r \mathcal{L}_{n,\lambda}(f - T_n)\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda\right) \\
&\le C\left(|f - T_n|^p_{H_p^{r,\alpha}} + |f - \mathcal{L}_{n,\lambda}(f)|^p_{H_{\overline{p}}^{r,\alpha}}\right) \le C|f - \mathcal{L}_{n,\lambda}(f)|^p_{H_{\overline{p}}^{r,\alpha}},
\end{aligned}$$

which proves the lemma. □

3. Direct and inverse theorems. Properties of the best approximation in $H_p^{r,\alpha}$

Let $0 < p \le \infty$, $0 < \alpha \le r$, $r, n \in \mathbb{N}$. Denote the best approximation in $H_p^{r,\alpha}$ by

$$E_n(f)_{H_p^{r,\alpha}} = \inf_{T \in \mathcal{T}_n} \|f - T\|_{H_p^{r,\alpha}}$$

and consider the following modulus of smoothness

$$\theta_{k,\alpha}(f, \delta)_p = \sup_{0 < h \le \delta} \frac{\omega_k(f, h)_p}{h^\alpha},$$

which was initially used for the investigation of approximation in the Hölder spaces (see, e.g., [**4**]).

We obtain the following Jackson-type theorem in terms of $\theta_{k,\alpha}(f, h)_p$.

THEOREM 3.1. *Let $f \in H_p^{r,\alpha}$, $0 < p \le \infty$, $0 < \alpha < \min(r, k)$ or $0 < \alpha = k = r$, and $r, k, n \in \mathbb{N}$. Then*

$$E_n(f)_{H_p^{r,\alpha}} \le C\theta_{k,\alpha}\left(f, \frac{1}{n}\right)_p, \tag{3.1}$$

where C is a constant independent of n and f.

PROOF. Let $\alpha < \min(r, k)$ and T_n, $n \in \mathbb{N}$, be polynomials of the best approximation of f in L_p. By Lemma 2.1 and Theorem A it suffices to find an estimation for $|f - T_n|_{H_p^{k,\alpha}}$.

We have

$$|f-T_n|_{H_p^{k,\alpha}}\le\left(\sup_{0<h<1/n}+\sup_{h\ge 1/n}\right)\frac{\|\Delta_h^k(f-T_n)\|_p}{h^\alpha}=S_1+S_2. \tag{3.2}$$

Using Theorem A we obtain

$$S_2\le Cn^\alpha\|f-T_n\|_p\le Cn^\alpha\omega_k(f,1/n)_p\le C\theta_{k,\alpha}(f,1/n)_p. \tag{3.3}$$

Moreover, for S_1 we estimate

$$\begin{aligned}S_1&\le C\left(\sup_{0<h<1/n}\frac{\|\Delta_h^k f\|_p}{h^\alpha}+\sup_{0<h<1/n}\frac{\|\Delta_h^k T_n\|_p}{h^\alpha}\right)\\&\le C\left(\theta_{k,\alpha}(f,1/n)_p+\sup_{0<h<1/n}\frac{\|\Delta_h^k T_n\|_p}{h^\alpha}\right).\end{aligned} \tag{3.4}$$

To estimate the last term in (3.4) we use Theorem B and Theorem A

$$\begin{aligned}\sup_{0<h<1/n}\frac{\|\Delta_h^k T_n\|_p}{h^\alpha}&\le Cn^\alpha\|\Delta_{1/n}^k T_n\|_p\le Cn^\alpha(\|f-T_n\|_p+\|\Delta_{1/n}^k f\|_p)\\&\le Cn^\alpha\omega_k(f,1/n)_p\le C\theta_{k,\alpha}(f,1/n)_p.\end{aligned} \tag{3.5}$$

Thus, combining (3.2)–(3.5), we obtain (3.1) in the case $\alpha<\min(r,k)$. The same scheme one can use in the case $r=\alpha=k$. □

The converse result is given by the following theorem.

THEOREM 3.2. *Let $f\in H_p^{r,\alpha}$, $0<p\le\infty$, $0<\alpha<\min(r,k)$ or $0<\alpha=k=r$, and $r,k,n\in\mathbb{N}$. Then*

$$\theta_{k,\alpha}\left(f,\frac1n\right)_p\le\frac{C}{n^{k-\alpha}}\left(\sum_{\nu=0}^n(\nu+1)^{(k-\alpha)p_1-1}E_\nu(f)_{H_p^{r,\alpha}}^{p_1}\right)^{\frac{1}{p_1}}, \tag{3.6}$$

where C is a constant independent of n and f.

PROOF. As above we consider only the case $\alpha<\min(r,k)$. Let $T_n\in\mathcal{T}_n$, $n\in\mathbb{N}$, be polynomials of the best approximation of f in $H_p^{r,\alpha}$. Choosing $m\in\mathbb{N}\cup\{0\}$ such that $2^m\le n<2^{m+1}$, we get

$$\theta_{k,\alpha}(f,1/n)_p^{p_1}\le\theta_{k,\alpha}(f-T_{2^{m+1}},1/n)_p^{p_1}+\theta_{k,\alpha}(T_{2^{m+1}},1/n)_p^{p_1}. \tag{3.7}$$

By the definition of the Hölder spaces and Lemma 2.1, we obtain

$$\theta_{k,\alpha}(f-T_{2^{m+1}},1/n)_p\le|f-T_{2^{m+1}}|_{H_p^{k,\alpha}}\le E_{2^{m+1}}(f)_{H_p^{k,\alpha}}\le CE_{2^{m+1}}(f)_{H_p^{r,\alpha}}. \tag{3.8}$$

By (2.4) in Theorem B, we get

$$\theta_{k,\alpha}(T_{2^{m+1}},1/n)_p\le Cn^{-(k-\alpha)}\|T_{2^{m+1}}^{(k)}\|_p. \tag{3.9}$$

Using again inequality (2.4) and Lemma 2.1, we gain

$$
\begin{aligned}
\|T^{(k)}_{2^{m+1}}\|_p^{p_1} &\le \|(T_1-T_0)^{(k)}\|_p^{p_1} + \sum_{\mu=0}^{m} \|(T_{2^{\mu+1}}-T_{2^\mu})^{(k)}\|_p^{p_1} \\
&\le C\left(\omega_k(T_1-T_0,1)_p^{p_1} + \sum_{\mu=0}^{m} 2^{\mu k p_1}\omega_k\left(T_{2^{\mu+1}}-T_{2^\mu}, 2^{-(\mu+1)}\right)_p^{p_1}\right) \\
&\le C\left(\|T_1-T_0\|^{p_1}_{H_p^{k,\alpha}} + \sum_{\mu=0}^{m} 2^{\mu(k-\alpha)p_1}\|T_{2^{\mu+1}}-T_{2^\mu}\|^{p_1}_{H_p^{k,\alpha}}\right) \qquad (3.10)\\
&\le C\left(E_0(f)^{p_1}_{H_p^{k,\alpha}} + \sum_{\mu=0}^{m} 2^{\mu(k-\alpha)p_1}E_{2^\mu}(f)^{p_1}_{H_p^{k,\alpha}}\right) \\
&\le C\left(\sum_{\nu=0}^{n}(\nu+1)^{(k-\alpha)p_1-1}E_\nu(f)^{p_1}_{H_p^{r,\alpha}}\right).
\end{aligned}
$$

Thus, combining (3.7)–(3.10), we get (3.6).

□

Now let us consider the problem on the precise order of decrease of the best approximation in $H_p^{r,\alpha}$.

