On approximation of functions by algebraic polynomials in Hölder spaces

Yurii Kolomoitsev[a,b,c,*], Tetiana Lomako[a,b,c], and Jürgen Prestin[a]

Abstract. We study approximation of functions by algebraic polynomials in the Hölder spaces corresponding to the generalized Jacobi translation and the Ditzian-Totik moduli of smoothness. By using modifications of the classical moduli of smoothness, we give improvements of the direct and inverse theorems of approximation and prove the criteria of the precise order of decrease of the best approximation in these spaces. Moreover, we obtain strong converse inequalities for some methods of approximation of functions. As an example, we consider approximation by the Durrmeyer-Bernstein polynomial operators.

1. Introduction

For a long time, there has been some interest in the approximation of functions in Hölder norms. This interest originated from the study of a certain class of integro-differential equations and from applications in error estimations for singular integral equations. Following the initial works of Kalandiya [**22**] and Prössdorf [**30**] some problems of approximation in Hölder spaces have been studied by Ioakimidis [**21**], Bloom and Elliott [**2**], [**16**], Bustamante and Roldan [**4**], and many others. One can find an interesting survey on this subject in [**3**], see also [**29**].

The first result about approximation in the Hölder spaces [**22**] was obtained in the case of approximation of functions by algebraic polynomials on an interval. Nevertheless, most interesting and sharp results have been obtained for approximation of periodic functions (see, for example, [**24**], [**25**] and [**36**]). One of the main reasons of this is the possibility of using some nice properties of the translation operators $f(x) \mapsto f(x+y)$, $x, y \in \mathbb{R}/2\pi\mathbb{Z}$, and well studied methods of Harmonic Analysis on the circle.

In this paper, we study approximation of functions by algebraic polynomials. Unlike the previous investigations, we consider the Hölder spaces generated by the generalized Jacobi translation (see [**35**]). Such approach allows us to apply the well-studied methods of Fourier-Jacobi harmonic analysis (see [**28**]) and deal with the problems which were solved earlier only for approximation of functions in the periodic Hölder spaces. In this way, we essentially improve the previously known results and obtain strong converse inequalities (see [**13**]) for some

1991 *Mathematics Subject Classification.* 41A10, 41A17, 41A25.

Key words and phrases. The Jacobi translation, Ditzian-Totik moduli of smoothness, Hölder spaces, best approximation, strong converse inequalities.

[a]Universität zu Lübeck, Institut für Mathematik, Ratzeburger Allee 160, 23562 Lübeck.

[b]Institute of Mathematics of NAS of Ukraine, Tereschenkivska 3, 01601 Kiev-4, Ukraine.

[c]Institute of Applied Mathematics and Mechanics, NAS of Ukraine, Dobrovol's'kogo 1, 84100 Slov'yans'k, Ukraine.

[*]Corresponding author.

E-mail address: kolomus1@mail.ru.

approximation methods in the Hölder spaces. As an example, we consider approximation by the Durrmeyer-Bernstein polynomial operators.

We also get several approximation results in the Hölder spaces corresponding to the Ditzian-Totik moduli of smoothness. We will see that these spaces are equivalent to the Hölder spaces corresponding to the generalized Jacobi translation in some sense. However, in the Hölder spaces corresponding to the Ditzian-Totik moduli of smoothness we can also deal with the case $0 < p < 1$.

The paper is organized as follows: In Section 2 we introduce the Hölder spaces corresponding to the generalized Jacobi translation and present the auxiliary results related to these spaces. In Section 3 we obtain some new properties of the best approximation and prove analogs of some classical theorems of approximation theory in the Hölder spaces. In Section 4 we prove the strong converse inequalities for approximation of functions by some linear summation methods of Fourier-Jacobi series. In Section 5 we consider similar problems in the Hölder spaces corresponding to the Ditzian-Totik moduli of smoothness. In Section 6 we improve some estimates of approximation by Bernstein operators in the Hölder spaces.

We denote by C some positive constants which may be different at each occurrence. As usual, $A(f,n) \asymp B(f,n)$ will mean that there exists a positive constant C such that $C^{-1}A(f,n) \le B(f,n) \le CA(f,n)$ for all f and n.

2. The Hölder spaces generated by the generalized Jacobi translation

Let $a, b > -1$. Denote by

$$w(x) = w_{a,b}(x) = (1-x)^a(1+x)^b$$

the Jacobi weight on $[-1,1]$. Let $L_{w,p} = L_p([-1,1];w)$, $1 \le p \le \infty$, be the space of all functions f measurable on $[-1,1]$ with the finite norm

$$\|f\|_{w,p} = \|f\|_{L_p([-1,1];w)} = \left(\int_{-1}^1 |f(x)|^p w(x)\mathrm{d}x\right)^{\frac{1}{p}}.$$

In the unweighted case, we will write $L_p = L_p[-1,1] = L_p([-1,1];w_{0,0})$, $\|f\|_p = \|f\|_{L_p[-1,1]} = \|f\|_{L_p([-1,1];w_{0,0})}$. For simplicity, we denote the space $C[-1,1]$ as $L_\infty[-1,1]$ and

$$\|f\|_\infty = \max_{x\in[-1,1]} |f(x)|.$$

For $a, b > -1$, denote by $P_k^{(a,b)}(x)$, $k = 0, 1, \ldots$, the system of Jacobi polynomials orthogonal on $[-1,1]$ such that

$$P_k^{(a,b)}(1) = \binom{k+a}{k}, \quad k = 0, 1, \ldots.$$

Let also $R_k^{(a,b)}$ be normalized Jacobi polynomials,

$$R_k^{(a,b)}(x) = \frac{P_k^{(a,b)}(x)}{P_k^{(a,b)}(1)}, \quad k = 0, 1, \ldots.$$

The expansion of a function $f \in L_{w,p}$, $1 \le p \le \infty$, $a, b > -1$, in the Fourier-Jacobi series has the form

$$f(x) \sim \sum_{k=0}^{\infty} c_k^{(a,b)}(f)\mu_k^{(a,b)} R_k^{(a,b)}(x), \tag{1}$$

with Fourier coefficients

$$c_k^{(a,b)}(f) = \int_{-1}^1 f(x) R_k^{(a,b)}(x) w(x)\mathrm{d}x, \quad k = 0, 1, \ldots,$$

and

$$\mu_k^{(a,b)} = \|R_k^{(a,b)}\|_{L_{w,2}}^{-2} \asymp k^{2a+1}.$$

The Fourier-Jacobi expansion is closely related to the generalized translation operator $T_h^{(a,b)}$, $0 < h < \pi$, acting on a function $f \in L_{w,p}$ with expansion (1) by the following formula

$$T_h^{(a,b)} f(x) \sim \sum_{k=0}^{\infty} c_k^{(a,b)}(f) \mu_k^{(a,b)} R_k^{(a,b)}(\cos h) R_k^{(a,b)}(x). \tag{2}$$

In particular, if $a = b = 0$, then one has

$$T_h^{(0,0)} f(x) = \frac{1}{\pi} \int_{-1}^{1} f\left(x \cos h + u\sqrt{(1-x^2)(1-u^2)}\right) \frac{\mathrm{d}u}{\sqrt{1-u^2}}.$$

Gasper [**17**] proved that for $a \ge b \ge -1/2$ and $0 < h < \pi$ the operator $T_h^{(a,b)}$ is positive. Therefore, we have

$$\|T_h^{(a,b)} f\|_{w,p} \le \|f\|_{w,p} \tag{3}$$

and

$$\|f - T_h^{(a,b)} f\|_{w,p} \to 0 \quad \text{as} \quad h \to 0. \tag{4}$$

In view of (3) and (4), everywhere below, we will suppose that $a \ge b \ge -1/2$ (see also Remark 4.8 below).

Let $r > 0$ and $0 < h < \pi$. The translation operator (2) allows us to naturally introduce the modulus of smoothness of the rth order by

$$\widetilde{\omega}_r(f,h)_{w,p} = \widetilde{\omega}_r^{(a,b)}(f,h)_{w,p} = \sup_{0<t\le h} \|\widetilde{\Delta}_t^r f\|_{w,p},$$

where

$$\widetilde{\Delta}_t^r = \widetilde{\Delta}_t^{r,(a,b)} = \left(I - T_t^{(a,b)}\right)^{r/2} = \sum_{k=0}^{\infty} (-1)^k \binom{r/2}{k} \left(T_t^{(a,b)}\right)^k$$

and I is the identity operator.

In what follows, we put by definition for $h \ge \pi$

$$\widetilde{\omega}_r(f,h)_{w,p} = \widetilde{\omega}_r(f,\pi)_{w,p} = \sup_{0<t<\pi} \|\widetilde{\Delta}_t^r f\|_{w,p}.$$

Now we are able to define the Hölder spaces with respect to the generalized Jacobi translation $T_h^{(a,b)}$. We will say that $f \in H_{w,p}^{r,\alpha}$ if $f \in L_{w,p}$ and

$$\|f\|_{H_{w,p}^{r,\alpha}} = \|f\|_{w,p} + |f|_{H_{w,p}^{r,\alpha}} < \infty, \tag{5}$$

where

$$|f|_{H_{w,p}^{r,\alpha}} = \sup_{h>0} \frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha}.$$

Some properties of these spaces can be found in [**35**].

2.1. Preliminary remarks and auxiliary results. Let $\mathcal{P}_n$ be the set of all algebraic polynomials of degree at most n. As usual, the error of the best approximation of a function $f \in L_{w,p}$ by algebraic polynomials of degree at most n is defined as follows:

$$E_n(f)_{w,p} = \inf_{P \in \mathcal{P}_{n-1}} \|f - P\|_{w,p}, \quad n \in \mathbb{N}.$$

An algebraic polynomial $P \in \mathcal{P}_{n-1}$ is called a polynomial of the best approximation of $f \in L_{w,p}$ if

$$\|f - P\|_{w,p} = E_n(f)_{w,p}.$$

Recall the Jackson-type theorem in $L_{w,p}$, see [**32**].

LEMMA 2.1. Let $f \in L_{w,p}$, $1 \le p \le \infty$, and $r > 0$. Then

$$E_n(f)_{w,p} \le C\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N},$$

where C is a constant independent of n and f.

The Jacobi polynomials are the eigenfunctions of the differential operator

$$\mathcal{D} = \mathcal{D}_w^1 = \frac{-1}{w(x)}\frac{\mathrm{d}}{\mathrm{d}x}w(x)(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}\,,$$

$$\mathcal{D}P_k^{(a,b)} = \lambda_k^{(a,b)}P_k^{(a,b)}, \quad \lambda_k^{(a,b)} = k(k+a+b+1).$$

If for $r > 0$ and a function $f \in L_{w,p}$, $1 \le p \le \infty$, there exists a function $g \in L_{w,p}$ such that its Fourier-Jacobi series has the form

$$g(x) \sim \sum_{k=1}^{\infty}\left(\lambda_k^{(a,b)}\right)^{r/2} c_k^{(a,b)}(f)\mu_k^{(a,b)}R_k^{(a,b)}(x),$$

then we use the notation $g = \mathcal{D}^r f$ and call $\mathcal{D}^r f$ the (fractional) derivative of order r of the function f.

