On two-dimensional classical and Hermite sampling

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Abstract

We investigate some modifications of the two-dimensional sampling series with a Gaussian function for wider classes of bandlimited functions including unbounded entire functions on $\mathbb{R}^2$ and analytic functions on a bivariate strip. The first modification is given for the two-dimensional version of the Whittaker-Kotelnikov-Shannon sampling (classical sampling) and the second is given for two-dimensional sampling involving values of all partial derivatives of order $\alpha \leq 2$ (Hermite sampling). These modifications improve the convergence rate of classical and Hermite sampling which will be of exponential type. Numerical examples are given to illustrate the advantages of the new method.

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1. Introduction

Denote by $L^p(\mathbb{R}^2)$, $1 \leq p \leq \infty$ the Banach space of all complex-valued Lebesgue measurable functions f with the usual norm $\|f\|_p$. The entire function $f(z)$ is of exponential type σ, $\sigma > 0$, if there exists a positive constant A such that

$$|f(z)| \leq A \exp\left(\sigma \sum_{j=1}^{2} |\Im z_j|\right), \quad z := (z_1, z_2) \in \mathbb{C}^2.$$

The Bernstein space, $B_{\sigma,p}\left(\mathbb{R}^2\right)$, $1 \leq p \leq \infty$, is the class of all entire functions of exponential type σ, which belong to $L^p(\mathbb{R}^2)$ when restricted to $\mathbb{R}^2$. According to Schwartz's theorem [18, p.109],

$$B_{\sigma,p}\left(\mathbb{R}^2\right) = \left\{f \in L^p\left(\mathbb{R}^2\right) : \operatorname{supp} \widehat{f} \subset [-\sigma, \sigma]^2\right\},$$

where $\widehat{f}$ is Fourier transform of f in the sense of generalized functions. The two-dimensional Whittaker-Kotelnikov-Shannon sampling (classical sampling) theorem states that if $f \in B_{\sigma,p}\left(\mathbb{R}^2\right)$, we can reconstruct via the following sampling expansion, see e.g. [9, 19, 20, 28],

$$f(x) = \sum_{k \in \mathbb{Z}^2} f\left(\frac{k\pi}{\sigma}\right) \prod_{j=1}^{2} \operatorname{sinc}\left(\sigma x_j - k_j \pi\right), \quad x \in \mathbb{R}^2, \tag{1.1}$$

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where $k := (k_1, k_2)$, $x = (x_1, x_2)$ and the sinc function is defined as

$$\operatorname{sinc} t = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0, \\ 1, & t = 0. \end{cases}$$

The series on the right-hand side of (1.1) converges absolutely and uniformly on $\mathbb{R}^2$, [28]. The classical sampling series (1.1) goes back to Parzen (1956), Peterson-Middleton (1962) and Gosselin (1963), see [9, 19, 20]. They introduced the sampling series (1.1) on the Paley-Wiener space, $B_{\sigma,2}\left(\mathbb{R}^2\right)$, and Wang-Fang (1996) extended their result to functions in the Bernstein space, $B_{\sigma,p}\left(\mathbb{R}^2\right)$, $1 < p < \infty$, [28].

Furthermore, there are many sampling expansions which involve samples of partial derivatives, e.g. multidimensional versions of Hermite sampling. To our knowledge the first multidimensional series using values from the function and its partial derivatives was introduced by Montgomery (1965), [17]. A more general form of two-dimensional sampling involving values of partial and mixed partial derivatives was given by Horng (1977), [12]. Fang and Li (2006) introduced a multidimensional version of the Hermite sampling theorem involving only samples from all the first partial derivatives for functions from $B_{2\sigma,p}\left(\mathbb{R}^2\right)$. They proved that, [8], if $f \in B_{2\sigma,p}\left(\mathbb{R}^2\right)$, we have the sampling series for all $x \in \mathbb{R}^2$

$$f(x) = \sum_{k\in\mathbb{Z}^2} \left\{ f\left(\frac{k\pi}{\sigma}\right) + \sum_{i=1}^{2} f_i'\left(\frac{k\pi}{\sigma}\right)\left(x_i - \frac{k_i\pi}{\sigma}\right) \right\} \prod_{j=1}^{2} \operatorname{sinc}^2\left(\sigma x_j - k_j\pi\right), \tag{1.2}$$

where $k := (k_1, k_2)$, $x = (x_1, x_2)$ and $f_i' = \partial f/\partial x_i$. The series (1.2) converges absolutely and uniformly on $\mathbb{R}^2$, [8].