THEOREM 3.3. *Let $f \in H_p^{r,\alpha}$, $0 < p \le \infty$, $0 < \alpha < r$, $\alpha \le s$, and $r, s \in \mathbb{N}$. There exists a constant $L > 0$ such that for any $n \in \mathbb{N}$*

$$
\theta_{s,\alpha}\left(f, \frac{1}{n}\right)_p \le L E_n(f)_{H_p^{r,\alpha}} \qquad (3.11)
$$

iff for some $k > s + 1/p_1 - 1$ there exists a constant $M > 0$ such that for any $h \in (0,1]$

$$
\theta_{s,\alpha}(f,h)_p \le M\theta_{k,\alpha}(f,h)_p. \qquad (3.12)
$$

REMARK 3.1. *Note that the constants M and L in Theorem 3.3 may depend on f.*

PROOF. To prove this theorem one can use the scheme of the proof of the corresponding result from [**24**] in the case $1 \le p \le \infty$ and from [**14**] for $0 < p < 1$. To do this one only needs to apply Theorem 3.1, Theorem 3.2, and the following two properties of $\theta_{k,\alpha}(f,\delta)_p$, which can be easily obtained from (2.6) and (2.7),

$$
\theta_{k,\alpha}(f,\delta)_p \le 2^{\frac{k-r}{p_1}}\theta_{r,\alpha}(f,\delta)_p, \qquad (3.13)
$$

$$
\theta_{k,\alpha}(f,\lambda\delta)_p \le k^{\frac{1}{p_1}-1}(\lambda+1)^{k-\alpha+\frac{1}{p_1}-1}\theta_{k,\alpha}(f,\delta)_p. \qquad (3.14)
$$

Let us present the proof only for the case $0 < p < 1$.

Let condition (3.12) be satisfied. Then from (3.13) and (3.14) we get

$$
\theta_{k,\alpha}(f,\lambda h)_p \le 2^{\frac{k-s}{p}} s^{\frac{1}{p}-1} M(\lambda+1)^{s-\alpha-1+\frac{1}{p}}\theta_{k,\alpha}(f,h)_p \qquad (3.15)
$$

for all $\lambda > 0$ and $h \in (0,1]$.

Let us prove that

$$
\frac{1}{n^{(k-\alpha)p}}\sum_{\nu=0}^{n}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}} \le CM^p\theta_{k,\alpha}\left(f,\frac{1}{n}\right)_p^p, \qquad (3.16)
$$

where C is some positive constant independent of f and n. Indeed, by using Theorem 3.1 and inequality (3.15), we obtain

$$\frac{1}{n^{(k-\alpha)p}}\sum_{\nu=0}^{n}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}} \le \frac{C}{n^{(k-\alpha)p}}\sum_{\nu=0}^{n}(\nu+1)^{(k-\alpha)p-1}\theta_{k,\alpha}\left(f,\frac{1}{\nu+1}\right)_p^p$$
$$\le \frac{CM^p}{n^{(k-s)p+p-1}}\theta_{k,\alpha}\left(f,\frac1n\right)_p^p\sum_{\nu=0}^{n}(\nu+1)^{(k-s)p+p-2}\le CM^p\theta_{k,\alpha}\left(f,\frac1n\right)_p^p.$$

Next, by using Theorem 3.2 and (3.16), we get that for all $m,n\in\mathbb{N}$

$$\theta_{k,\alpha}\left(f,\frac{1}{mn}\right)_p^p \le \frac{C}{(mn)^{(k-\alpha)p}}\sum_{\nu=0}^{mn}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}}$$
$$= \frac{C}{(mn)^{(k-\alpha)p}}\left(\sum_{\nu=n+1}^{mn}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}}+\sum_{\nu=0}^{n}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}}\right) \tag{3.17}$$
$$\le C\left(\frac{1}{(mn)^{(k-\alpha)p}}\sum_{\nu=n+1}^{mn}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}}+\frac{M^p}{m^{(k-\alpha)p}}\theta_{k,\alpha}\left(f,\frac1n\right)_p^p\right).$$

Inequality (3.17) implies that

$$\sum_{\nu=n+1}^{mn}(\nu+1)^{(k-\alpha)p-1}E_\nu(f)^p_{H_p^{r,\alpha}}\ge\frac{(mn)^{(k-\alpha)p}}{C}\theta_{k,\alpha}\left(f,\frac{1}{mn}\right)_p^p-M^pn^{(k-\alpha)p}\theta_{k,\alpha}\left(f,\frac1n\right)_p^p$$

and, by using the monotonicity of $E_n(f)_{H_p^{r,\alpha}}$ and (3.15), we derive

$$E_n(f)^p_{H_p^{r,\alpha}}\sum_{\nu=n+1}^{mn}(\nu+1)^{(k-\alpha)p-1}\ge(Cm^{(k-s)p+p-1}-M^p)n^{(k-\alpha)p}\theta_{k,\alpha}\left(f,\frac1n\right)_p^p.$$

Thus, choosing m appropriately we can find a positive constant $C=C_{k,s,p,\alpha,M}$ such that

$$E_n(f)^p_{H_p^{r,\alpha}}\ge C\theta_{k,\alpha}\left(f,\frac1n\right)_p^p.$$

From the last inequality and (3.12) we obtain (3.11).

The reverse direction is an immediate consequence of Theorem 3.1, which finishes the proof. □

Now let us establish connection between the errors of approximation in the spaces $H_p^{r,\alpha}$ and L_p.

LEMMA 3.1. *Let $f\in H_p^{r,\alpha}$, $0<p\le\infty$, $0<\alpha\le r$, and $r,n\in\mathbb{N}$. Then*

$$C_1n^\alpha E_n(f)_p\le E_n(f)_{H_p^{r,\alpha}}\le C_2\left(n^\alpha E_n(f)_p+\left(\sum_{\nu=n}^{\infty}\nu^{\alpha p_1-1}E_\nu(f)_p^{p_1}\right)^{\frac{1}{p_1}}\right), \tag{3.18}$$

where C_1 and C_2 are some positive constants independent of n and f.

PROOF. Let T_n, $n\in\mathbb{N}$, be such that $\|f-T_n\|_{H_p^{r,\alpha}}=E_n(f)_{H_p^{r,\alpha}}$. Then, by using Theorem A, we obtain

$$n^\alpha E_n(f)_p\le Cn^\alpha\omega_r(f-T_n,1/n)_p$$
$$\le C\sup_{0<h\le 1/n}\frac{\|\Delta_h^r(f-T_n)\|_p}{h^\alpha}\le CE_n(f)_{H_p^{r,\alpha}}.$$

Now let us prove the estimate from above. Let T_n, $n\in\mathbb{N}$, be the polynomials of the best approximation of f in L_p and let $m\in\mathbb{N}$ be such that $2^{m-1}\le n<2^m$. We can write

$$f=T_{2^m}+\sum_{\nu=m}^{\infty}U_{2^\nu}\quad\text{in}\quad L_p, \tag{3.19}$$

where $U_{2^\nu}=T_{2^{\nu+1}}-T_{2^\nu}$. Using (3.19) we have

$$|f-T_n|_{H_p^{r,\alpha}}^{p_1}\le |T_{2^m}-T_n|_{H_p^{r,\alpha}}^{p_1}+\sum_{\nu=m}^{\infty}|T_{2^{\nu+1}}-T_{2^\nu}|_{H_p^{r,\alpha}}^{p_1}=S_1+S_2. \tag{3.20}$$