Most results about approximation in $L_{w,p}$ have been formulated in terms of the generalized K-functionals related to the differential operator $\mathcal{D}^r$ (see [**10**]):

$$\widetilde{K}_r(f,h)_{w,p} = \inf_g\left\{\|f-g\|_{w,p} + h^r\|\mathcal{D}^r g\|_{w,p}\right\}.$$

There is the following natural connection between moduli of smoothness and K-functionals (see [**32**]):

LEMMA 2.2. Let $f \in L_{w,p}$, $1 \le p \le \infty$, and $r > 0$. Then

$$\widetilde{K}_r(f,h)_{w,p} \asymp \widetilde{\omega}_r(f,h)_{w,p}, \quad 0 < h < \pi. \tag{6}$$

We will often use the following two lemmas. The first one is the Stechkin-Nikolskii type inequality (see [**32**]):

LEMMA 2.3. Let $1 \le p \le \infty$, $n \in \mathbb{N}$, $0 < h < \pi/n$, and $r > 0$. Then for any polynomial $P_n \in \mathcal{P}_n$ we have

$$h^r\|\mathcal{D}^r P_n\|_{w,p} \asymp \widetilde{\omega}_r(P_n,h)_{w,p},$$

where $\asymp$ is a two-sided inequality with absolute constants independent of P_n and h. Moreover, if $P_n \in \mathcal{P}_{n-1}$ is a polynomial of the best approximation of a function $f \in L_{w,p}$, then

$$\|\widetilde{\Delta}_h^r P_n\|_{w,p} \le C\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{w,p},$$

where C is a constant independent of P_n, h, and f.

LEMMA 2.4. Let $1 \le p \le \infty$ and $0 < \alpha < r < k$. Then the norms of a function in $H_{w,p}^{r,\alpha}$ and $H_{w,p}^{k,\alpha}$ are equivalent.

PROOF. To prove the lemma we can use the scheme of the proof of Theorem 10.1 in [**8**, Ch. 2]. For this we only need to use the inequality

$$\widetilde{\omega}_k(f,h)_{w,p} \le 2^{k-r}\widetilde{\omega}_r(f,h)_{w,p}, \quad k > r, \quad h > 0,$$

which can be easily obtained from (3), and the Marchaud inequality

$$\widetilde{\omega}_r(f,h)_{w,p} \le Ch^r\int_h^{\pi}\frac{\widetilde{\omega}_k(f,u)_{w,p}}{u^{r+1}}\mathrm{d}u, \quad k > r, \quad 0 < h < \pi.$$

Note that the last inequality follows from the corresponding inequality for the K-functionals in [**9**] and Lemma 2.2. □

3. Properties of the best approximation. Direct and inverse theorems

Denote the error of the best approximation in the Hölder space $H^{r,\alpha}_{w,p}$ by

$$E_n(f)_{H^{r,\alpha}_{w,p}} = \inf_{P\in\mathcal{P}_{n-1}} \|f-P\|_{H^{r,\alpha}_{w,p}}, \quad n\in\mathbb{N}.$$

As above, an algebraic polynomial $P\in\mathcal{P}_{n-1}$ is called a polynomial of the best approximation of $f\in H^{r,\alpha}_{w,p}$ if

$$\|f-P\|_{H^{r,\alpha}_{w,p}} = E_n(f)_{H^{r,\alpha}_{w,p}}.$$

Let us establish a connection between the errors of the best approximation in the spaces $H^{r,\alpha}_{w,p}$ and $L_{w,p}$.

Theorem 3.1. Let $f\in H^{r,\alpha}_{w,p}$, $1\le p\le\infty$, and $0<\alpha\le r$. Then

$$C^{-1}n^\alpha E_n(f)_{w,p} \le E_n(f)_{H^{r,\alpha}_{w,p}} \le C\left(n^\alpha E_n(f)_{w,p} + \sum_{\nu=n+1}^{\infty}\nu^{\alpha-1}E_\nu(f)_{w,p}\right), \quad n\in\mathbb{N}, \tag{7}$$

where C is a positive constant independent of n and f.

Proof. Let $P_n\in\mathcal{P}_{n-1}$, $n\in\mathbb{N}$, be the polynomials of the best approximation of $f\in H^{r,\alpha}_{w,p}$. Then by Lemma 2.1, we obtain the lower bound by

$$\begin{aligned} n^\alpha E_n(f)_{w,p} &\le Cn^\alpha\widetilde{\omega}_r(f-P_n,1/n)_{w,p}\\ &\le C\sup_{0<h\le 1/n}\frac{\widetilde{\omega}_r(f-P_n,h)_{w,p}}{h^\alpha} \le CE_n(f)_{H^{r,\alpha}_{w,p}}. \end{aligned}$$

Let us prove the upper estimate in (7). Now, let $P_n\in\mathcal{P}_{n-1}$, $n\in\mathbb{N}$, be the polynomials of the best approximation of $f\in L_{w,p}$. Let $m\in\mathbb{N}$ such that $2^{m-1}\le n<2^m$. Assuming that $\sum_{\nu=1}^\infty \nu^{\alpha-1}E_\nu(f)_{w,p}<\infty$, we can write

$$f = P_{2^m} + \sum_{\nu=m}^{\infty} U_{2^\nu} \quad \text{in} \quad L_{w,p}, \tag{8}$$

where $U_{2^\nu} = P_{2^{\nu+1}} - P_{2^\nu}$. From (8) we have

$$|f-P_n|_{H^{r,\alpha}_{w,p}} \le |P_{2^m}-P_n|_{H^{r,\alpha}_{w,p}} + \sum_{\nu=m}^{\infty}|P_{2^{\nu+1}}-P_{2^\nu}|_{H^{r,\alpha}_{w,p}} = S_1+S_2. \tag{9}$$

By Lemma 2.3 and (3), we obtain

$$\begin{aligned} S_1 &\le \left(\sup_{0<h<2^{-m}} + \sup_{h\ge 2^{-m}}\right)\frac{\widetilde{\omega}_r(P_{2^m}-P_n,h)_{w,p}}{h^\alpha}\\ &\le C2^{\alpha m}\left(\widetilde{\omega}_r(P_{2^m}-P_n,2^{-m})_{w,p} + \|P_{2^m}-P_n\|_{w,p}\right)\\ &\le C2^{\alpha m}\|P_{2^m}-P_n\|_{w,p} \le Cn^\alpha E_n(f)_{w,p}. \end{aligned} \tag{10}$$

Again, by Lemma 2.3 and (3), we get

$$\begin{aligned} S_2 &\le \sum_{\nu=m}^{\infty}\sup_{0<h\le 2^{-\nu-1}} h^{-\alpha}\widetilde{\omega}_r(U_{2^\nu},h)_{w,p} + \sum_{\nu=m}^{\infty}\sup_{h\ge 2^{-\nu-1}} h^{-\alpha}\widetilde{\omega}_r(U_{2^\nu},h)_{w,p}\\ &\le C\left(\sum_{\nu=m}^{\infty}2^{\alpha\nu}\widetilde{\omega}_r(U_{2^\nu},2^{-\nu-1})_{w,p} + \sum_{\nu=m}^{\infty}2^{\alpha\nu}\sup_{h\ge 2^{-\nu-1}}\widetilde{\omega}_r(U_{2^\nu},h)_{w,p}\right)\\ &\le C\sum_{\nu=m}^{\infty}2^{\alpha\nu}\|U_{2^\nu}\|_{w,p} \le C\sum_{\nu=m}^{\infty}2^{\alpha\nu}E_{2^\nu}(f)_{w,p} \le C\sum_{\mu=n}^{\infty}\mu^{\alpha-1}E_\mu(f)_{w,p}. \end{aligned} \tag{11}$$

Thus, combining (9)–(11), we obtain the upper estimate in (7). □

Corollary 3.2. Let $f \in H^{r,\alpha}_{w,p}$, $1 \le p \le \infty$, $0 < \alpha \le r$, and $\gamma > 0$. Then the following assertions are equivalent:

(*i*) $E_n(f)_{H^{r,\alpha}_{w,p}} = \mathcal{O}(n^{-\gamma})$ as $n \to \infty$,

(*ii*) $E_n(f)_{w,p} = \mathcal{O}(n^{-\gamma-\alpha})$ as $n \to \infty$.

By using the upper inequality in (7) and Lemma 2.1, one can prove that under the conditions of Theorem 3.1 for any $k > 0$

$$E_n(f)_{H^{r,\alpha}_{w,p}} \le C \int_0^{1/n} \frac{\widetilde{\omega}_k(f,t)_{w,p}}{t^{\alpha+1}} \mathrm{d}t, \quad n \in \mathbb{N}, \tag{12}$$

where C is a constant independent of f and n (see also Theorem 5.7 below).

In the case $\alpha < r$, one can obtain a sharper estimate by using the following modulus of smoothness

$$\tilde{\theta}_{k,\alpha}(f,t)_{w,p} = \sup_{0<h\le t} \frac{\widetilde{\omega}_k(f,h)_{w,p}}{h^\alpha}, \quad 0 < \alpha \le k.$$

The similar moduli of smoothness have initially been used for the investigation of approximation in Hölder spaces (see, for example, [**3**] and [**4**]).

We prove the following Jackson-type theorem in terms of $\widetilde{\theta}_{k,\alpha}(f,h)_{w,p}$.

Theorem 3.3. Let $f \in H^{r,\alpha}_{w,p}$, $1 \le p \le \infty$, $0 < \alpha < \min(r,k)$ or $0 < \alpha = k = r$. Then

$$E_n(f)_{H^{r,\alpha}_{w,p}} \le C \widetilde{\theta}_{k,\alpha}\left(f, \frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N}, \tag{13}$$

where C is a constant independent of n and f.

Proof. First let $\alpha < \min(r,k)$ and $P_n \in \mathcal{P}_{n-1}$, $n \in \mathbb{N}$, be polynomials of the best approximation of $f \in L_{w,p}$. By Lemma 2.1 and Lemma 2.4, it suffices to find an estimation for $|f - P_n|_{H^{k,\alpha}_{w,p}}$. We have

$$|f - P_n|_{H^{k,\alpha}_{w,p}} \le \left(\sup_{0<h<1/n} + \sup_{h \ge 1/n} \right) \frac{\widetilde{\omega}_k(f - P_n, h)_{w,p}}{h^\alpha} = S_1 + S_2. \tag{14}$$

By (3) and Lemma 2.1, we get

$$S_2 \le C n^\alpha \|f - P_n\|_{w,p} \le C n^\alpha \widetilde{\omega}_k(f, 1/n)_{w,p} \le C \widetilde{\theta}_{k,\alpha}(f, 1/n)_{w,p}. \tag{15}$$

We also obtain

$$\begin{aligned} S_1 &\le \sup_{0<h<1/n} \frac{\widetilde{\omega}_k(f,h)_{w,p}}{h^\alpha} + \sup_{0<h<1/n} \frac{\widetilde{\omega}_k(P_n,h)_{w,p}}{h^\alpha} \\ &\le \widetilde{\theta}_{k,\alpha}(f,1/n)_{w,p} + \sup_{0<h<1/n} \frac{\widetilde{\omega}_k(P_n,h)_{w,p}}{h^\alpha}. \end{aligned} \tag{16}$$

To estimate the last term in (16) we use Lemma 2.3, (3), and Lemma 2.1:

$$\begin{aligned} \sup_{0<h<1/n} \frac{\widetilde{\omega}_k(P_n,h)_{w,p}}{h^\alpha} &\le C n^\alpha \widetilde{\omega}_k(P_n, 1/n)_{w,p} \le C n^\alpha \left(\|f - P_n\|_{w,p} + \widetilde{\omega}_k(f, 1/n)_{w,p} \right) \\ &\le C n^\alpha \widetilde{\omega}_k(f, 1/n)_{w,p} \le n^\alpha \widetilde{\theta}_{k,\alpha}(f, 1/n)_{w,p}. \end{aligned} \tag{17}$$

Thus, combining (14)–(17), we prove the theorem in the case $\alpha < \min(r,k)$. One can use the same scheme to prove the theorem in the case $k = r = \alpha$.