In this paper, we introduce a modification for classical and Hermite sampling by using a bivariate Gaussian function

$$G(z) = \exp\left(-(z_1^2 + z_2^2)\right), \quad z = (z_1, z_2) \in \mathbb{C}^2.$$

We call the modification of classical and Hermite sampling with Gaussian function classical Gauss and Hermite-Gauss formulas, respectively. As we see in the sequel of this paper, these formulas have the following advantages:

- They extend to classes of functions that need not belong to $L^p(\mathbb{R}^2)$. They may even be unbounded on $\mathbb{R}^2$.
- They give a highly better convergence rate. In the following table we summarize the comparison between the convergence rates of the sampling formulas.

Formula	without Gaussian	with Gaussian
Classical sampling	$N^{-1/p}$	$\mathrm{e}^{-\alpha N}/\sqrt{N}$
Hermite sampling	$N^{-1/p}$	$\mathrm{e}^{-\beta N}/\sqrt{N}$

Table 1: The rate of convergence, $\beta > \alpha > 0$.

The modification of classical sampling series with a bandlimited multiplier can reconstruct the functions exactly, see e.g. [26, formula 5]. Here the Gaussian multiplier is not bandlimited and the classical Gauss and Hermite-Gauss formulas cannot reconstruct the functions exactly. These modifications will be used in approximating eigenpairs of two-parameter boundary value problems, [5]. Recently, the one-dimensional classical sampling series with Gaussian function has been used in approximating eigenvalues of one-parameter boundary value problems, see e.g. [2, 3]. The modification of one-dimensional classical sampling series with Gaussian function was studied by many authors in recent years. Qian and his co-authors (2002–2006) studied these formulas in a series of papers [22, 23, 24] using a Fourier-analytic approach. Also, Schmeisser and Stenger (2007) and Tanaka, Sugihara and Murota (2008) studied this approach by complex-analytic methods, [26, 27]. Furthermore, the modification of one-dimensional Hermite sampling series is studied in [4].

We organize this paper as follows: Section 2 is devoted to the bounds of the truncation error of two-dimensional sampling (classical and Hermite). In Section 3, we present a modification of the two-dimensional version of classical sampling on the class of entire functions and some further results and we extend the estimates to the class of analytic functions on a strip. In Section 4, we proceed analogously with the two-dimensional version of Hermite sampling. Numerical examples with figures are given in Section 5.

2. Bounds for the truncation error

The truncation error of multidimensional classical sampling has been studied under the assumption that f satisfies a decay condition, cf. [14, 15]. Ye studied the truncation error of multidimensional classical sampling series based on localized sampling without decay assumption, [29]. For any positive number N, he truncated the series (1.1) as follows:

$$T_{\sigma,N}[f](x) := \sum_{k\in\mathbf{Z}_N^2(x)} f\left(\frac{k\pi}{\sigma}\right)\prod_{j=1}^{2}\operatorname{sinc}\left(\sigma x_j - k_j\pi\right), \quad x\in\mathbb{R}^2,$$

where

$$\mathbf{Z}_N^2(x) := \left\{k\in\mathbb{Z}^2 : -N < \frac{\sigma x_j}{\pi} - k_j \le N, \quad j = 1,2\right\}. \tag{2.1}$$

That is, if we want to estimate f, we only sum over values of f on a part of $(\pi/\sigma)\mathbb{Z}^2$ near x. In what follows, we often use the same symbol C for possibly different positive constants. Let $B_{\sigma,p}\left(\mathbb{R}^2\right)$, $1\le p\le\infty$, then for any $x\in\mathbb{R}^2$ [29, Theorem 1]

$$|f(x) - T_{\sigma,N}[f](x)| \le C\|f\|_p\left(\frac{\sigma}{\pi}\right)^{2/p} N^{-1/p}.$$

As far as we know, there are no studies on the truncation error of multidimensional sampling of Hermite type. Therefore, we use the technique of [29] to find a bound for the truncation error of the series in (1.2). We truncate this series (1.2) as follows:

$$\mathcal{T}_{\sigma,N}[f](x) = \sum_{k\in\mathbf{Z}_N^2(x)}\left\{f\left(\frac{k\pi}{\sigma}\right) + \sum_{i=1}^{2} f_i'\left(\frac{k\pi}{\sigma}\right)\left(x_i - \frac{k_i\pi}{\sigma}\right)\right\}\prod_{j=1}^{2}\operatorname{sinc}^2\left(\sigma x_j - k_j\pi\right). \tag{2.2}$$

In the next lemma we introduce an auxiliary result which will be used in the estimate of the bound of $|f(x) - \mathcal{T}_{\sigma,N}[f](x)|$.