By using Theorem B, we obtain

$$\begin{aligned} S_1&\le\left(\sup_{0<h<2^{-m}}+\sup_{h\ge 2^{-m}}\right)\frac{\|\Delta_h^r(T_{2^m}-T_n)\|_p^{p_1}}{h^{\alpha p_1}}\\ &\le C2^{\alpha p_1 m}\left(\|\Delta_{2^{-m}}^r(T_{2^m}-T_n)\|_p^{p_1}+\|T_{2^m}-T_n\|_p^{p_1}\right)\\ &\le C2^{\alpha p_1 m}\|T_{2^m}-T_n\|_p^{p_1}\le Cn^{\alpha p_1}E_n(f)_p^{p_1}. \end{aligned} \tag{3.21}$$

Again, by using Theorem B, we have

$$\begin{aligned} S_2&\le\sum_{\nu=m}^{\infty}\sup_{0<h\le 2^{-\nu-1}}h^{-\alpha p_1}\|\Delta_h^rU_{2^\nu}\|_p^{p_1}+\sum_{\nu=m}^{\infty}\sup_{h\ge 2^{-\nu-1}}h^{-\alpha p_1}\|\Delta_h^rU_{2^\nu}\|_p^{p_1}\\ &\le C\sum_{\nu=m}^{\infty}2^{\alpha p_1\nu}\|\Delta_{2^{-\nu-1}}^rU_{2^\nu}\|_p^{p_1}+\sum_{\nu=m}^{\infty}2^{\alpha p_1\nu}\sup_{h\ge 2^{-\nu-1}}\|\Delta_h^rU_{2^\nu}\|_p^{p_1}\\ &\le C\sum_{\nu=m}^{\infty}2^{\alpha p_1\nu}\|U_{2^\nu}\|_p^{p_1}\le C\sum_{\nu=m}^{\infty}2^{\alpha p_1\nu}E_{2^\nu}(f)_p^{p_1}\le C\sum_{\mu=n}^{\infty}\mu^{\alpha p_1-1}E_\mu(f)_p^{p_1}. \end{aligned} \tag{3.22}$$

Thus, combining (3.20)–(3.22), we obtain the upper estimate in (3.18). □

One can see from the above lemma that $E_n(f)_{H_p^{r,\alpha}}$ can tend to zero very fast. But at the same time if for some function $f\in L_p$ we have $\theta_{r,\alpha}(f,\delta)_p=o(\delta^{r-\alpha})$, then $f\equiv\text{const}$. Besides, if $1\le p\le\infty$ and $f\not\equiv\text{const}$, then $\theta_{r,r}(f,\delta)_p\ge C>0$ (see Lemma 2.1). Thus, estimates (3.1) and (3.6) are far from being sharp, because of the failure of $\theta_{r,\alpha}(f,\delta)_p$. We introduce the new "modulus of smoothness", which, as we think, will be more natural and useful in the Jackson-type theorem in the Hölder spaces $H_p^{r,\alpha}$. At least, the idea of this modulus of smoothness works for the strong converse inequalities in the Hölder spaces (see the next section).

Let $0<p\le\infty$, $0<\alpha\le r$, and $r,k\in\mathbb{N}$. Denote

$$\psi_{k,r,\alpha}(f,\delta)_p:=\sup_{0<h\le\delta}\frac{\omega_k(\Delta_h^rf,\delta)_p}{h^\alpha}.$$

It is easy to see that for $f\in L_p$, $r,k\in\mathbb{N}$, $0<\alpha\le r$, and $\delta>0$

$$\theta_{k+r,\alpha}(f,\delta)_p\le\psi_{k,r,\alpha}(f,\delta)_p\le C\theta_{\min(k,r),\alpha}(f,\delta)_p, \tag{3.23}$$

where C is a constant independent of f and δ.

Now let us consider the following improvement of Theorem 3.1. Actually, in view of (3.23), it is an improvement only in the case $\alpha=r$.

THEOREM 3.4. *Let $f\in H_p^{r,\alpha}$, $0<p\le\infty$, $0<\alpha\le r$, and $k,r,n\in\mathbb{N}$. Then*

$$E_n(f)_{H_p^{r,\alpha}}\le C\begin{cases}\psi_{k,r,\alpha}(f,1/n)_p\,, & 1\le p\le\infty,\\ \left(\int_0^{1/n}\left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^p\frac{\mathrm{d}t}{t}\right)^{\frac1p}, & 0<p<1.\end{cases}\tag{3.24}$$

In particular, for all $0<p\le\infty$ we have

$$E_n(f)_{H_p^{r,\alpha}}\le C\left(\int_0^{1/n}\left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1}\frac{\mathrm{d}t}{t}\right)^{\frac{1}{p_1}},\tag{3.25}$$

where C is a constant independent of f and n.

PROOF. Inequality (3.24) in the case $0<p<1$ is a direct consequence of Lemma 3.1 and Theorem A, that is we have

$$E_n(f)_{H_p^{r,\alpha}}^p\le C\left(n^{\alpha p}E_n(f)_p^p+\sum_{\nu=n}^\infty \nu^{\alpha p-1}E_\nu(f)_p^p\right)\le C\int_0^{1/n}\left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^p\frac{\mathrm{d}t}{t}.\tag{3.26}$$

Let us consider the case $1\le p\le\infty$. Let V_n, $n\in\mathbb{N}$, be linear polynomial operators of the form (1.1) such that

$$\|f-V_n(f)\|_p\le C_{k+r}\omega_{k+r}(f,1/n)_p.\tag{3.27}$$

By [**5**, pp. 204-205] such operators always exists.

Thus, by using (3.27), we have

$$\begin{aligned}E_n(f)_{H_p^{r,\alpha}}&\le C\left(\|f-V_n(f)\|_p+|f-V_n(f)|_{H_p^{r,\alpha}}\right)\\&\le C\left(\omega_{r+k}(f,1/n)_p+|f-V_n(f)|_{H_p^{r,\alpha}}\right).\end{aligned}\tag{3.28}$$

It is evident that one only needs to estimate the second term of the right-hand side in (3.28). We have

$$|f-V_n(f)|_{H_p^{r,\alpha}}\le\left(\sup_{0<h<1/n}+\sup_{h\ge1/n}\right)\frac{\|\Delta_h^r(f-V_n(f))\|_p}{h^\alpha}=S_1+S_2.\tag{3.29}$$

Again by using (3.27), we get

$$\begin{aligned}S_2&\le Cn^\alpha\|f-V_n(f)\|_p\le Cn^\alpha\omega_{r+k}(f,1/n)_p=Cn^\alpha\sup_{0<\delta\le1/n}\|\Delta_\delta^k\Delta_\delta^r f\|_p\\&\le Cn^\alpha\sup_{0<h\le1/n}\sup_{0<\delta\le1/n}\|\Delta_\delta^k\Delta_h^r f\|_p\le C\psi_{k,r,\alpha}\left(f,\frac1n\right)_p.\end{aligned}\tag{3.30}$$

To estimate S_1 we use the equality $\Delta_h^rV_n(f)=V_n(\Delta_h^rf)$ and once again (3.27). Thus, we have

$$S_1=\sup_{0<h\le1/n}\frac{\|\Delta_h^rf-V_n(\Delta_h^rf)\|_p}{h^\alpha}\le C\psi_{k,r,\alpha}\left(f,\frac1n\right)_p.\tag{3.31}$$

Hence, combining (3.29)–(3.31), we get (3.24) for $1\le p\le\infty$, and (3.24) is proved.

To prove (3.25) one can use (3.33). □

It turns out that under some natural condition on a function f the modulus of smoothness $\psi_{k,r,\alpha}(f,1/n)_p$ is equivalent to the corresponding integral in (3.25).