□

By using the standard scheme we also obtain the following inverse result:

THEOREM 3.4. Let $f \in H^{r,\alpha}_{w,p}$, $1 \le p \le \infty$, $0 < \alpha < \min(r,k)$ or $0 < \alpha = k = r$. Then

$$\widetilde{\theta}_{k,\alpha}\left(f, \frac{1}{n}\right)_{w,p} \le \frac{C}{n^{k-\alpha}} \sum_{\nu=1}^{n} \nu^{k-\alpha-1} E_\nu(f)_{H^{r,\alpha}_{w,p}}, \quad n \in \mathbb{N}, \tag{18}$$

where C is a constant independent of n and f.

PROOF. As in Theorem 3.3, we consider only the case $\alpha < \min(r,k)$. Let $P_n \in \mathcal{P}_{n-1}$, $n \in \mathbb{N}$, be the polynomials of the best approximation of $f \in H^{r,\alpha}_{w,p}$. For any $m \in \mathbb{N} \cup \{0\}$ we get

$$\widetilde{\theta}_{k,\alpha}(f, 1/n)_{w,p} \le \widetilde{\theta}_{k,\alpha}(f - P_{2^{m+1}}, 1/n)_{w,p} + \widetilde{\theta}_{k,\alpha}(P_{2^{m+1}}, 1/n)_{w,p}. \tag{19}$$

By the definition of the Hölder space and Lemma 2.4, we obtain

$$\widetilde{\theta}_{k,\alpha}(f - P_{2^{m+1}}, 1/n)_{w,p} \le C|f - P_{2^{m+1}}|_{H^{r,\alpha}_{w,p}} \le C E_{2^{m+1}}(f)_{H^{r,\alpha}_{w,p}}. \tag{20}$$

By Lemma 2.2, Lemma 2.3, and Lemma 2.4, we gain

$$\begin{aligned}
\widetilde{\theta}_{k,\alpha}(P_{2^{m+1}}, 1/n)_{w,p} &\le C \sup_{0<h\le 1/n} \frac{\widetilde{K}_k(P_{2^{m+1}}, h)_{w,p}}{h^\alpha} \le \frac{C}{n^{k-\alpha}} \|\mathcal{D}^k P_{2^{m+1}}\|_{w,p} \\
&\le \frac{C}{n^{k-\alpha}} \left(\|\mathcal{D}^k P_2 - \mathcal{D}^k P_1\|_{w,p} + \sum_{\nu=1}^{m} \|\mathcal{D}^k P_{2^{\nu+1}} - \mathcal{D}^k P_{2^\nu}\|_{w,p} \right) \\
&\le \frac{C}{n^{k-\alpha}} \left(\widetilde{\omega}_k(P_2 - P_1, 1)_{w,p} + \sum_{\nu=1}^{m} 2^{\nu k} \widetilde{\omega}_k \left(P_{2^{\nu+1}} - P_{2^\nu}, 2^{-(\nu+1)} \right)_{w,p} \right) \\
&\le \frac{C}{n^{k-\alpha}} \left(\|P_2 - P_1\|_{H^{r,\alpha}_{w,p}} + \sum_{\nu=1}^{m} 2^{\nu(k-\alpha)} \|P_{2^{\nu+1}} - P_{2^\nu}\|_{H^{r,\alpha}_{w,p}} \right) \\
&\le \frac{C}{n^{k-\alpha}} \left(E_1(f)_{H^{r,\alpha}_{w,p}} + \sum_{\nu=1}^{m} 2^{\nu(k-\alpha)} E_{2^\nu}(f)_{H^{r,\alpha}_{w,p}} \right).
\end{aligned} \tag{21}$$

Since $2^{\nu(k-\alpha)} E_{2^\nu}(f)_{H^{r,\alpha}_{w,p}} \le 2 \sum_{\mu=2^{\nu-1}+1}^{2^\nu} \mu^{k-\alpha-1} E_\mu(f)_{H^{r,\alpha}_{w,p}}$, $\nu \ge 1$, we obtain from (21) that

$$\widetilde{\theta}_{k,\alpha}(P_{2^{m+1}}, 1/n)_{w,p} \le \frac{C}{n^{k-\alpha}} \left(E_1(f)_{H^{k,\alpha}_{w,p}} + \sum_{\mu=2}^{2^m} \mu^{k-\alpha-1} E_\mu(f)_{H^{r,\alpha}_{w,p}} \right). \tag{22}$$

Choose m such that $2^m \le n < 2^{m+1}$. Then (19), (20), and (22) yield (18). □

Now let us consider the problem concerning the sharp order of decrease of the best approximation in the spaces $H^{r,\alpha}_{w,p}$.

THEOREM 3.5. Let $f \in H^{r,\alpha}_{w,p}$, $1 \le p \le \infty$, $0 < \alpha < r$, and $s \ge \alpha$. Then the following assertions are equivalent:
(i) there exists a constant $L > 0$ such that

$$\widetilde{\theta}_{s,\alpha}\left(f, \frac{1}{n}\right)_{w,p} \le L E_n(f)_{H^{r,\alpha}_{w,p}}, \quad n \in \mathbb{N}, \tag{23}$$

(ii) for some $k > s$ there exists a constant $M > 0$ such that

$$\widetilde{\theta}_{s,\alpha}(f, h)_{w,p} \le M \widetilde{\theta}_{k,\alpha}(f, h)_{w,p}, \quad h > 0. \tag{24}$$

Proof. To prove this theorem we follow the scheme of the proof of the corresponding result in [**31**], see also [**34**, Ch. 4]. For this purpose, we need Theorem 3.3, Theorem 3.4, and the following inequalities:

$$\widetilde{\theta}_{k,\alpha}(f,\delta)_{w,p}\le 2^{k-r}\widetilde{\theta}_{r,\alpha}(f,\delta)_{w,p},\quad k>r>0,\quad \delta>0, \tag{25}$$

$$\widetilde{\theta}_{k,\alpha}(f,n\delta)_{w,p}\le Cn^{k-\alpha}\widetilde{\theta}_{k,\alpha}(f,\delta)_{w,p},\quad n\in\mathbb{N},\quad k,\delta>0, \tag{26}$$

where C is a constant independent of f, δ, and n. These inequalities can easily be obtained from inequalities (3) and (6) (see, for example, [**8**, Ch. 2, §7] and [**28**]).

Let condition (24) be satisfied. Then from (25) and (26) we get

$$\widetilde{\theta}_{k,\alpha}(f,n\delta)_{w,p}\le CMn^{s-\alpha}\widetilde{\theta}_{k,\alpha}(f,\delta)_{w,p},\quad n\in\mathbb{N},\quad \delta>0. \tag{27}$$

Therefore, $\widetilde{\theta}_{k,\alpha}(f,\lambda\delta)_{w,p}\le CM(1+\lambda)^{s-\alpha}\widetilde{\theta}_{k,\alpha}(f,\delta)_{w,p}$ for all $\delta,\lambda>0$.

Let us prove that

$$\frac{1}{n^{k-\alpha}}\sum_{\nu=1}^{n}\nu^{k-\alpha-1}E_\nu(f)_{H^{r,\alpha}_{w,p}}\le CM\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}. \tag{28}$$

Indeed, by Theorem 3.3 and inequality (27), we obtain

$$\begin{aligned}\frac{1}{n^{k-\alpha}}\sum_{\nu=1}^{n}\nu^{k-\alpha-1}E_\nu(f)_{H^{r,\alpha}_{w,p}}&\le\frac{C}{n^{k-\alpha}}\sum_{\nu=1}^{n}\nu^{k-\alpha-1}\widetilde{\theta}_{k,\alpha}\left(f,\frac1\nu\right)_{w,p}\\&\le\frac{CM}{n^{k-s}}\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}\sum_{\nu=1}^{n}\nu^{k-s-1}\le CM\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}.\end{aligned}$$

Next, by Theorem 3.4 and (28), we get that for all $m,n\in\mathbb{N}$

$$\begin{aligned}\widetilde{\theta}_{k,\alpha}\left(f,\frac{1}{mn}\right)_{w,p}&\le\frac{C}{(mn)^{k-\alpha}}\sum_{\nu=1}^{mn}\nu^{k-\alpha-1}E_\nu(f)_{H^{r,\alpha}_{w,p}}\\&=\frac{C}{(mn)^{k-\alpha}}\left(\sum_{\nu=n+1}^{mn}\nu^{k-\alpha-1}E_\nu(f)_{H^{r,\alpha}_{w,p}}+\sum_{\nu=1}^{n}\nu^{k-\alpha-1}E_\nu(f)_{H^{r,\alpha}_{w,p}}\right)\\&\le C\left(\frac{1}{(mn)^{k-\alpha}}\sum_{\nu=n+1}^{mn}\nu^{k-\alpha-1}E_\nu(f)^p_{H^{r,\alpha}_{w,p}}+\frac{M}{m^{k-\alpha}}\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}\right).\end{aligned} \tag{29}$$

Inequality (29) implies that

$$\sum_{\nu=n+1}^{mn}\nu^{k-\alpha-1}E_\nu(f)_{H^{r,\alpha}_{w,p}}\ge\frac{(mn)^{k-\alpha}}{C}\widetilde{\theta}_{k,\alpha}\left(f,\frac{1}{mn}\right)_{w,p}-Mn^{k-\alpha}\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}$$

from which, by using the monotonicity of $E_n(f)_{H^{r,\alpha}_{w,p}}$ and (27), we derive

$$E_n(f)_{H^{r,\alpha}_{w,p}}\sum_{\nu=n+1}^{mn}\nu^{k-\alpha-1}\ge(Cm^{k-s}-M)n^{k-\alpha}\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}.$$

Thus, choosing m appropriately, we can find a positive constant $C=C_{k,s,M}$ such that

$$E_n(f)_{H^{r,\alpha}_{w,p}}\ge C\widetilde{\theta}_{k,\alpha}\left(f,\frac1n\right)_{w,p}.$$

From the last inequality and (24) we obtain (23).