Lemma 2.1. *Let $p,q>1$ such that $\frac{1}{p}+\frac{1}{q}=1$ and let $\sigma>0$. Then for any $(x_1,x_2)\in\mathbb{R}^2$*

$$\left(\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|\prod_{j=1}^{2}\operatorname{sinc}\left(\sigma x_j - k_j\pi\right)\right|^q\right)^{1/q} \le C_{p\sigma}N^{-1/p}, \tag{2.3}$$

where $C_{p\sigma}$ is a positive constant depending on p,σ only.

Proof. From the definition (2.1) of $\mathbf{Z}_N^2(x)$ we obtain

$$\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|\prod_{j=1}^{2}\operatorname{sinc}\,(\sigma x_j-k_j\pi)\right|^q \le \sum_{\left|\frac{\sigma x_1}{\pi}-k_1\right|>N}|\operatorname{sinc}\,(\sigma x_1-k_1\pi)|^q\sum_{k_2=-\infty}^{\infty}|\operatorname{sinc}\,(\sigma x_2-k_2\pi)|^q + \sum_{k_1=-\infty}^{\infty}|\operatorname{sinc}\,(\sigma x_1-k_1\pi)|^q\sum_{\left|\frac{\sigma x_2}{\pi}-k_2\right|>N}|\operatorname{sinc}\,(\sigma x_2-k_2\pi)|^q\,. \tag{2.4}$$

Combining inequality of Splettstößer et al. [25], see also [11, pp. 114-115],

$$\sum_{k=-\infty}^{\infty}|\operatorname{sinc}\,(\sigma x-k\pi)|^q\le p^q$$

and the inequality (cf. [30, p. 415])

$$\sum_{\left|\frac{\sigma x}{\pi}-k\right|>N}|\operatorname{sinc}\,(\sigma x-k\pi)|^q\le C^qN^{-q/p}$$

with (2.4) implies (2.3). □

Lemma 2.2. *Let $f\in B_{\sigma,p}\left(\mathbb{R}^2\right)$, $1\le p\le\infty$, then for any $x\in\mathbb{R}^2$ we have*

$$|f(x)-\mathcal{T}_{\sigma,N}[f](x)|\le C\|f\|_p\left(\frac{\sigma}{\pi}\right)^{2/p}N^{-1/p}. \tag{2.5}$$

Proof. Since $B_{\sigma,p}\left(\mathbb{R}^2\right)\subset B_{2\sigma,p}\left(\mathbb{R}^2\right)$, we can apply the expansion (1.2). Together with (2.2) and the triangle inequality we obtain

$$|f(x)-\mathcal{T}_{\sigma,N}[f](x)|\le\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|f\left(\frac{k\pi}{\sigma}\right)\prod_{j=1}^{2}\operatorname{sinc}\,(\sigma x_j-k_j\pi)\right| +\frac{1}{\sigma}\sum_{i=1}^{2}\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|f_i'\left(\frac{k\pi}{\sigma}\right)\prod_{j=1}^{2}\operatorname{sinc}\,(\sigma x_j-k_j\pi)\right|, \tag{2.6}$$

where we used the fact that $|\operatorname{sinc}t|\le1$ for $t\in\mathbb{R}$. Applying Hölder's inequality for all terms on the right-hand side of (2.6), we obtain

$$|f(x)-\mathcal{T}_{\sigma,N}[f](x)|\le\left(\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}\left(\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|\prod_{j=1}^{2}\operatorname{sinc}\,(\sigma x_j-k_j\pi)\right|^q\right)^{1/q} + \frac{1}{\sigma}\sum_{i=1}^{2}\left(\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|f_i'\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}\left(\sum_{k\in\mathbb{Z}^2\setminus\mathbf{Z}_N^2(x)}\left|\prod_{j=1}^{2}\operatorname{sinc}\,(\sigma x_j-k_j\pi)\right|^q\right)^{1/q} \tag{2.7}$$