LEMMA 3.2. *Let $f \in L_p$, $0 < p \le \infty$, $0 < \alpha \le r$, and $k, r \in \mathbb{N}$. Suppose that there exists a positive constant C independent of f and δ such that*

$$\left(\int_0^\delta \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p_1}} \le C \frac{\omega_{r+k}(f,\delta)_p}{\delta^\alpha}, \quad 0 < \delta < 1. \tag{3.32}$$

Then

$$\left(\int_0^\delta \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p_1}} \asymp \psi_{k,r,\alpha}(f,\delta)_p, \quad 0 < \delta < 1. \tag{3.33}$$

REMARK 3.2. *1) Note that the estimate from below in (3.33), which is used in the proof of Theorem 3.4, holds without assumption (3.32).*

2) Note that condition (3.32) is equivalent to

$$\int_0^\delta \frac{\omega_{r+k}(f,t)_p}{t^\alpha} \frac{\mathrm{d}t}{t} \le C \frac{\omega_{r+k}(f,\delta)_p}{\delta^\alpha}, \quad 0 < \delta < 1,$$

where C is a constant independent of f and δ (see [**31**, Corollary 4.10]*).*

PROOF. The estimation from above can be easily obtained from (3.32) and from the corresponding estimation for $\delta^{-\alpha}\omega_{r+k}(f,\delta)_p$ in (3.30).

Let us prove the estimation from below. Let T_{2^ν}, $\nu \in \mathbb{Z}_+$, be polynomials of the best approximation of f in L_p. Let $n \in \mathbb{Z}_+$ be such that $2^{-(n+1)} \le \delta < 2^{-n}$. We have

$$\psi_{k,r,\alpha}(f,\delta)_p^{p_1} \le \psi_{k,r,\alpha}(f,2^{-n})_p^{p_1} \le \psi_{k,r,\alpha}(T_{2^n},2^{-n})_p^{p_1} + \psi_{k,r,\alpha}(f-T_{2^n},2^{-n})_p^{p_1}. \tag{3.34}$$

To estimate the first term in the last inequality we use Theorem A and Theorem B. We obtain

$$\begin{aligned}\psi_{k,r,\alpha}(T_{2^n},2^{-n})_p &\le C2^{\alpha n}\|\Delta_{2^{-n}}^{r+k}T_{2^n}\|_p \le C2^{\alpha n}\left(\|f-T_{2^n}\|_p + \|\Delta_{2^{-n}}^{r+k}f\|_p\right)\\ &\le C2^{\alpha n}\omega_{r+k}(f,2^{-n})_p \le C\delta^{-\alpha}\omega_{r+k}(f,\delta/2)_p\\ &\le C\left(\int_{\delta/2}^\delta \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p_1}} \le C\left(\int_0^\delta \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1} \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p_1}}.\end{aligned} \tag{3.35}$$

To estimate the second term in (3.34) we use (3.22), (3.26), and (2.7). We conclude

$$\begin{aligned}\psi_{k,r,\alpha}(f-T_{2^n},2^{-n})_p^{p_1} &\le C|f-T_{2^n}|_{H_p^{r,\alpha}}^{p_1} \le C\sum_{\nu=n}^\infty 2^{\alpha p_1 \nu} E_{2^\nu}(f)_p^{p_1}\\ &\le C\int_0^{2^{-n}} \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1} \frac{\mathrm{d}t}{t} \le C\int_0^\delta \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^{p_1} \frac{\mathrm{d}t}{t}.\end{aligned} \tag{3.36}$$

Thus, combining (3.34)–(3.36), we obtain the estimate from below in (3.33). □

From Theorem 3.1 and inequality (3.23), Theorem 3.4 and Lemma 3.2 we deduce the following assertion.

COROLLARY 3.1. *Let $f \in H_p^{r,\alpha}$, $0 < p \le \infty$, $0 < \alpha \le r$, and $k, r \in \mathbb{N}$. For $0 < p < 1$ and $\alpha = r$ suppose also that f satisfies condition (3.32). Then*

$$E_n(f)_{H_p^{r,\alpha}} \le C\psi_{k,r,\alpha}(f,1/n)_p, \tag{3.37}$$

where C is a constant independent of f and n.

Corollary 3.1 implies that one can replace the integral by the modulus $\psi_{k,r,\alpha}(f,1/n)_p$ in (3.24) when $\alpha<r$ and $0<p<1$. We do not know whether it is possible in the case $\alpha=r$. But we are inclined to believe that the answer is negative. Indeed, it is well-known that for any $f\in L_p$, $1<p<\infty$, and for any T_n, $n\in\mathbb{N}$, such that $T_n\to f$ and $T_n'\to g$ in L_p we have

$$\|f-T_n\|_{H_p^{1,1}}\to 0 \quad \text{as} \quad n\to\infty$$

(see Lemma 2.2 and [**18**, Lemma 4.5.5]). In the case $0<p<1$ the situation is totally different. In particular, the following proposition holds:

Proposition 3.1. *Let $0<p<1$. There exist a function $f\in L_p$ and polynomials T_n, $n\in\mathbb{N}$, such that $T_n\to f$ and $T_n'\to g$ in L_p, but*

$$\|f-T_n\|_{H_p^{1,1}}\ge C_f>0 \quad \textit{for sufficiently large} \quad n.$$

Proof. We will use an example from [**8**]. Let

$$f(x)=\begin{cases} x, & x\in[0,\pi),\\ 2\pi-x, & x\in[\pi,2\pi]\end{cases}$$

and $f(x)=f(x+2\pi)$. Let also

$$g_n(x)=\begin{cases} \dfrac{k}{n}, & \dfrac{k}{n}\le x<\dfrac{k+1}{n}-\dfrac{1}{n^2},\\[2mm] \dfrac{k}{n}+\left(x-\dfrac{k+1}{n}+\dfrac{1}{n^2}\right)n, & \dfrac{k+1}{n}-\dfrac{1}{n^2}\le x<\dfrac{k+1}{n}\end{cases}$$

for $k=0,1,\dots,n-1$, $g_n(x)=1-g_n(x-1)$ for $1<x\le 2$, and $\varphi_n(x)=\pi g_n(x/\pi)$ for $x\in[0,2\pi]$.

We will need the following inequalities:

$$\omega_1(\varphi_n,1/n)_p\le Cn^{-1}\|\varphi_n'\|_p\le Cn^{-\frac1p}. \tag{3.38}$$

One can find the first inequality in [**16**], but the second one can be verified by simple calculation. It is also easy to see that

$$\|f-\varphi_n\|_p=\mathcal{O}\left(1/n\right).$$

Let T_n, $n\in\mathbb{N}$, be polynomials of the best approximation of φ_n in L_p, $0<p<1$. By using (2.3) and (3.38), we have

$$\begin{aligned}\|f-T_n\|_p&\le C(\|f-\varphi_n\|_p+\|\varphi_n-T_n\|_p)\\ &\le C(n^{-1}+\omega_1(\varphi_n,1/n)_p)\le Cn^{-1}.\end{aligned} \tag{3.39}$$

Taking into account Theorem B and (3.38), we get

$$\|T_n'\|_p\le Cn\omega_1(\varphi_n,1/n)_p\le Cn^{1-\frac1p}. \tag{3.40}$$

Thus, by using (2.4) and (3.40), we obtain

$$\begin{aligned}\|f-T_n\|_{H_p^{1,1}}^p&\ge \sup_{0<h\le 1/n}\frac{\|\Delta_h^1(f-T_n)\|_p^p}{h^p}\\ &\ge \sup_{0<h\le 1/n}\frac{\|\Delta_h^1 f\|_p^p}{h^p}-\sup_{0<h\le 1/n}\frac{\|\Delta_h^1 T_n\|_p^p}{h^p}\ge \pi^p-C\|T_n'\|_p^p\ge\pi^p-Cn^{p-1}.\end{aligned}$$

□

The following two theorems can be obtained in the same manner as Theorem 3.2 and Theorem 3.3.