The reverse direction is an immediate consequence of Theorem 3.3. □

One can observe that in Theorem 3.1 or in (12) the best approximation $E_n(f)_{H^{r,\alpha}_{w,p}}$ can tend to zero very fast. But at the same time, if for a function $f \in L_{w,p}$, we have $\widetilde{\theta}_{r,\alpha}(f,\delta)_{w,p} = o(\delta^{r-\alpha})$, then $f \equiv \text{const}$ by (26). Thus, estimates (13) and (18) are not sharp in some sense because of the failure of $\widetilde{\theta}_{r,\alpha}(f,\delta)_{w,p}$ in the case $\alpha = r$. We introduce a "modulus of smoothness" which, as we suppose, will be more natural and useful for approximation in the Hölder spaces. At least, the idea of this "modulus of smoothness" works for the strong converse inequalities in the next section.

Let $1 \le p \le \infty$, $0 < \alpha \le r$, and $k > 0$. Denote

$$\widetilde{\psi}_{k,r,\alpha}(f,\delta)_{w,p} = \sup_{0<h\le\delta} \frac{\widetilde{\omega}_k(\widetilde{\Delta}_h^r f,\delta)_{w,p}}{h^\alpha}. \tag{30}$$

Concerning the properties of (30) we only mention that for any $f \in L_{w,p}$, $0 < \alpha \le r$, and $k > 0$

$$\widetilde{\theta}_{k+r,\alpha}(f,\delta)_{w,p} \le \widetilde{\psi}_{k,r,\alpha}(f,\delta)_{w,p} \le C\widetilde{\theta}_{r,\alpha}(f,\delta)_{w,p} \tag{31}$$

where $0 < \delta < \pi$ and the constant C depends only on k and r.

Indeed, by the definitions of moduli of smoothness and inequality (3), we obtain

$$\begin{aligned}\widetilde{\theta}_{k+r,\alpha}(f,\delta)_{w,p} &= \sup_{0<h\le\delta}\sup_{0<t\le h} \frac{\|\widetilde{\Delta}_t^{k+r} f\|_{w,p}}{h^\alpha} = \sup_{0<h\le\delta}\sup_{0<t\le h}\left(\frac{t}{h}\right)^\alpha \frac{\|\widetilde{\Delta}_t^k\widetilde{\Delta}_t^r f\|_{w,p}}{t^\alpha}\\ &\le \sup_{0<t\le\delta}\frac{\|\widetilde{\Delta}_t^k\widetilde{\Delta}_t^r f\|_{w,p}}{t^\alpha} \le \sup_{0<t\le\delta}\frac{\sup_{0<u\le t}\|\widetilde{\Delta}_u^k\widetilde{\Delta}_t^r f\|_{w,p}}{t^\alpha}\\ &= \sup_{0<t\le\delta}\frac{\widetilde{\omega}_k(\widetilde{\Delta}_t^r f,\delta)_{w,p}}{t^\alpha} = \widetilde{\psi}_{k,r,\alpha}(f,\delta)_{w,p}.\end{aligned}$$

At the same time, by (3)

$$\sup_{0<h\le\delta}\frac{\widetilde{\omega}_k(\widetilde{\Delta}_h^r f,\delta)_{w,p}}{h^\alpha} \le C\sup_{0<h\le\delta}\frac{\|\widetilde{\Delta}_h^r f\|_{w,p}}{h^\alpha} \le C\sup_{0<h\le\delta}\frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha} = C\widetilde{\theta}_{r,\alpha}(f,\delta)_{w,p},$$

which gives the second inequality in (31).

By using the modulus of smoothness (30), we obtain the following improvement of Theorem 3.3 in the case $\alpha = r$:

THEOREM 3.6. *Let $f \in H^{r,\alpha}_{w,p}$, $1 \le p \le \infty$, $0 < \alpha \le r$, and $k > 0$. Then*

$$E_n(f)_{H^{r,\alpha}_{w,p}} \le C\widetilde{\psi}_{k,r,\alpha}\left(f,\frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N}, \tag{32}$$

where C is a constant independent of f and n.

PROOF. We will need the following de la Vallée Poussin-type means

$$V_n(f)(x) = \sum_{\nu=0}^n v\left(\frac{\nu}{n}\right) c_\nu^{(a,b)}(f)\mu_\nu^{(a,b)} R_\nu^{(a,b)}(x),$$

where $v \in C^\infty(\mathbb{R})$, $v(x) = 1$ for $|x| \le 1/2$ and $v(x) = 0$ for $|x| \ge 1$. It is well-known (see [**32**] or [**28**]) that there exists a constant C such that for any $f \in L_{w,p}$, $1 \le p \le \infty$,

$$\|f - V_n(f)\|_{w,p} \le C\widetilde{\omega}_{r+k}\left(f,\frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N}. \tag{33}$$

This yields that

$$
\begin{aligned}
E_n(f)_{H^{r,\alpha}_{w,p}} &\le \|f-V_n(f)\|_{w,p}+|f-V_n(f)|_{H^{r,\alpha}_{w,p}}\\
&\le C\widetilde{\omega}_{r+k}\left(f,\frac1n\right)_{w,p}+|f-V_n(f)|_{H^{r,\alpha}_{w,p}}.
\end{aligned}\tag{34}
$$

It is evident (see the last inequality in (36), below) that one only needs to estimate the second term of the right-hand side in (34). We have

$$
|f-V_n(f)|_{H^{r,\alpha}_{w,p}}\le\left(\sup_{0<h<1/n}+\sup_{h\ge 1/n}\right)\frac{\widetilde{\omega}_r(f-V_n(f),h)_{w,p}}{h^\alpha}=S_1+S_2.\tag{35}
$$

By inequalities (33) and (3), we get

$$
\begin{aligned}
S_2&\le Cn^\alpha\|f-V_n(f)\|_{w,p}\le Cn^\alpha\widetilde{\omega}_{r+k}\left(f,\frac1n\right)_{w,p}=Cn^\alpha\sup_{0<\delta\le 1/n}\|\widetilde{\Delta}_\delta^k\widetilde{\Delta}_\delta^r f\|_{w,p}\\
&\le Cn^\alpha\sup_{0<h\le 1/n}\sup_{0<\delta\le 1/n}\|\widetilde{\Delta}_\delta^k\widetilde{\Delta}_h^r f\|_{w,p}\le C\widetilde{\psi}_{k,r,\alpha}\left(f,\frac1n\right)_{w,p}.
\end{aligned}\tag{36}
$$

In order to estimate S_1 we use the equality $\widetilde{\Delta}_h^r V_n(f)=V_n(\widetilde{\Delta}_h^r f)$ and once again (33). Hence,

$$
S_1\le\sup_{0<h\le 1/n}\frac{\|\widetilde{\Delta}_h^r f-V_n(\widetilde{\Delta}_h^r f)\|_{w,p}}{h^\alpha}\le C\widetilde{\psi}_{k,r,\alpha}\left(f,\frac1n\right)_{w,p}.\tag{37}
$$

Thus, combining (34)–(37), we get (32).

□

The following two theorems can be obtained in the same manner as Theorem 3.4 and Theorem 3.5 above. However, we already have non-trivial inequalities in the case $\alpha=r$.

THEOREM 3.7. Let $f\in H^{r,\alpha}_{w,p}$, $1\le p\le\infty$, $0<\alpha\le r$, and $k>0$. Then

$$
\widetilde{\psi}_{k,r,\alpha}\left(f,\frac1n\right)_{w,p}\le\frac{C}{n^k}\sum_{\nu=1}^n\nu^{k-1}E_\nu(f)_{H^{r,\alpha}_{w,p}},\quad n\in\mathbb{N},
$$

where C is a constant independent of f and n.

THEOREM 3.8. Let $f\in H^{r,\alpha}_{w,p}$, $1\le p\le\infty$, $0<\alpha\le r$, and $s>0$. Then the following assertion are equivalent:
(i) there exists a constant $L>0$ such that

$$
\widetilde{\psi}_{s,r,\alpha}\left(f,\frac1n\right)_{w,p}\le LE_n(f)_{H^{r,\alpha}_{w,p}},\quad n\in\mathbb{N},
$$

(ii) for some $k>s$ there exists a constant $M>0$ such that

$$
\widetilde{\psi}_{s,r,\alpha}(f,h)_{w,p}\le M\widetilde{\psi}_{k,r,\alpha}(f,h)_{w,p},\quad h>0.
$$

4. Strong converse inequalities in the Hölder spaces $H^{r,\alpha}_{w,p}$

To formulate the main theorem in this section we will need some auxiliary notations. For that purpose let us introduce the general modulus of smoothness.

We will say that $\omega=\omega(\cdot,\cdot)_{w,p}\in\Omega_{w,p}=\Omega(L_{w,p},\mathbb{R}_+)$, $1\le p\le\infty$, if for any $f,g\in L_{w,p}$

(i) $\omega(f,\delta)_{w,p}\le C\|f\|_{w,p}$, $\delta>0$,

(ii) $\omega(f+g,\delta)_{w,p}\le C\left(\omega(f,\delta)_{w,p}+\omega(g,\delta)_{w,p}\right)$, $\delta>0$,

where C is a constant independent of f, g, and δ.

As a function ω we can take, for example, the modulus of smoothness $\widetilde{\omega}_k(f,\delta)_{w,p}$ of arbitrary order k, or the corresponding K-functional or its realization (see [**6**]).

By using $\omega \in \Omega_{w,p}$, we define the new "modulus of smoothness" related to the Hölder space $H^{r,\alpha}_{w,p}$ by

$$\omega(f,\delta)_{H^{r,\alpha}_{w,p}} = \omega(f,\delta)_{w,p} + \sup_{0<h<\pi} \frac{\omega(\widetilde{\Delta}^r_h f,\delta)_{w,p}}{h^\alpha}. \tag{38}$$

Note that, if we take $\omega(f,\delta)_{w,p} = \|f\|_{w,p}$, then (38) defines the norm in the Hölder spaces which was introduced in (5), see also Lemma 4.2 below. But if $\omega(f,\delta)_{w,p} = \widetilde{\omega}_k(f,\delta)_{w,p}$, then formula (38) provides the definition of the corresponding modulus of smoothness $\widetilde{\omega}_k(f,\delta)_{H^{r,\alpha}_{w,p}}$ in the Hölder spaces $H^{r,\alpha}_{w,p}$.

Remark 4.1. It is easy to see that Theorem 3.6, Theorem 3.7, and Theorem 3.8 remain valid if we replace the modulus of smoothness $\widetilde{\psi}_{k,r,\alpha}(f,1/n)_{w,p}$ by $\widetilde{\omega}_k(f,\delta)_{H^{r,\alpha}_{w,p}}$ in the corresponding theorem.

Lemma 4.2. Let $1 \le p \le \infty$ and $0 < \alpha \le r$. Then

$$\sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h f\|_{w,p}}{h^\alpha} = \sup_{h>0} \frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha} = |f|_{H^{r,\alpha}_{w,p}}. \tag{39}$$

Proof. Indeed, it is obvious that

$$\sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h f\|_{w,p}}{h^\alpha} \le \sup_{h>0} \frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha}. \tag{40}$$

Thus, to show (39) we have only to verify the converse inequality.