with $\frac{1}{p}+\frac{1}{q}=1$. Because of $f\in B_{\sigma,p}\left(\mathbb{R}^2\right)$, we have $f_i'\in B_{\sigma,p}\left(\mathbb{R}^2\right)$, $i=1,2$, and (see [18, pp. 123-124])

$$\left(\sum_{k\in\mathbb{Z}^2}\left|f\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p}\le c\left(\frac{\sigma}{\pi}\right)^{2/p}\|f\|_p, \tag{2.8}$$

$$\left(\sum_{k\in\mathbb{Z}^2}\left|f_i'\left(\frac{k\pi}{\sigma}\right)\right|^p\right)^{1/p} \le c\left(\frac{\sigma}{\pi}\right)^{2/p}\|f_i'\|_p \le c\,\sigma\left(\frac{\sigma}{\pi}\right)^{2/p}\|f\|_p, \tag{2.9}$$

where we used the Bernstein inequality, [18, p.116], in the last step of (2.9). Combining (2.7)–(2.9) with (2.3), we get (2.5) and the proof is completed. □

3. Classical Gauss sampling

We define $E^2_\sigma(\varphi)$, $\sigma \ge 0$, to be the class of entire functions of two variables satisfying the following condition

$$|f(z)| \le \varphi\left(|\Re z_1|, |\Re z_2|\right)\,\exp\left(\sigma\sum_{j=1}^{2}|\Im z_j|\right), \quad z := (z_1, z_2) \in \mathbb{C}^2, \tag{3.1}$$

where φ is a non-negative function on $\mathbb{R}^2_+$ and non-decreasing for both of the variables $|\Re z_j|$, $j = 1, 2$. Clearly, the space $E^2_\sigma(\varphi)$ is larger than the Bernstein space $B_{\sigma,p}\left(\mathbb{R}^2\right)$. Moreover, $B_{\sigma,p}\left(\mathbb{R}^2\right) = E^2_\sigma(C) \cap L^p(\mathbb{R}^2)$ such that the intersection is understood as $f \in E^2_\sigma(C)$ and $f|_{\mathbb{R}^2} \in L^p(\mathbb{R}^2)$ and C is anon-negative constance. For $h \in (0, \pi/\sigma]$, we define $\alpha := (\pi - h\sigma)/2$. Let E^2 be the class of all entire functions on $\mathbb{C}^2$ and let us define the localization operator $\mathcal{G}_{h,N} : E^2_\sigma(\varphi) \to E^2 \cap L^p(\mathbb{R}^2)$ as follows:

$$\mathcal{G}_{h,N}[f](z) := \sum_{k\in\mathbb{Z}^2_N(z)} f(kh)\prod_{j=1}^{2}\operatorname{sinc}\left(\pi h^{-1}z_j - k_j\pi\right)\exp\left(-\frac{\alpha\left(z_j - k_jh\right)^2}{Nh^2}\right), \tag{3.2}$$

where $z \in \mathbb{C}^2$ and

$$\mathbb{Z}^2_N(z) := \left\{k \in \mathbb{Z}^2 : \left|\lfloor h^{-1}\Re z_j + 1/2\rfloor - k_j\right| \le N, \quad j = 1, 2\right\}. \tag{3.3}$$

The operator $\mathcal{G}_{h,N}$ provides a piecewise analytic approximation for functions from $E^2_\sigma(\varphi)$ on each of the bivariate strips

$$\left\{z \in \mathbb{C}^2 : \left(n_j - \frac{1}{2}\right)h \le \Re z_j \le \left(n_j + \frac{1}{2}\right)h, \quad j = 1, 2\right\}, \qquad (n_1, n_2) \in \mathbb{Z}^2. \tag{3.4}$$

Denote the expansion (1.1) by $\mathcal{L}_\sigma[f](z)$ such that $\mathcal{L}_\sigma : B_{\sigma,p}\left(\mathbb{R}^2\right) \to B_{\sigma,p}\left(\mathbb{R}^2\right)$. In the following result we show that the operator $\mathcal{L}_\sigma$ is a limit for the operators $\mathcal{G}_{h,N}$.