THEOREM 3.5. *Let* $f \in H_p^{r,\alpha}$, $0 < p \le \infty$, $0 < \alpha \le r$, $r, k \in \mathbb{N}$, *and* $n \in \mathbb{N}$. *Then*

$$\psi_{k,r,\alpha}\left(f, \frac{1}{n}\right)_p \le \frac{C}{n^k}\left(\sum_{\nu=0}^{n}(\nu+1)^{kp_1-1}E_\nu(f)_{H_p^{r,\alpha}}^{p_1}\right)^{\frac{1}{p_1}},$$

where C *is a constant independent of* f *and* n.

THEOREM 3.6. *Let* $f \in H_p^{r,\alpha}$, $0 < p \le \infty$, $r, s \in \mathbb{N}$, $0 < \alpha \le r$. *Suppose also that* f *satisfies condition (3.32) in the case* $0 < p < 1$ *and* $\alpha = r$. *There exists a constant* $L > 0$ *such that for any* $n \in \mathbb{N}$

$$\psi_{s,r,\alpha}\left(f, \frac{1}{n}\right)_p \le LE_n(f)_{H_p^{r,\alpha}}, \tag{3.41}$$

iff for some $k > s + 1/p_1 - 1$ *there exists a constant* $M > 0$ *such that for any* $h \in (0,1]$

$$\psi_{s,r,\alpha}(f,h)_p \le M\psi_{k,r,\alpha}(f,h)_p. \tag{3.42}$$

In Theorem 3.6 we found the lower estimate for $E_n(f)_{H_p^{r,\alpha}}$ under conditions (3.42) and (3.32). It turns out that for $E_n(f)_{H_p^{r,\alpha}}$ with $0 < p \le 1$ there exists a non-trivial estimate from below for all $f \in H_p^{r,\alpha}$ in terms of the special differences.

PROPOSITION 3.2. *If* $f \in H_p^{r,\alpha}$, $0 < p \le 1$, $0 < \alpha \le r$, *and* $r \in \mathbb{N}$, *then for all* $n \in \mathbb{N}$

$$n^{1-\frac{1}{p}}\sup_{h>0}\frac{\|\widetilde{(\Delta_h^r f)}_n\|_p}{h^\alpha} \le C_pE_n(f)_{H_p^{r,\alpha}},$$

where

$$\widetilde{f}_n(\lambda) = \frac{1}{4n+1}\sum_{j=0}^{4n} f(t_j+\lambda), \quad t_j = t_{j,n} = \frac{2\pi j}{4n+1}.$$

PROOF. Let

$$E_n^0(f)_p = \inf\left\{\|f-T\|_p \,:\, T \in \mathcal{T}_n, \quad \int_{\mathbb{T}} T(t)\mathrm{d}t = 0\right\}.$$

It turns out that

$$c_pn^{1-\frac{1}{p}}\|\widetilde{f}_n\|_p \le E_n^0(f)_p \le C_p\left(n^{1-\frac{1}{p}}\|\widetilde{f}_n\|_p + E_{n/2}(f)_p\right).$$

The estimation from above can be found in [**13**]. To estimate $E_n^0(f)_p$ from below let us note that for any $T_n \in \mathcal{T}_n$, $\int_{\mathbb{T}} T(t)\mathrm{d}t = 0$, we have

$$\widetilde{f}_n(\lambda) = \frac{1}{4n+1}\sum_{j=0}^{4n}(f(t_j+\lambda) - T_n(t_j+\lambda)).$$

Applying this equality, we obtain

$$\|\widetilde{f}_n\|_p^p \le \frac{C}{(4n+1)^p}\sum_{j=0}^{4n}\int_{\mathbb{T}}|f(t_j+\lambda)-T_n(t_j+\lambda)|^p\mathrm{d}\lambda \le Cn^{1-p}\|f-T_n\|_p^p.$$

Now, let $T_n \in \mathcal{T}_n$, $n \in \mathbb{N}$, be polynomials of the best approximation in $H_p^{r,\alpha}$. Thus, from the above inequality we have

$$n^{1-\frac{1}{p}}\sup_{h>0}\frac{\|\widetilde{(\Delta_h^r f)}_n\|_p}{h^\alpha} \le C\sup_{h>0}\frac{E_n^0(\Delta_h^r f)_p}{h^\alpha} \le C\sup_{h>0}\frac{\|\Delta_h^r f - \Delta_h^r T_n\|_p}{h^\alpha} \le CE_n(f)_{H_p^{r,\alpha}},$$

which proves the proposition. □

4. Two-sided estimates of approximation by linear polynomial methods in $H_p^{r,\alpha}$

To formulate the main theorems in this section we need some auxiliary notations. For that purpose let us introduce the general modulus of smoothness.

DEFINITION 4.1. *We will say that* $w = w(\cdot,\cdot)_p \in \Omega_p = \Omega(L_p, \mathbb{R}_+)$, $0 < p \le \infty$, *if*
1) for $f \in L_p$ *and for any* $\delta > 0$ *we have*

$$w(f,\delta)_p \le C\|f\|_p; \tag{4.1}$$

2) for $f, g \in L_p$ *and for any* $\delta > 0$ *we have*

$$w(f+g,\delta)_p \le C(w(f,\delta)_p + w(g,\delta)_p), \tag{4.2}$$

where C *is a constant independent of* f, g, *and* δ.

As a function w we can take, for example, the classical modulus of smoothness $\omega_k(f,\delta)_p$ of arbitrary order k, or a corresponding K-functional or its realization (see [**7**]), but one can also use more artificial objects, which were introduced and studied in [**15**].

DEFINITION 4.2. *For a function* $w \in \Omega_p$ *we define a "modulus of smoothness" related to the Hölder space* $H_p^{r,\alpha}$ *as follows*

$$w(f,\delta)_{H_p^{r,\alpha}} = w(f,\delta)_p + \sup_{h>0} \frac{w(\Delta_h^r f,\delta)_p}{h^\alpha}. \tag{4.3}$$

Let us consider some examples. It turns out that if we take $w(f,\delta)_p = \|f\|_p$, then (4.3) defines the norm in Hölder spaces, which was introduced in (1.3). However, if $w(f,\delta)_p = \omega_k(f,\delta)_p$, then formula (4.3) provides the definition of the corresponding modulus of smoothness in Hölder spaces $H_p^{r,\alpha}$, see the right-hand side of formula (1.7), and if $w(f,1/n) = \|f - \mathcal{L}_{n,\lambda}(f)\|_{\overline{p}}$, then in (4.3) we will have formula (2.8).

Now we are ready to formulate our main result in this section.