Let $0 < \delta \le h < \pi$, then

$$\|\widetilde{\Delta}^r_\delta f\|_{w,p} \le \delta^\alpha \sup_{0<t\le\delta} \frac{\|\widetilde{\Delta}^r_t f\|_{w,p}}{t^\alpha} \le h^\alpha \sup_{0<t\le\pi} \frac{\|\widetilde{\Delta}^r_t f\|_{w,p}}{t^\alpha}.$$

Therefore, for any $0 < h < \pi$ we get

$$\frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha} \le \sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h f\|_{w,p}}{h^\alpha}. \tag{41}$$

By the definition of $\widetilde{\omega}_r(f,h)_{w,p}$, we also have

$$\sup_{h\ge\pi} \frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha} = \frac{\widetilde{\omega}_r(f,\pi)_{w,p}}{\pi^\alpha} = \sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h f\|_{w,p}}{\pi^\alpha} \le \sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h f\|_{w,p}}{h^\alpha}.$$

The last inequality together with (41) implies

$$\sup_{h>0} \frac{\widetilde{\omega}_r(f,h)_{w,p}}{h^\alpha} \le \sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h f\|_{w,p}}{h^\alpha}. \tag{42}$$

Finally, combining (40) and (42), we obtain (39). □

Now we are ready to formulate the main result of this section.

Theorem 4.3. Let $1 \le p \le \infty$, $0 < \alpha \le r$, and $\omega \in \Omega_{w,p}$. Let $\{\mathcal{L}_n\}$ be bounded linear operators in $L_{w,p}$ such that

$$T^{(a,b)}_h(\mathcal{L}_n(f)) = \mathcal{L}_n\left(T^{(a,b)}_h f\right), \quad f \in L_{w,p}, \quad n \in \mathbb{N}, \quad h \in (0,\pi). \tag{43}$$

If the following equivalence holds for any $f \in L_{w,p}$

$$\|f - \mathcal{L}_n(f)\|_{w,p} \asymp \omega\left(f, \frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N}, \tag{44}$$

then we have for any $f \in H^{r,\alpha}_{w,p}$

$$\|f-\mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}} \asymp \omega\left(f,\frac{1}{n}\right)_{H^{r,\alpha}_{w,p}}, \quad n\in\mathbb{N}. \tag{45}$$

PROOF. In view of (43), for any $f \in L_{w,p}$ one has

$$\mathcal{L}_n(\widetilde{\Delta}_h^r f) = \widetilde{\Delta}_h^r \mathcal{L}_n(f). \tag{46}$$

Thus, by (44), (46), and (39) we get

$$\begin{aligned}\sup_{0<h<\pi}\frac{\omega(\widetilde{\Delta}_h^r f,1/n)_{w,p}}{h^\alpha} &\asymp \sup_{0<h<\pi}\frac{\|\widetilde{\Delta}_h^r f-\mathcal{L}_n(\widetilde{\Delta}_h^r f)\|_{w,p}}{h^\alpha}\\ &= \sup_{0<h<\pi}\frac{\|\widetilde{\Delta}_h^r f-\widetilde{\Delta}_h^r\mathcal{L}_n(f)\|_{w,p}}{h^\alpha} = |f-\mathcal{L}_n(f)|_{H^{r,\alpha}_{w,p}},\end{aligned}$$

which together with (44) yields (45). □

REMARK 4.4. It is easy to see that in the assertion of Theorem 4.3 one can simultaneously replace two-sided inequality (44) by $\|f-\mathcal{L}_n(f)\|_{w,p} \le C\omega(f,1/n)_{w,p}$ and two-sided inequality (45) by $\|f-\mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}} \le C\omega(f,1/n)_{H^{r,\alpha}_{w,p}}$. The same is true with the sign "$\ge$" in place of "$\le$".

It turns out that, in general, strong converse inequalities do not hold in terms of $\widetilde{\theta}_{r,\alpha}(f,\delta)_{w,p}$. However, we have the following result:

PROPOSITION 4.5. Let $1\le p\le\infty$ and $0<\alpha\le r$. Suppose $\{\mathcal{L}_n\}$ are bounded polynomial operators from $L_{w,p}$ to $\mathcal{P}_{n-1}$ and the following equivalence holds for any $f\in L_{w,p}$

$$\|f-\mathcal{L}_n(f)\|_{w,p} \asymp \widetilde{\omega}_r\left(f,\frac{1}{n}\right)_{w,p}, \quad n\in\mathbb{N}.$$

Then for any $f\in H^{r,\alpha}_{w,p}$

$$n^\alpha\widetilde{\omega}_r\left(f,\frac{1}{n}\right)_{w,p} + \|f-\mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}} \asymp \widetilde{\theta}_{r,\alpha}\left(f,\frac{1}{n}\right)_{w,p}, \quad n\in\mathbb{N}. \tag{47}$$

Note that the estimate from above in (47) can be obtained by repeating the proof of Theorem 3.3. Concerning the estimate from below it turns out that without the first term on the left-hand side of (47) these estimates do not hold (see Proposition 4.6 below).

PROOF. It is sufficient only to prove the estimate from below. Let $h\in(0,1/n)$ be fixed and $P\in\mathcal{P}_{n-1}$, $n\in\mathbb{N}$, be polynomials of the best approximation in $H^{r,\alpha}_{w,p}$. By Lemma 2.3, we obtain

$$\begin{aligned}h^{-\alpha}\widetilde{\omega}_r(f,h)_{w,p} &\le h^{-\alpha}\left(\widetilde{\omega}_r(f-P,h)_{w,p}+\widetilde{\omega}_r(P,h)_{w,p}\right)\\ &\le h^{-\alpha}\left(\widetilde{\omega}_r(f-P,h)_{w,p}+Ch^r\|\mathcal{D}^rP\|_{w,p}\right)\\ &\le h^{-\alpha}\widetilde{\omega}_r(f-P,h)_{w,p}+Cn^{-r+\alpha}\|\mathcal{D}^rP\|_{w,p}\\ &\le \|f-P\|_{H^{r,\alpha}_{w,p}}+Cn^\alpha\widetilde{\omega}_r(P,1/n)_{w,p}\\ &\le \|f-P\|_{H^{r,\alpha}_{w,p}}+Cn^\alpha\widetilde{\omega}_r(f,1/n)_{w,p}\\ &\le \|f-\mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}}+Cn^\alpha\widetilde{\omega}_r(f,1/n)_{w,p}.\end{aligned}$$

Proposition 4.5 is proved. □

Now we show that the first term on the left-hand side of (47) cannot be dropped.

PROPOSITION 4.6. Let $1 \le p \le \infty$ and $0 < \alpha \le r$. Suppose that $\{\mathcal{L}_n\}$ are bounded linear operators in $L_{w,p}$ satisfying (43) and the following inequality holds for any $f \in L_{w,p}$

$$\|f - \mathcal{L}_n(f)\|_{w,p} \le C\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N}, \tag{48}$$

where C is some constant independent of f and n. Then for any non-trivial function f, i.e. $\widetilde{\omega}_r(f,\pi)_{w,p} \ne 0$, $\mathcal{D}^r f \in H^{r,\alpha}_{w,p}$, and for any sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0+$ we have

$$\frac{\widetilde{\theta}_{r,\alpha}(f,1/n)_{w,p}}{\varepsilon_n n^\alpha \widetilde{\omega}_r(f,1/n)_{w,p} + \|f - \mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}}} \to \infty \quad \text{as} \quad n \to \infty.$$

PROOF. Taking into account the inequality $\widetilde{\theta}_{r,\alpha}(f,1/n)_{w,p} \ge n^\alpha \widetilde{\omega}_r(f,1/n)_{w,p}$, we only need to prove

$$\frac{\|f - \mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}}}{n^\alpha \widetilde{\omega}_r(f,1/n)_{w,p}} \to 0 \quad \text{as} \quad n \to \infty.$$

Suppose to the contrary that there exists a constant $C > 0$ and a sequence of natural numbers $\{n_k\}$, $n_k \to \infty$ as $k \to \infty$, such that

$$n_k^\alpha \widetilde{\omega}_r(f,1/n_k)_{w,p} \le C\|f - \mathcal{L}_{n_k}(f)\|_{H^{r,\alpha}_{w,p}}. \tag{49}$$

By Lemma 2.2, we have

$$\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{w,p} \le \frac{C}{n^r}\|\mathcal{D}^r f\|_{w,p}. \tag{50}$$

Hence, by Lemma 4.2 and (50), we obtain

$$\begin{aligned}
\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{H^{r,\alpha}_{w,p}} &= \widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{w,p} + \sup_{0<h<\pi} \frac{\widetilde{\omega}_r(\widetilde{\Delta}^r_h f, \frac{1}{n})_{w,p}}{h^\alpha} \\
&\le \frac{C}{n^r}\left(\|\mathcal{D}^r f\|_{w,p} + \sup_{0<h<\pi} \frac{\|\mathcal{D}^r \widetilde{\Delta}^r_h f\|_{w,p}}{h^\alpha}\right) \\
&= \frac{C}{n^r}\left(\|\mathcal{D}^r f\|_{w,p} + \sup_{0<h<\pi} \frac{\|\widetilde{\Delta}^r_h \mathcal{D}^r f\|_{w,p}}{h^\alpha}\right) \\
&= \frac{C}{n^r}\|\mathcal{D}^r f\|_{H^{r,\alpha}_{w,p}}.
\end{aligned} \tag{51}$$

Inequalities (51) together with (48) and Remark 4.4 imply that

$$\|f - \mathcal{L}_n(f)\|_{H^{r,\alpha}_{w,p}} \le C\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{H^{r,\alpha}_{w,p}} \le \frac{C}{n^r}\|\mathcal{D}^r f\|_{H^{r,\alpha}_{w,p}}. \tag{52}$$

Thus, combining (49) and (52), we obtain

$$\widetilde{\omega}_r\left(f, 1/n_k\right)_{w,p} \le C {n_k}^{-r-\alpha}\|\mathcal{D}^r f\|_{H^{r,\alpha}_{w,p}}. \tag{53}$$

At the same time it is easy to verify (see, e.g. [**28**] and (6)) that

$$\frac{\widetilde{\omega}_r\left(f, 1/n\right)_{w,p}}{n^r} \le C\widetilde{\omega}_r\left(f, \frac{1}{n}\right)_{w,p}, \quad n \in \mathbb{N}. \tag{54}$$

Finally, combining (53) and (54), we derive $0 < C < 1/n_k^\alpha$ which is a contradiction. □

Example. As an application of Theorem 4.3, let us consider the Durrmeyer-Bernstein polynomial operators M_n, which are denoted for $f \in L_p[0,1]$, $1 \le p \le \infty$, by

$$M_n(f,x) = \sum_{k=0}^{n} P_{n,k}(x)(n+1)\int_0^1 P_{n,k}(y)f(y)\mathrm{d}y,$$

where

$$P_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}.$$

In [**5**], it was proved that for any $f \in L_p[0,1]$

$$\|f - M_n(f)\|_{L_p[0,1]} \asymp \inf_g \left\{ \|f-g\|_{L_p[0,1]} + \frac{1}{n}\left\| \frac{\mathrm{d}}{\mathrm{d}x} x(1-x) \frac{\mathrm{d}}{\mathrm{d}x} g \right\|_{L_p[0,1]} \right\}, \quad n \in \mathbb{N}.$$