Lemma 3.1. *Let φ be a constant function and $h := \pi/\sigma$. Then we have*

$$\lim_{N\to\infty}\mathcal{G}_{h,N}f = \mathcal{L}_\sigma f = f, \quad \textit{for all } f \in B_{\sigma,p}\left(\mathbb{R}^2\right).$$

Proof. Since $h = \pi/\sigma$, we have $\alpha = 0$. Letting $\alpha = 0$ in (3.2) and taking $N \to \infty$ implies the expansion (1.1). □

Let C_1 be an arbitrary rectifiable piecewise smooth Jordan curve enclosing a simply connected region of the complex variable w_1 which contains the points $w_1 = z_1$ and $w_1 = k_1h$ in its interior. Similarly, C_2 is an arbitrary rectifiable piecewise smooth Jordan curve enclosing a simply connected region of the complex variable w_2 which contains the points $w_2 = z_2$ and $w_2 = k_2h$ in its interior where $(k_1, k_2) \in \mathbb{Z}^2_N(z)$. Let us consider the kernel function

$$\mathcal{K}(z, w) := f(w)S(z, w)\prod_{j=1}^{2}\frac{\exp\left(-\frac{\alpha\,(z_j - w_j)^2}{Nh^2}\right)}{(w_j - z_j)\sin(\pi h^{-1}w_j)},$$

where $w := (w_1, w_2)$, $z := (z_1, z_2)$ and

$$S(z,w) := \sin(\pi h^{-1} z_1)\sin(\pi h^{-1} w_2) + \sin(\pi h^{-1} z_2)\sin(\pi h^{-1} w_1) - \sin(\pi h^{-1} z_1)\sin(\pi h^{-1} z_2).$$

This kernel has a singularity of order one at all the points of the sets $\{(z_1, \mathbb{C}), (\mathbb{C}, z_2) : z_1, z_2 \in \mathbb{C}\}$ and $\{(k_1 h, \mathbb{C}), (\mathbb{C}, k_2 h) : k_1, k_2 \in \mathbb{Z}\}$. These sets are subsets of $\mathbb{C}^2$ and understood as the Cartesian product of the w_j-planes for $j = 1, 2$. In the following result we show that the error of approximation of functions from $E^2_\sigma(\varphi)$ by the operator $\mathcal{G}_{h,N}$ can be written as the integral of the kernel $\mathcal{K}$ over the curves C_1 and C_2.

Lemma 3.2. *Let $f \in E^2_\sigma(\varphi)$. Then we have for all $z \in \mathbb{C}^2$*

$$f(z) - \mathcal{G}_{h,N}[f](z) = \frac{-1}{4\pi^2} \oint_{C_2} \oint_{C_1} \mathcal{K}(z,w)\, \mathrm{d}w_1 \mathrm{d}w_2. \tag{3.5}$$

Proof. We consider z_1, z_2 and w_2 to be arbitrary fixed complex parameters and we regard to the kernel $\mathcal{K}(z,w)$ as a function of w_1. Applying the classical Cauchy integral formula on w_1-plane, see e.g. [1, p. 141], [16, Chapter 3] , we obtain

$$\frac{1}{2\pi \mathrm{i}} \oint_{C_1} \mathcal{K}(z,w)\, \mathrm{d}w_1 = \mathrm{Res}\,(\mathcal{K}; (z_1, w_2)) + \sum_{|\lfloor h^{-1}\Re z_1 + 1/2 \rfloor - k_1| \le N} \mathrm{Res}\,(\mathcal{K}; (k_1 h, w_2)), \tag{3.6}$$

where $\mathrm{Res}\,(\mathcal{K}; (\cdot,\cdot))$ is denoted to the residue of the function $\mathcal{K}$ at the point $(\cdot,\cdot)$. Now we consider the right-hand side of (3.6) as a function of the arbitrary fixed complex parameters w_2 and z_1, z_2. Applying the classical Cauchy integral formula on w_2-plane, we get

$$\frac{-1}{4\pi^2} \oint_{C_2} \oint_{C_1} \mathcal{K}(z,w)\, \mathrm{d}w_1 \mathrm{d}w_2 = \mathrm{Res}\,(\mathcal{K}; (z_1, z_2)) + \sum_{k \in \mathbb{Z}^2_N(z)} \mathrm{Res}\,(\mathcal{K}; (k_1 h, k_2 h)). \tag{3.7}$$

The residue at each point satisfies

$$\mathrm{Res}\,(\mathcal{K}; (z_1, z_2)) = f(z), \tag{3.8}$$

and for $k \in \mathbb{Z}^2_N(z)$

$$\sum_{k \in \mathbb{Z}^2_N(z)} \mathrm{Res}\,(\mathcal{K}; (k_1 h, k_2 h)) = -\mathcal{G}_{h,N}[f](z). \tag{3.9}$$

Combining (3.7)–(3.9) implies (3.5). □

In the following theorem we estimate the integral in (3.5) to find an error bound for $|f(z) - \mathcal{G}_{h,N}[f](z)|$.