THEOREM 4.1. *Let* $0 < p \le \infty$, $0 < \alpha \le r$, *and* $r \in \mathbb{N}$. *Let* $\{\mathcal{L}_{n,\lambda}\}$ *be bounded in* L_p, $w_{\mathcal{L}} \in \Omega_p$, *and let us assume that the following equivalence holds for any* $f \in L_p$ *and* $n \in \mathbb{N}$:

$$\|f - \mathcal{L}_{n,\lambda}(f)\|_{\overline{p}} \asymp w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_p. \tag{4.4}$$

Then for any $f \in H_p^{r,\alpha}$ *and* $n \in \mathbb{N}$ *we have*

$$\|f - \mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}} \asymp w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_{H_p^{r,\alpha}} + \begin{cases} E_n(f)_{H_p^{r,\alpha}}, & 0 < p < 1, \\ 0, & 1 \le p \le \infty. \end{cases} \tag{4.5}$$

PROOF. We start from the case $0 < p < 1$.

Let us first prove the lower bound for $\|f - \mathcal{L}_{\lambda,n}(f)\|_{H_{\overline{p}}^{r,\alpha}}$. It is evident that

$$E_n(f)_{H_p^{r,\alpha}} \le \|f - \mathcal{L}_{\lambda,n}(f)\|_{H_{\overline{p}}^{r,\alpha}}. \tag{4.6}$$

Thus, by (4.6) and (4.4) we only need to prove that

$$\sup_{h>0} \frac{w_{\mathcal{L}}(\Delta_h^r f, 1/n)_p}{h^\alpha} \le \|f - \mathcal{L}_{\lambda,n}(f)\|_{H_{\overline{p}}^{r,\alpha}}. \tag{4.7}$$

By properties (4.1) and (4.2) we obtain

$$\begin{aligned} w_{\mathcal{L}}(\Delta_h^r f, 1/n)_p^p &= \frac{1}{2\pi}\int_{\mathbb{T}} w_{\mathcal{L}}(\Delta_h^r f, 1/n)_p^p \mathrm{d}\lambda \\ &\le C\left(\|\Delta_h^r (f - \mathcal{L}_{n,\lambda}(f))\|_{\overline{p}}^p + \int_{\mathbb{T}} w_{\mathcal{L}}(\Delta_h^r \mathcal{L}_{n,\lambda}(f), 1/n)_p^p \mathrm{d}\lambda\right) \end{aligned} \tag{4.8}$$

and, by (4.4) we arrive at

$$\int_{\mathbb{T}} w_{\mathcal{L}}(\Delta_h^r \mathcal{L}_{n,\lambda}(f), 1/n)_p^p \mathrm{d}\lambda \le C \int_{\mathbb{T}}\int_{\mathbb{T}} \|\Delta_h^r \mathcal{L}_{n,\lambda}(f) - \mathcal{L}_{n,\beta}(\Delta_h^r \mathcal{L}_{n,\lambda}(f))\|_p^p \mathrm{d}\lambda \mathrm{d}\beta. \tag{4.9}$$

Note that for any $\lambda, \beta \in \mathbb{R}$ we have

$$\mathcal{L}_{n,\beta}(\Delta_h^r \mathcal{L}_{n,\lambda}(f)) = \Delta_h^r \mathcal{L}_{n,\beta}(\mathcal{L}_{n,\lambda}(f)) \tag{4.10}$$

and

$$\mathcal{L}_{n,\lambda} - \mathcal{L}_{n,\beta} \circ \mathcal{L}_{n,\lambda} = (\mathcal{L}_{n,\lambda} - I) + (I - \mathcal{L}_{n,\beta}) + \mathcal{L}_{n,\beta} \circ (I - \mathcal{L}_{n,\lambda}), \tag{4.11}$$

where I is the identity operator.

Thus, by (4.9), (4.10) and (4.11), we obtain

$$\begin{aligned} \int_{\mathbb{T}} w_{\mathcal{L}}(\Delta_h^r \mathcal{L}_{n,\lambda}(f), 1/n)_p^p \mathrm{d}\lambda \le C\Big(&\|\Delta_h^r (\mathcal{L}_{n,\lambda}(f) - f)\|_{\overline{p}}^p \\ &+ \|\Delta_h^r (f - \mathcal{L}_{n,\beta}(f))\|_{\overline{p}}^p + \int_{\mathbb{T}}\int_{\mathbb{T}} \|\Delta_h^r \mathcal{L}_{n,\beta}(f - \mathcal{L}_{n,\lambda}(f))\|_p^p \mathrm{d}\lambda \mathrm{d}\beta\Big). \end{aligned} \tag{4.12}$$

By using Lemma 2.3 and Lemma 2.5, we also conclude

$$\begin{aligned} \sup_{h>0} \int_{\mathbb{T}}\int_{\mathbb{T}} \frac{\|\Delta_h^r \mathcal{L}_{n,\beta}(f - \mathcal{L}_{n,\lambda}(f))\|_p^p}{h^{\alpha p}} \mathrm{d}\lambda \mathrm{d}\beta &\le \int_{\mathbb{T}}\int_{\mathbb{T}} |\mathcal{L}_{n,\beta}(f - \mathcal{L}_{n,\lambda}(f))|_{H_p^{r,\alpha}}^p \mathrm{d}\beta \mathrm{d}\lambda \\ &\le C \int_{\mathbb{T}} \sup_{h \ge 1/n} \frac{\omega_r(f - \mathcal{L}_{n,\lambda}(f), h)_p^p}{h^{\alpha p}} \mathrm{d}\lambda \\ &\le C \int_{\mathbb{T}} |f - \mathcal{L}_{n,\lambda}(f)|_{H_p^{r,\alpha}}^p \mathrm{d}\lambda \le C |f - \mathcal{L}_{n,\lambda}(f)|_{H_{\overline{p}}^{r,\alpha}}^p. \end{aligned} \tag{4.13}$$

Thus, combining (4.12) and (4.13), we have

$$\sup_{h>0} \int_{\mathbb{T}} \frac{w_{\mathcal{L}}(\Delta_h^r \mathcal{L}_{n,\lambda}(f), 1/n)_p^p}{h^{\alpha p}} \mathrm{d}\lambda \le C |f - \mathcal{L}_{\lambda,n}(f)|_{H_{\overline{p}}^{r,\alpha}}^p.$$

The last inequality together with (4.8) implies (4.7).

Now, let us prove the upper bound. It is sufficient to prove that

$$|f - \mathcal{L}_{n,\lambda}(f)|_{H_{\overline{p}}^{r,\alpha}} \le C \left(\sup_{h>0} \frac{w_{\mathcal{L}}(\Delta_h^r f, 1/n)_p}{h^\alpha} + \inf_{T \in \mathcal{T}_n} |f - T|_{H_p^{r,\alpha}} \right). \tag{4.14}$$

Let $T_n \in \mathcal{T}_n$ be an arbitrary polynomial. Then

$$\|\Delta_h^r (f - \mathcal{L}_{n,\lambda}(f))\|_{\overline{p}} \le C \left(\|\Delta_h^r (f - T_n)\|_p + \|\Delta_h^r (T_n - \mathcal{L}_{n,\lambda}(T_n))\|_{\overline{p}} + \|\Delta_h^r \mathcal{L}_{n,\lambda}(f - T_n)\|_{\overline{p}} \right). \tag{4.15}$$

By Lemma 2.3, we have

$$|\mathcal{L}_{n,\lambda}(f - T_n)|_{H_{\overline{p}}^{r,\alpha}} \le C |f - T_n|_{H_p^{r,\alpha}}. \tag{4.16}$$

Note also that

$$\Delta_h^r \mathcal{L}_{n,\lambda}(T_n) = \mathcal{L}_{n,\lambda}(\Delta_h^r T_n). \tag{4.17}$$

Thus, by using (4.17) and (4.4), we have for any $h > 0$

$$\|\Delta_h^r (T_n - \mathcal{L}_{n,\lambda}(T_n))\|_{\overline{p}} = \|\Delta_h^r T_n - \mathcal{L}_{n,\lambda}(\Delta_h^r T_n))\|_{\overline{p}} \le C w_{\mathcal{L}}(\Delta_h^r T_n, 1/n)_p \tag{4.18}$$

and, by using (4.1) and (4.2), we obtain

$$w_{\mathcal{L}}(\Delta_h^r T_n, 1/n)_p \le C \left(w_{\mathcal{L}}(\Delta_h^r f, 1/n)_p + \|\Delta_h^r (f - T_n)\|_p \right). \tag{4.19}$$

Combining (4.15), (4.16), (4.18), and (4.19), we get (4.14).