By Lemma 2.2, after the affine transform $[-1,1] \mapsto [0,1]$, we get the following two-sided estimate:

$$\|f - M_n(f)\|_{L_p[0,1]} \asymp \widetilde{\omega}_2\left(f, \frac{1}{\sqrt{n}}\right)_{L_p[0,1]}, \quad n \in \mathbb{N}, \tag{55}$$

where $\widetilde{\omega}_2\,(f,h)_{L_p[0,1]}$ is the corresponding modification of $\widetilde{\omega}_2\,(f,h)_p$ related to the interval $[0,1]$. Note that the modulus $\widetilde{\omega}_2\,(f,h)_{L_p[0,1]}$ can be computed by the following formula:

$$\widetilde{\omega}_2\,(f,h)_{L_p[0,1]} = \sup_{0<\delta<h} \|\bar{\Delta}_\delta f\|_{L_p[0,1]},$$

where

$$\bar{\Delta}_h f(x) = f(x) - \frac{1}{\pi}\int_{-1}^{1} f\left(2\sin^2\frac{h}{2} - x\cos h - u\sqrt{(3+2x-x^2)(1-u^2)}\right)\frac{\mathrm{d}u}{\sqrt{1-u^2}}. \tag{56}$$

It is easy to see that the operators M_n, $n \in \mathbb{N}$, satisfy condition (43) (see, for example, [**8**, Ch. 10, § 8]). Thus, by Theorem 4.3 and (55), we obtain the following strong converse inequality:

$$\|f - M_n(f)\|_{H_p^{r,\alpha}[0,1]} \asymp \widetilde{\omega}_2\left(f, \frac{1}{\sqrt{n}}\right)_{H_p^{r,\alpha}[0,1]}, \quad n \in \mathbb{N}. \tag{57}$$

From inequality (57), Theorem 3.1, and Remark 4.1 one can deduce

Corollary 4.7. Let $1 \le p \le \infty$, $0 < \alpha \le r$, and $0 < \gamma < 2$. Then the following conditions are equivalent:

(i) $\|f - M_n(f)\|_{H_p^{r,\alpha}[0,1]} = \mathcal{O}\left(n^{-\gamma/2}\right), \quad n \to \infty,$

(ii) $\widetilde{\omega}_2\,(f,\delta)_{H_p^{r,\alpha}[0,1]} = \mathcal{O}\left(\delta^\gamma\right), \quad \delta \to 0,$

(iii) $E_n(f)_{H_p^{r,\alpha}[0,1]} = \mathcal{O}\left(n^{-\gamma}\right), \quad n \to \infty,$

(iv) $E_n(f)_{L_p[0,1]} = \mathcal{O}\left(n^{-\gamma-\alpha}\right), \quad n \to \infty.$

Remark 4.8. If $\alpha \ge \beta > -1$, $\alpha + \beta > -1$, and k and r are even numbers, then all results of Subsection 2.1, Section 3, and Section 4 remain true (see [**1**], [**17**] and the remark in [**32**]).

5. The Hölder spaces with respect to the Ditzian-Totik moduli of smoothness (second approach)

5.1. Preliminary remarks and auxiliary results. In this section, we use another approach to the problem of approximation of functions by algebraic polynomials in the Hölder spaces. The main point of this approach is to consider the Hölder spaces generated by the Ditzian-Totik moduli of smoothness, in which we can deal with the case $0 < p < 1$.

For simplicity, we consider the unweighted case. However, most of our results are valid with some Jacobi weights, too.

Now let us introduce the necessary notations. We denote by I an interval of the real line and by φ an admissible function with respect to I in the sense of Ditzian-Totik (see [**14**, p. 8]). Let $L_p(I)$, $0 < p < \infty$, be the usual Lebesgue spaces with the (quasi-)norm $\|f\|_p = \left(\int_I |f(x)|^p \mathrm{d}x\right)^{1/p}$ and $L_\infty(I) = C(I)$ with the norm $\|f\|_\infty = \max_{x\in I} |f(x)|$.

For a function $f \in L_p(I)$, $0 < p \le \infty$, and $r \in \mathbb{N}$, the Ditzian-Totik modulus of smoothness is given by

$$\omega_r^\varphi(f,\delta)_p = \sup_{|h|\le\delta} \|\Delta_{h\varphi}^r f\|_p,$$

where φ is an admissible function for an interval I in the sense of Ditzian-Totik and

$$\Delta_{h\varphi(x)}^r f(x) = \begin{cases} \displaystyle\sum_{k=0}^{r} (-1)^k \binom{r}{k} f\left(x + \left(\frac{r}{2} - k\right) h\varphi(x)\right), & x \pm \frac{r}{2} h\varphi(x) \in I, \\ 0, & \text{otherwise.} \end{cases}$$

Now we are able to define the Hölder spaces with respect to the Ditzian-Totik modulus of smoothness. We will say that $f \in H_p^{r,\alpha,\varphi}(I)$, $I \subset \mathbb{R}$, $0 < \alpha \le r$, $r \in \mathbb{N}$, $0 < p \le \infty$, if $f \in L_p(I)$ and

$$\|f\|_{H_p^{r,\alpha,\varphi}} = \|f\|_{H_p^{r,\alpha,\varphi}(I)} = \|f\|_{L_p(I)} + |f|_{H_p^{r,\alpha,\varphi}(I)} < \infty,$$

where

$$|f|_{H_p^{r,\alpha,\varphi}} = |f|_{H_p^{r,\alpha,\varphi}(I)} = \sup_{0<h<1} \frac{\omega_r^\varphi(f,h)_p}{h^\alpha}.$$

Proposition 5.1. Let $f \in H_p^{r,\alpha,\varphi}[-1,1]$, $1 < p < \infty$, $0 < \alpha \le r$, $r \in \mathbb{N}$, and $\varphi(x) = \sqrt{1-x^2}$. Then $f \in H_p^{r,\alpha}$ and

$$\|f\|_{H_p^{r,\alpha,\varphi}} \asymp \|f\|_{H_p^{r,\alpha}},$$

where $H_p^{r,\alpha} = H_{p,w_{0,0}}^{r,\alpha}$ is related to the Jacobi translation $T_h^{(0,0)}$.

Proof. From Theorem 7.1 in [**6**], it follows that there exists a constant C such that for any $f \in L_p$ and $t \in (0,t_0)$

$$C^{-1} K_r^\varphi(f,t)_p \le \widetilde{K}_r(f,t)_p \le C\left(K_r^\varphi(f,t)_p + t^r\|f\|_p\right),$$

where $K_r^\varphi(f,t)_p = \inf_g\{\|f-g\|_p + t^r\|\varphi^r g^{(r)}\|_p\}$ is the K-functional related to a function φ.

It is well-known (see [**14**, Ch. 2]) that

$$K_r^\varphi(f,t)_p \asymp \omega_r^\varphi(f,t)_p, \quad t \in (0,t_0). \tag{58}$$

Thus, (6) and (58) imply Proposition 5.1. □

Remark 5.2. Proposition 5.1 does not hold for $p = 1$ or $p = \infty$. Nevertheless, for any $0 < \alpha \le 2$ the inequalities

$$\|f\|_{H_\infty^{2,\alpha,\varphi}} \le C\|f\|_{H_\infty^{2,\alpha}}$$

and

$$\|f\|_{H_1^{2,\alpha}} \le C\|f\|_{H_1^{2,\alpha,\varphi}}$$

are valid (see [**6**, Remark 7.9]).

Proposition 5.1 implies that results on approximation in the spaces $H_p^{r,\alpha}$ can be transferred to $H_p^{r,\alpha,\varphi}$ in the case $1<p<\infty$. For example, from (57) we obtain for $1<p<\infty$ and $\varphi(x)=\sqrt{x(1-x)}$ the two-sided estimate

$$\|f-M_n(f)\|_{H_p^{r,\alpha,\varphi}[0,1]}\asymp\widetilde{\omega}_2\left(f,\frac{1}{\sqrt{n}}\right)_{H_p^{r,\alpha}[0,1]}.$$

Moreover, taking into account that

$$\|f-M_n(f)\|_{L_p[0,1]}\asymp\omega_2^\varphi\left(f,\frac{1}{\sqrt{n}}\right)_{L_p[0,1]}+\frac{1}{n}\|f\|_{L_p[0,1]}$$

(see (8.11) and (8.14) in [**10**]) we obtain that

$$\|f-M_n(f)\|_{H_p^{r,\alpha,\varphi}[0,1]}\asymp\omega_2^\varphi\left(f,\frac{1}{\sqrt{n}}\right)_{L_p[0,1]}$$
$$+\sup_{0<h<1}h^{-\alpha}\left(\omega_2^\varphi\left(\bar{\Delta}_h^r f,\frac{1}{\sqrt{n}}\right)_{L_p[0,1]}+\frac{1}{n}\|\bar{\Delta}_h^r f\|_{L_p[0,1]}\right),$$

where $\bar{\Delta}$ was defined by (56).

Below, we collect auxiliary results, which correspond to the similar results from Section 2.1. In what follows we let $I=[-1,1]$, $\varphi(x)=\sqrt{1-x^2}$, and $p_1=\min(p,1)$.

Lemma 5.3. (See [**26**] and [**15**]). Let $0<p\le\infty$, $k,r\in\mathbb{N}$, and $0<\alpha<r<k$. Then the (quasi-)norms of a function in the spaces $H_p^{r,\alpha,\varphi}$ and $H_p^{k,\alpha,\varphi}$ are equivalent.

Recall the Jackson-type theorem for the Ditzian-Totik modulus of smoothness in L_p-spaces (see in [**14**] for $1\le p\le\infty$ and in [**7**] for $0<p<1$).

Lemma 5.4. Let $f\in L_p$, $0<p\le\infty$, and $k\in\mathbb{N}$. Then

$$E_n(f)_p\le C\omega_k^\varphi\left(f,\frac{1}{n}\right)_p,\quad n>4k,$$

where C is a constant independent of n and f.

The following Stechkin-Nikolskii type inequality corresponds to Lemma 2.3 (see [**20**]).

Lemma 5.5. Let $0<p\le\infty$, $n\in\mathbb{N}$, $0<h\le 1/n$, and $r\in\mathbb{N}$. Then for any algebraic polynomial $P_n\in\mathcal{P}_n$ we have

$$h^r\|\varphi^r P_n^{(r)}\|_p\asymp\omega_r^\varphi(P_n,h)_p,$$

where $\asymp$ is a two-sided inequality with absolute constants independent of P_n and h. Moreover, if P_n is a polynomial of the best approximation of a function $f\in L_p$, then

$$\omega_r^\varphi(P_n,h)_p\le C\omega_r^\varphi\left(f,\frac{1}{n}\right)_p,$$

where C is a constant independent of P_n, h, and f.