To prove the theorem in the case $1 \le p \le \infty$ one can take into account Lemma 2.4 and the equality

$$\|\Delta_h^r(f-\mathcal{L}_n(f))\|_p = \|\Delta_h^r f - \mathcal{L}_n(\Delta_h^r f)\|_p,$$

which yields the desired result. □

By using the same arguments as in the above proof of Theorem 4.1 one can easily show the following theorem for $p \ge 1$.

THEOREM 4.2. *Let $1 \le p \le \infty$, $0 < \alpha \le r$, and $r \in \mathbb{N}$. Let $\{\mathcal{L}_n\}$ be bounded in L_p, $w_{\mathcal{L}} \in \Omega_p$, and let us assume that the following equivalence holds for any $f \in L_p$ and $n \in \mathbb{N}$:*

$$\|f-\mathcal{L}_n(f)\|_p \asymp w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_p.$$

Then for any $f \in H_p^{r,\alpha}$ and $n \in \mathbb{N}$ we have

$$\|f-\mathcal{L}_n(f)\|_{H_p^{r,\alpha}} \asymp w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_{H_p^{r,\alpha}}.$$

In the case $0 < p < 1$ the problem is more complicated, which is based on the difficulty of sharp estimates for $E_n(f)_{H_p^{r,\alpha}}$. But some relationship can be presented.

By using Theorem 4.1, Theorem 3.4, and Proposition 3.2 we can prove the following result.

COROLLARY 4.1. *Let $0 < p < 1$, $0 < \alpha \le r$, $r, k \in \mathbb{N}$, and $w_{\mathcal{L}} \in \Omega_p$. Suppose also that $\{\mathcal{L}_{n,\lambda}\}$ is bounded in L_p and that the following equivalence holds for any $f \in L_p$ and $n \in \mathbb{N}$:*

$$\|f-\mathcal{L}_{n,\lambda}(f)\|_{\overline{p}} \asymp w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_p. \tag{4.20}$$

Then for any $f \in H_p^{r,\alpha}$ and $n \in \mathbb{N}$ we have the following two-sided estimate:

$$w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_{H_p^{r,\alpha}} + n^{1-\frac{1}{p}} \sup_{h>0} \frac{\|\widetilde{(\Delta_h^r f)}_n\|_p}{h^\alpha} \le C\|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}}, \tag{4.21}$$

$$C\|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}} \le w_{\mathcal{L}}\left(f, \frac{1}{n}\right)_{H_p^{r,\alpha}} + \left(\int_0^{1/n} \left(\frac{\omega_{r+k}(f,t)_p}{t^\alpha}\right)^p \frac{\mathrm{d}t}{t}\right)^{\frac{1}{p}}. \tag{4.22}$$

REMARK 4.1. *Note that the second term in the right-hand side of (4.21)–(4.22) can be replaced by $\theta_{r,\alpha}(f,1/n)_p$ (see Theorem 3.1) or by $\psi_{k,r,\alpha}(f,1/n)_p$, $k \in \mathbb{N}$, if $r > \alpha$ or if condition (3.32) holds for the function f (see Theorem 3.4 and Corollary 3.1).*

For some classes of functions and for the classical moduli of smoothness one can also obtain sharp two-sided estimates. Indeed, by using Theorem 4.1 and Corollary 3.1 and taking into account that $\psi_{k,r,\alpha}(f,h)_p \le \omega_k(f,h)_{H_p^{r,\alpha}}$, we get the following result.

COROLLARY 4.2. *Let $0 < p < 1$, $0 < \alpha \le r$, $r \in \mathbb{N}$, and $k \in \mathbb{N}$. Suppose also that $\{\mathcal{L}_{n,\lambda}\}$ is bounded in L_p and that the following equivalence holds for any $f \in L_p$ and $n \in \mathbb{N}$:*

$$\|f-\mathcal{L}_{n,\lambda}(f)\|_{\overline{p}} \asymp \omega_k\left(f, \frac{1}{n}\right)_p.$$

If $r > \alpha$ or $f \in L_p$ satisfy condition (3.32), then we have the following equivalence:

$$\|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}} \asymp \omega_k\left(f, \frac{1}{n}\right)_{H_p^{r,\alpha}}.$$

REMARK 4.2. *Note that by Lemma 2.5 one can replace* $\|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}}$ *by* $\left(\int_{\mathbb{T}} |f-\mathcal{L}_{n,\lambda}(f)|_{H_p^{r,\alpha}}^p \mathrm{d}\lambda\right)^{1/p}$ *in Theorem 4.1, Corollary 4.1, and Corollary 4.2.*

5. Some corollaries and concluding remarks

Let us now discuss two-sided inequalities like (4.5) and (4.20) for $\theta_{r,\alpha}(f,\delta)_p$. It turns out that in terms of $\theta_{r,\alpha}(f,\delta)_p$ such inequalities, in general, do not hold. However, we can prove the following results, which can be of interest for some particular functions f.

THEOREM 5.1. *Let* $0<p\le\infty$, $0\le\alpha\le r$, *and* $r\in\mathbb{N}$. *Suppose also that* $\{\mathcal{L}_{n,\lambda}\}$ *is bounded in* L_p *and that the following equivalence holds for any* $f\in L_p$ *and* $n\in\mathbb{N}$*:*

$$\|f-\mathcal{L}_{n,\lambda}(f)\|_{\overline{p}} \asymp \omega_r\left(f,\frac{1}{n}\right)_p.$$

Then the following equivalence holds for any $f\in H_p^{r,\alpha}$ *and* $n\in\mathbb{N}$*:*

$$n^\alpha\omega_r\left(f,\frac{1}{n}\right)_p + \|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}} \asymp \theta_{r,\alpha}\left(f,\frac{1}{n}\right)_p. \tag{5.1}$$

Before proving Theorem 5.1 let us note that the estimate from above (5.1) is a simple corollary from Theorem 4.1 and Theorem 3.1. Concerning the estimate from below it turns out that these estimates do not hold without the first term in the left-hand side of (5.1) (see Proposition 5.1).

PROOF. As it was mentioned above it is sufficient only to prove the estimation from below. Let $h\in(0,1/n)$ be fixed and $T_n\in\mathcal{T}_n$, $n\in\mathbb{N}$, be polynomials of the best approximation in $H_p^{r,\alpha}$. By using Theorem B, we obtain

$$\begin{aligned}
h^{-\alpha}\|\Delta_h^r f\|_p &\le Ch^{-\alpha}\left(\|\Delta_h^r(f-T_n)\|_p + \|\Delta_h^r T_n\|_p\right)\\
&\le C\left(h^{-\alpha}\|\Delta_h^r(f-T_n)\|_p + n^\alpha\|\Delta_{1/n}^r T_n\|_p\right)\\
&\le C\left(h^{-\alpha}\|\Delta_h^r(f-T_n)\|_p + n^\alpha\|\Delta_{1/n}^r (f-T_n)\|_p + n^\alpha\|\Delta_{1/n}^r f\|_p\right)\\
&\le C\left(\|f-T_n\|_{H_p^{r,\alpha}} + n^\alpha\omega_r(f,1/n)_p\right)\\
&\le C\left(\|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}} + n^\alpha\omega_r(f,1/n)_p\right).
\end{aligned}$$

Theorem 5.1 is proved. □

Now we show that the first term in the left-hand side of (5.1) cannot be dropped.