5.2. Properties of the best approximation in $H_p^{r,\alpha,\varphi}$. Direct and inverse theorems. In this subsection we present results which correspond to the similar results from Section 3. The proofs can easily be obtained by using the schemes of proofs for the corresponding results from Section 3 and the above auxiliary results.

As above, we denote the error of the best approximation in the Hölder space $H_p^{r,\alpha,\varphi}$ by

$$E_n(f)_{H_p^{r,\alpha,\varphi}}=\inf_{P\in\mathcal{P}_{n-1}}\|f-P\|_{H_p^{r,\alpha,\varphi}},\quad n\in\mathbb{N}.$$

A polynomial $P \in \mathcal{P}_{n-1}$ is called a polynomial of the best approximation of $f \in H_p^{r,\alpha,\varphi}$ if

$$\|f - P\|_{H_p^{r,\alpha,\varphi}} = E_n(f)_{H_p^{r,\alpha,\varphi}}.$$

As above, we have the following connection between the errors of the best approximation in the spaces $H_p^{r,\alpha,\varphi}$ and L_p:

THEOREM 5.6. Let $f \in H_p^{r,\alpha,\varphi}$, $0 < p \le \infty$, $0 < \alpha \le r$, and $r \in \mathbb{N}$. Then

$$C^{-1} n^\alpha E_n(f)_p \le E_n(f)_{H_p^{r,\alpha,\varphi}} \le C \left(n^\alpha E_n(f)_p + \left(\sum_{\nu=n}^{\infty} \nu^{\alpha p_1 - 1} E_\nu(f)_p^{p_1} \right)^{\frac{1}{p_1}} \right), \quad n \in \mathbb{N},$$

where C is a positive constant independent of n and f.

Theorem 5.6 and some standard arguments give us the following direct estimate:

THEOREM 5.7. Let $f \in H_p^{r,\alpha,\varphi}$, $0 < p \le \infty$, $0 < \alpha \le r$, and $k, r \in \mathbb{N}$. Then

$$E_n(f)_{H_p^{r,\alpha,\varphi}} \le C \left(\int_0^{1/n} \left(\frac{\omega_k^\varphi(f,t)_p}{t^\alpha} \right)^{p_1} \frac{\mathrm{d}t}{t} \right)^{\frac{1}{p_1}}, \quad n \in \mathbb{N}, \tag{59}$$

where C is a constant independent of f and n.

Some slight improvements of (59) can be obtained by using the modulus of smoothness

$$\theta_{k,\alpha}^\varphi(f,\delta)_p = \sup_{0<h\le\delta} \frac{\omega_k^\varphi(f,h)_p}{h^\alpha}.$$

THEOREM 5.8. Let $f \in H_p^{r,\alpha,\varphi}$, $0 < p \le \infty$, $0 < \alpha < \min(r,k)$ or $0 < \alpha = k = r$, and $r, k \in \mathbb{N}$. Then

$$E_n(f)_{H_p^{r,\alpha,\varphi}} \le C \theta_{k,\alpha}^\varphi \left(f, \frac{1}{n}\right)_p, \quad n > 4k,$$

$$\theta_{k,\alpha}^\varphi \left(f, \frac{1}{n}\right)_p \le \frac{C}{n^{k-\alpha}} \left(\sum_{\nu=1}^{n} \nu^{(k-\alpha)p_1 - 1} E_\nu(f)_{H_p^{r,\alpha,\varphi}}^{p_1} \right)^{\frac{1}{p_1}}, \quad n \in \mathbb{N},$$

where C is a constant independent of n and f.

Note that Theorem 5.8 was obtained in [**4**] in the case $1 \le p \le \infty$ and $\alpha < k = r$.

Now let us consider the problem of the precise order of decrease of the best approximation in $H_p^{r,\alpha,\varphi}$. The proof of Theorem 5.9 below is similar to the proof of Theorem 3.5, see also the proof of related results in the case $0 < p < 1$ in [**23**]. We only note that instead of (25) we will use the inequality

$$\theta_{k,\alpha}^\varphi(f,\delta)_p \le C \theta_{r,\alpha}^\varphi(f,\delta)_p, \quad k > r, \quad \delta \in (0, \delta_0).$$

This inequality can be obtained from Lemma 5.4, Lemma 5.5, and the following Bernstein type inequality (see [**11**])

$$\|\varphi^r P_n'\|_p \le Cn \|\varphi^{r-1} P_n\|_p, \quad 0 < p \le \infty, \quad 1 \le r \le n,$$

where a constant C is independent of n. Concerning an analog of inequality (26) we have in the case $1 \le p \le \infty$, that

$$\theta_{k,\alpha}^\varphi(f, n\delta)_p \le C n^{k-\alpha} \theta_{k,\alpha}^\varphi(f,\delta)_p, \tag{60}$$

where a constant C is independent of f and $\delta \in (0, \delta_0)$. Inequality (60) is a simple corollary from the corresponding inequality for Ditzian-Totik moduli of smoothness

$$\omega_k^\varphi(f, n\delta)_p \le C n^k \omega_k^\varphi(f,\delta)_p$$

(see, e.g., [**14**, Ch. 2]). In the case $0<p<1$, repeating step by step the proof of Lemma 5.2 in [**12**] (see also Corollary 5.5 in [**12**]) and using the equality (see, e.g. [**27**, p. 187–188])

$$\Delta_{n\delta\varphi(x)}^r f(x)=\sum_{\nu=0}^{r(n-1)} A_{\nu,n}^{(k)}\,\Delta_{\delta\varphi(x)}^r f\left(x+\left(\nu-\frac{r(n-1)}{2}\right)\delta\varphi(x)\right),$$

where $0<A_{\nu,n}^{(k)}\le n^{k-1}$ and $x\pm rn\delta\varphi(x)/2\in(-1,1)$, we can prove that for every $f\in L_p$, $0<p<1$, and $k,n\in\mathbb{N}$ one obtains

$$\omega_k^\varphi(f,n\delta)_p\le Cn^{\frac3p+2(k-1)}\omega_k^\varphi(f,\delta)_p\,.$$

Therefore, in the case $0<p<1$ we get

$$\theta_{k,\alpha}^\varphi(f,n\delta)_p\le Cn^{\frac3p+2(k-1)-\alpha}\theta_{k,\alpha}^\varphi(f,\delta)_p\,.$$

THEOREM 5.9. Let $f\in H_p^{r,\alpha,\varphi}$, $0<p\le\infty$, $0<\alpha<r$, $s\ge\alpha$, and $r,s\in\mathbb{N}$. Then the following assertions are equivalent:
(i) there exists a constant $L>0$ such that

$$\theta_{s,\alpha}^\varphi\left(f,\frac1n\right)_p\le LE_n(f)_{H_p^{r,\alpha,\varphi}},\quad n\in\mathbb{N},$$

(ii) for some

$$k>\begin{cases} s, & 1\le p\le\infty,\\ 3/p+2(s-1), & 0<p<1,\end{cases}$$

there exists a constant $M>0$ such that

$$\theta_{s,\alpha}^\varphi(f,h)_p\le M\theta_{k,\alpha}^\varphi(f,h)_p,\quad h>0.$$

6. Improvements of some estimates of approximation by Bernstein operators in Hölder spaces

Now let us consider the approximation of functions by polynomial operators for which we cannot apply the methods from Section 4. These are, for example, the Bernstein, Kantorovich, and Szasz-Mirakyan polynomial operators. We restrict ourself to the Bernstein operators

$$B_n(f,x)=\sum_{k=0}^n f\left(\frac kn\right)\binom nk x^k(1-x)^{n-k}.$$

Everywhere below $I=[0,1]$, $\varphi(x)=\sqrt{x(1-x)}$, $\|f\|=\|f\|_\infty$, $\omega_r^\varphi(f,h)=\omega_r^\varphi(f,h)_\infty$, $\theta_{r,\alpha}^\varphi(f,h)=\theta_{r,\alpha}^\varphi(f,h)_\infty$, and $H^{r,\alpha,\varphi}=H_\infty^{r,\alpha,\varphi}$.

It is well-known (see [**33**]) that for any $f\in C(I)$ the following two-sided estimate holds:

$$\|f-B_n(f)\|\asymp\omega_2^\varphi\left(f,\frac1{\sqrt n}\right). \tag{61}$$

In [**4**] it was proved that for any $f\in C(I)$ and $0<\alpha<2$

$$\|f-B_n(f)\|_{H^{2,\alpha,\varphi}}\le C\theta_{2,\alpha}^\varphi\left(f,\frac1{\sqrt n}\right), \tag{62}$$

where C is a constant independent of f and n.

Clearly, inequality (62) is not sharp. Indeed, if $f^{(r-1)}$ is locally absolutely continuous and $\varphi^r f^{(r)}\in C(I)$, then

$$\omega_r^\varphi(f,h)\le C_rh^r\|\varphi^rf^{(r)}\| \tag{63}$$

(see [**14**, p. 63]). Moreover, the following result was proved in [**19**]:

LEMMA 6.1. Let $f \in C^k(I)$ and $k = 1, 2, \ldots, n-1$. Then

$$\|(f - B_n(f))^{(k)}\| \le \|f^{(k)} - B_{n-k}(f^{(k)})\| + \min\left\{1, \frac{(k-1)^2}{n}\right\}\|f^{(k)}\| + \omega_1\left(f^{(k)}, \frac{k}{n}\right), \tag{64}$$

where $\omega_1(f,h) = \sup\{|f(x) - f(y)|,\, |x-y| < h,\, x, y \in I\}$.

Thus, by (63), (64), and (61), we obtain for $f \in C^2(I)$, $n \ge 3$, and $0 < \alpha \le 2$

$$\begin{aligned} \sup_{0<h<1} \frac{\omega_2^\varphi(f - B_n(f), h)}{h^\alpha} &\le C\|\varphi^2(f'' - B_n''(f))\| \le C\|f'' - B_n''(f)\| \\ &\le C\left(\|f'' - B_{n-2}(f'')\| + \omega_1\left(f'', \frac{2}{n}\right) + \frac{1}{n}\|f''\|\right) \\ &\le C\left(\omega_2^\varphi\left(f'', \frac{1}{\sqrt{n-2}}\right) + \omega_1\left(f'', \frac{2}{n}\right) + \frac{1}{n}\|f''\|\right). \end{aligned} \tag{65}$$

If in addition, $f \in C^4(I)$, then from (65) we get

$$\|f - B_n(f)\|_{H^{2,\alpha,\varphi}} = \mathcal{O}\left(\frac{1}{n}\right), \quad n \to \infty.$$

At the same time, inequality (62) yields the following less sharp estimate:

$$\|f - B_n(f)\|_{H^{2,\alpha,\varphi}} = \mathcal{O}\left(\frac{1}{n^{1-\alpha/2}}\right), \quad n \to \infty.$$

Let us present here an improvement of (62).