PROPOSITION 5.1. *Let* $0<p\le\infty$, $0<\alpha\le r$, *and* $r\in\mathbb{N}$. *Suppose that* $\{\mathcal{L}_{n,\lambda}\}$ *is bounded in* L_p *and that the following inequality holds for any* $f\in L_p$ *and* $n\in\mathbb{N}$*:*

$$\|f-\mathcal{L}_{n,\lambda}(f)\|_{\overline{p}} \le C\omega_r\left(f,\frac{1}{n}\right)_p,$$

where C *is some constant independent of* f *and* n*. Then, for any non-constant function* f, $f^{(r-1)}\in AC$, $f^{(r)}\in H_{p^*}^{r,\alpha}$ *with* $p^*=\max(1,p)$, *and for any sequence* $\{\varepsilon_n\}$ *with* $\varepsilon_n\to 0+$ *there holds*

$$\frac{\theta_{r,\alpha}(f,1/n)_p}{\varepsilon_n n^\alpha\omega_r(f,1/n)_p + \|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}}} \to\infty \quad \text{as} \quad n\to\infty.$$

PROOF. By using the inequality $\theta_{r,\alpha}(f,1/n)_p \ge n^\alpha \omega_r(f,1/n)_p$ it is sufficient only to prove that

$$\frac{\|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}}}{n^\alpha \omega_r(f,1/n)_p} \to 0 \quad \text{as} \quad n\to\infty. \tag{5.2}$$

Suppose that there exists a constant $A_f>0$ such that for any $n\in\mathbb{N}$

$$n^\alpha \omega_r(f,1/n)_p \le A_f \|f-\mathcal{L}_{n,\lambda}(f)\|_{H_{\overline{p}}^{r,\alpha}}. \tag{5.3}$$

Recall, that if $f^{(r-1)}\in AC$ and $f^{(r)}\in L_p$, $1\le p\le\infty$, then

$$\omega_r(f,\delta)_p \le C_{r,p}\delta^r \|f^{(r)}\|_p \tag{5.4}$$

and

$$\omega_{2r}(f,\delta)_p \le C\delta^r \omega_r(f^{(r)},\delta)_p \le C\delta^{r+\alpha}\|f^{(r)}\|_{H_p^{r,\alpha}} \tag{5.5}$$

(see [**5**, p. 53]). Thus, by using Hölder's inequality, and (5.4), we get that for any $h>0$

$$\omega_r(\Delta_h^r f,\delta)_p \le C\omega_r(\Delta_h^r f,\delta)_{p^*} \le C\delta^r \|\Delta_h^r f^{(r)}\|_{p^*}. \tag{5.6}$$

By using (3.25) and (5.5) we also have for any $n\in\mathbb{N}$

$$E_n(f)_{H_p^{r,\alpha}} \le Cn^{-r}\|f^{(r)}\|_{H_{p^*}^{r,\alpha}}. \tag{5.7}$$

Thus, combining (5.3), (5.6), (5.7) with (4.5), we finally obtain

$$\omega_r\left(f,1/n\right)_p \le CA_f n^{-\alpha}\left(\omega_r(f,1/n)_{H_p^{r,\alpha}} + E_n(f)_{H_p^{r,\alpha}}\right) \le CA_f n^{-r-\alpha}\|f^{(r)}\|_{H_{p^*}^{r,\alpha}}.$$

However, if $f^{(r-1)}\in AC$ and $f\not\equiv \text{const}$, then the last statement is impossible, see Lemma 1.5 in [**29**] and Theorem 3.5 in [**8**].

Thus, we have proved (5.2) and hence our proposition. □

The following two corollaries can be obtained by a standard scheme (see, for example, [**3**], [**21**], and [**20**]). In particular, by Theorem 3.4 we obtain the following result.

COROLLARY 5.1. *Let* $0<p\le\infty$, $0<\beta\le\alpha\le r$, $r\in\mathbb{N}$, *and* $n\in\mathbb{N}$. *Then the following inequalities hold for any* $f\in H_p^{r,\alpha}$:

$$E_n(f)_{H_p^{r,\beta}} \le \frac{C}{n^{\alpha-\beta}}\|f\|_{H_p^{r,\alpha}}$$

and

$$E_n(f)_{H_p^{r,\beta}} \le \frac{C}{n^{\alpha-\beta}}E_n(f)_{H_p^{r,\alpha}}.$$

COROLLARY 5.2. *Let* $0<p\le\infty$, $0<\beta\le\alpha\le r$, $r,k\in\mathbb{N}$, $f\in H_p^{r,\alpha}$, *and* $0<\gamma<k$. *Suppose also f satisfies condition (3.32) in the case* $0<p<1$ *and* $\alpha=r$. *Then the following assertions are equivalent:*

(*i*) $\psi_{k,r,\alpha}(f,h)_p=\mathcal{O}(h^\gamma), \quad h\to+0,$
(*ii*) $E_n(f)_{H_p^{r,\alpha}}=\mathcal{O}\left(n^{-\gamma}\right), \quad n\to\infty,$
(*iii*) $E_n(f)_p=\mathcal{O}\left(n^{-\gamma-\alpha}\right), \quad n\to\infty,$
(*iv*) $E_n(f)_{H_p^{r,\beta}}=\mathcal{O}\left(n^{-\gamma-\alpha+\beta}\right), \quad n\to\infty.$
If, in addition, $\alpha+\gamma<k$, *then*
(*v*) $\omega_k(f,h)_p=\mathcal{O}(h^{\gamma+\alpha}), \quad h\to+0.$

PROOF. The equivalence (*i*) ⇔ (*ii*) follows form Theorem 3.5 and Corollary 3.1. The equivalences (*ii*) ⇔ (*iii*) ⇔ (*iv*) follows form Lemma 3.1. The equivalence (*iii*) ⇔ (*v*) is standard, see [**5**, Ch. 7, §3], see also Theorem A and [**10**, Theorem 5] for the case $0<p<1$. □

REMARK 5.1. *In all above results we suppose that $r \geq \alpha$. This assumption is essential. Indeed, if $\alpha > r$, then the quantity $E_n(f)_{H_p^{r,\alpha}}$ is not well-defined. For example, let the function $f \not\equiv const$ be such that $\omega_r(f,h)_p = \mathcal{O}(h^r)$. Then, by using Theorem B, we have for any $0 < h \leq 1/n$*

$$C_1 h^{r-\alpha} \|T_n^{(r)}\|_p - C_1 \|f\|_{H_p^{r,\alpha}} \leq E_n(f)_{H_p^{r,\alpha}}.$$

That is $E_n(f)_{H_p^{r,\alpha}} = \infty$.

REMARK 5.2. *Some of the above results remain true in more general Hölder spaces $H_p^{r,\omega}$, with the norm*

$$\|f\|_{H_p^{r,\omega}} = \|f\|_p + \sup_{h>0} \frac{\|\Delta_h^r f\|_p}{\omega(h)},$$

where the function ω is some modulus of continuity such that the function $\frac{\omega(h)}{h^\alpha}$ is monotonically decreasing, see e.g. [**21**].

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