PROPOSITION 6.2. Let $f \in H^{2,\alpha,\varphi}$, $0 < \alpha \le 2$, $k \ge 2$, and $0 \le \gamma \le 1/2$. Then

$$\|f - B_n(f)\|_{H^{2,\alpha,\varphi}} \le C_k\left(\theta_{k,\alpha}^\varphi\left(f, \frac{1}{n^\gamma}\right) + n^{\alpha\gamma}\omega_2^\varphi\left(f, \frac{1}{\sqrt{n}}\right)\right). \tag{66}$$

The proof of this proposition is standard. One should only take into account Lemma 5.3.

Let us show that (66) is an improvement of (62). Indeed, let $\omega_k^\varphi(f,h) = \mathcal{O}(h^{\eta_k})$, where $k \ge 2$. Then, by choosing $\gamma = \eta_2/\eta_k$, we get

$$\|f - B_n(f)\|_{H^{2,\alpha,\varphi}} = \mathcal{O}\left(n^{-\frac{1}{2}\left(\eta_2 - \alpha\frac{\eta_2}{\eta_k}\right)}\right).$$

At the same time, (62) provides only $\mathcal{O}(n^{-\frac{1}{2}(\eta_2 - \alpha)})$.

As one can see, inequality (66) represents only a slight improvement of (62). Because, for $f \in C^4(I)$ we already have $\mathcal{O}(n^{-1})$, but (66) even for $f \in C^N(I)$, $N \ge 4$, yields $\mathcal{O}\left(n^{-1+\alpha/N}\right)$.

By using a combination of Proposition 6.2 and Lemma 6.1, one can obtain stronger results for smooth functions.

Let us consider the more classical Hölder spaces $C^{r,\alpha}$ with the norm

$$\|f\|_{C^{r,\alpha}} = \|f\| + \sup_{0<h<1} \frac{\|\Delta_h^r f\|}{h^\alpha}$$

instead of $H^{2,\alpha,\varphi}$.

THEOREM 6.3. Let $f \in C^r(I)$, $0 < \alpha \le r$, $r \in \mathbb{N}$, $\gamma \ge 0$, and $n \ge r^2 + 1$. Then

$$\begin{aligned} \|f - B_n(f)\|_{C^{r,\alpha}} \le \frac{1}{n^{\gamma(r-\alpha)}}&\left(C\omega_2^\varphi\left(f^{(r)}, \frac{1}{\sqrt{n}}\right) + \omega_1\left(f^{(r)}, \frac{r}{n}\right)\right. \\ &\left. + \frac{(r-1)^2}{n}\|f^{(r)}\|\right) + Cn^{\gamma\alpha}\omega_2^\varphi\left(f, \frac{1}{\sqrt{n}}\right), \end{aligned}$$

where C is a constant independent of f and n and ω_1 was defined in Lemma 6.1.

Proof. We have

$$\sup_{0<h<1}\frac{\|\Delta_h^r(f-B_n(f))\|}{h^\alpha}\le\left(\sup_{0<h<1/n^\gamma}+\sup_{1/n^\gamma\le h<1}\right)\frac{\|\Delta_h^r(f-B_n(f))\|}{h^\alpha}=S_1+S_2. \tag{67}$$

By (61), it is easy to see that

$$S_2\le Cn^{\gamma\alpha}\|f-B_n(f)\|\le Cn^{\gamma\alpha}\omega_2^\varphi\left(f,\frac{1}{\sqrt{n}}\right). \tag{68}$$

By using (63) with $\varphi\equiv 1$, Lemma 6.1, and (61), we obtain

$$\begin{aligned}S_1&\le\frac{C}{n^{\gamma(r-\alpha)}}\|(f-B_n(f))^{(r)}\|\\&\le\frac{C}{n^{\gamma(r-\alpha)}}\left(\omega_2^\varphi\left(f^{(r)},\frac{1}{\sqrt{n}}\right)+\omega_1\left(f^{(r)},\frac{r}{n}\right)+\frac{(r-1)^2}{n}\|f^{(r)}\|\right).\end{aligned} \tag{69}$$

Thus, combining (67), (68), and (69), we have proved the theorem. □

In this sense Theorem 6.3 is also an improvement of the main results of [**19**] and [**18**].

Acknowledgements. We would like to thank the referees for their valuable comments and remarks. The authors were supported by FP7-People-2011-IRSES Project number 295164 (EUMLS: EU-Ukrainian Mathematicians for Life Sciences). The first author was partially supported by Project 15-1bb\19, "Metric Spaces, Harmonic Analysis of Functions and Operators and Singular and Non-classic Problems for Differential Equations", Donetsk National University (Vinnitsa, Ukraine).

References

[1] R. Askey and S. Wainger, A convolution structure for Jacobi series, Amer. J. Math. **91**, 463–485 (1969).
[2] W. R. Bloom and D. Elliott, The modulus of continuity of the remainder in the approximation of Lipschitz functions, J. Approx. Theory **31**, 59-66 (1981).
[3] J. Bustamante and M. A. Jiménez, Trends in Hölder approximation, in: Proceedings of the Fifth International Conference on Approximation and Optimization in the Caribbean (Guadeloupe, France, 1999); M. Lassonde (Ed.), Approximation, Optimization and Mathematical Economics (Springer, Berlin, 2001), 81–95.
[4] J. Bustamante and C. Roldan, Direct and inverse results in Hölder norms, J. Approx. Theory **138**, 112–132 (2006).
[5] W. Chen, Z. Ditzian, and K. Ivanov, Strong converse inequality for the Bernstein-Durrmeyer operator, J. Approx. Theory **75**(1), 25–43 (1993).
[6] F. Dai and Z. Ditzian, Littlewood-Paley theory and sharp Marchaud inequality, Acta Math. Szeged **71**(1–2), 65–90 (2005).
[7] R. A. DeVore, D. Leviatan, and X. M. Yu, Polynomial approximation in L_p, $0<p<1$, Constr. Approx. **8**(2), 187–201 (1992).
[8] R. A. DeVore and G. G. Lorentz, Constructive Approximation (Springer-Verlag, New York, 1993).
[9] Z. Ditzian, Fractional derivatives and best approximation, Acta Math. Hungar. **81**(4), 323–348 (1998).
[10] Z. Ditzian, Polynomial approximation and $\omega_\varphi^r(f,t)$. Twenty years later. Surveys in Approximation Theory **3**, 106–151 (2007).
[11] Z Ditzian, D. Jiang, and D. Leviatan, Inverse theorem for best polynomial approximation in L_p, $0<p<1$, Proc. Amer. Math. Soc. **120**(1), 151-154 (1994).
[12] Z. Ditzian, V. Hristov, and K. Ivanov, Moduli of smoothness and K-functional in L_p, $0<p<1$, Constr. Approx. **11**, 67–83 (1995).
[13] Z. Ditzian and K. Ivanov, Strong converse inequalities, J. Analyse Math. **61**, 61–111 (1993).
[14] Z. Ditzian and V. Totik, Moduli of Smoothness (Springer-Verlag, Berlin-New York, 1987).
[15] Z. Ditzian and V. Totik, Remarks on Besov spaces and Best Polynomial Approximation, Proc. Amer. Math. Soc. **104**(4), 1059–1066 (1988).
[16] D. Elliot, On the Hölder semi-norm of the remainder in polynomial approximation, Bull. Austral. Math. Soc. **49** (1994), 421-426.

[17] G. Gasper, Positivity and the convolution structure for Jacobi series, Ann. of Math. **93**(1), 112–118 (1971).

[18] H. Gonska, J. Prestin, and G. Tachev, A new estimate for Hölder approximation by Bernstein operators, Appl. Math. Lett. **26**, 43–45 (2013).

[19] H. Gonska, J. Prestin, G. Tachev, and D. X. Zhou, Simultaneous approximation by Bernstein operators in Hölder norms, Math. Nachr. **286**, 349–359 (2013).

[20] Y. Hu and Y. Liu, On equivalence of moduli of smoothness of polynomials in L_p, $0 < p \le \infty$, J. Approx. Theory **136**(2), 182–197 (2005).

[21] N. I. Ioakimidis, An improvement of Kalandiyas theorem, J. Approx. Theory **38**, 354–356 (1983).

[22] A. I. Kalandiya, A direct method for the solution of the wind equation and its applications in elasticity theory, Mat. Sb. **42**, 249-272 (1957) (in Russian).

[23] Yu. S. Kolomoitsev, On moduli of smoothness and Fourier multipliers in L_p, $0 < p < 1$, Ukrainian Mathematical Journal **59**(9), 1364–1384 (2007).

[24] Yu. Kolomoitsev, J. Prestin, Sharp estimates of approximation of periodic functions in Hölder spaces, J. Approx. Theory **200**, 68–91 (2015).

[25] L. Leindler, A. Meir, and V. Totik, On approximation of functions in Lipschitz norms, Acta Math. Hung. **45**(3-4), 441-443 (1985).

[26] S. Li, The equivalence relations between the Ditzian-Totik moduli of smoothness and best polynomial approximation in the Besov spaces, J. Math. Anal. Appl. **215**(1), 1–14 (1997).

[27] P. P. Petrushev and V. A. Popov, Rational Approximation of Real Functions, Cambridge University Press, Cambridge, UK, 1987.

[28] S. S. Platonov, Fourier-Jacobi harmonic analysis and approximation of functions, Izv. Ross. Akad. Nauk, Ser. Mat. **78**(1), 117–166 (2014).

[29] J. Prestin, S. Prössdorf, Error estimates in generalized trigonometric Hölder-Zygmund norms, Z. Anal. Anwend. **9**(4), 343–349 (1990).

[30] S. Prössdorf, Zur Konvergenz der Fourierreihen Hölderstetiger Funktionen, Math. Nachr. **69**, 7–14 (1975).

[31] R. Rathore, The Problem of A. F. Timan on the precise order of decrease of the best approximations, J. Approx. Theory **77**(2), 153–166 (1994).

[32] Kh. P. Rustamov, Moduli of smoothness of higher orders related to the Fourier-Jacobi expansion and the approximation of functions by algebraic polynomials. (English. Russian original) Dokl. Math. **52**(2), 244–247 (1995); translation from Dokl. Akad. Nauk, Ross. Akad. Nauk **344**(5), 593–596 (1995).

[33] V. Totik, Approximation by Bernstein polynomials, Amer. J. Math. **116**, 995–1018 (1994).

[34] R. M. Trigub and E. S. Belinsky, Fourier Analysis and Appoximation of Functions (Kluwer, 2004).

[35] A. Weinmann, R. Lasser, Lipschitz spaces with respect to Jacobi translation, Math. Nachrichten **284**, 2312–2326 (2011).

[36] V. V. Zhuk, Approximating periodic functions in Hölder type metrics by the Fourier sums and the Riesz means, J. of Math. Sci. (New York) **150**(3), 2045-2055 (2008).

Institute of Applied Mathematics and Mechanics, NAS of Ukraine, Dobrovol's'kogo 1, 84100 Slov'yans'k, Ukraine

E-mail address: kolomus1@mail.ru

Institute of Applied Mathematics and Mechanics, NAS of Ukraine, Dobrovol's'kogo 1, 84100 Slov'yans'k, Ukraine

E-mail address: tlomako@yandex.ru

Universität zu Lübeck, Institut für Mathematik, Ratzeburger Allee 160, 23562 Lübeck

E-mail address: prestin@math.uni-luebeck.de