Theorem 3.3. *Let $f \in E^2_\sigma(\varphi)$. Then we have for all $|\Im z_j| < N$, $j = 1, 2$*

$$|f(z) - \mathcal{G}_{h,N}[f](z)| \le 2\,\varphi\,(b(z_1), b(z_2))\, \mathcal{A}_N(z)\, \frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi \alpha N}}, \tag{3.10}$$

where $b(z_i) := |\Re z_i| + h(N+1)$, $i = 1, 2$,

$$\mathcal{A}_N(z) := \sum_{i=1}^{2} \mathrm{e}^{\sigma |\Im z_{3-i}|} \left|\sin(\pi h^{-1} z_i)\right| \theta_N\left(h^{-1}\Im z_i\right) + 2\prod_{j=1}^{2} \left|\sin(\pi h^{-1} z_j)\right| \theta_N\left(h^{-1}\Im z_j\right),$$

and

$$\begin{aligned} \theta_N(t) &:= \cosh(2\alpha\, t) + \frac{2\mathrm{e}^{\alpha\, t^2/N}}{\sqrt{\pi\alpha N}\left(1 - (t/N)^2\right)} + \frac{1}{2}\left[\frac{\mathrm{e}^{2\alpha\, t}}{\mathrm{e}^{2\pi(N-t)} - 1} + \frac{\mathrm{e}^{-2\alpha\, t}}{e^{2\pi(N+t)} - 1}\right] \\ &= \cosh(2\alpha\, t) + O\left(N^{-1/2}\right), \quad \text{as} \quad N \to \infty. \end{aligned} \tag{3.11}$$

Proof. Let R_j be the rectangle with vertices at $\pm h\left(N+\frac{1}{2}\right)+hN_{h^{-1}z_j}+\mathrm{i}h(\Im z_j \pm N)$ where $N_{z_j} := \lfloor \Re z_j + \frac{1}{2} \rfloor$. The concept here is to estimate the integral in (3.5) with $C_j := R_j$, $j=1,2$. With the use of the Cauchy integral formula in one dimension the integral in (3.5) may be expanded to obtain the representation

$$\begin{aligned} f(z)-\mathcal{G}_{h,N}[f](z) \;=\; & \frac{\sin(\pi h^{-1}z_1)}{2\pi \mathrm{i}} \oint_{R_1} \frac{\exp\left(-\frac{\alpha\,(z_1-w_1)^2}{Nh^2}\right) f(w_1,z_2)}{(w_1-z_1)\sin(\pi h^{-1}w_1)}\,\mathrm{d}w_1 \\ & + \frac{\sin(\pi h^{-1}z_2)}{2\pi \mathrm{i}} \oint_{R_2} \frac{\exp\left(-\frac{\alpha\,(z_2-w_2)^2}{Nh^2}\right) f(z_1,w_2)}{(w_2-z_2)\sin(\pi h^{-1}w_2)}\,\mathrm{d}w_2 \\ & + \frac{1}{4\pi^2}\prod_{m=1}^{2}\sin(\pi h^{-1}z_m)\oint_{R_2}\oint_{R_1} f(w_1,w_2)\prod_{j=1}^{2}\frac{\exp\left(-\frac{\alpha\,(z_j-w_j)^2}{Nh^2}\right)}{(w_j-z_j)\sin(\pi h^{-1}w_j)}\,\mathrm{d}w_1\mathrm{d}w_2. \end{aligned} \tag{3.12}$$

Since $f \in E^2_\sigma(\varphi)$, we have from (3.1) for all points (w_1,w_2) in the bivariate rectangle $R_1 \times R_2$

$$|f(w_1,w_2)| \le \varphi\left(b(z_1),b(z_2)\right)\prod_{j=1}^{2}\mathrm{e}^{\sigma|\Im w_j|}, \tag{3.13}$$

where $b(z_j) = |\Re z_j| + h(N+1)$. If z_1 or z_2 is a fixed point, we have

$$|f(z_1,w_2)| \le \varphi\left(|\Re z_1|, b(z_2)\right)\mathrm{e}^{\sigma|\Im z_1|}\mathrm{e}^{\sigma|\Im w_2|}, \quad w_2 \in R_2. \tag{3.14}$$

$$|f(w_1,z_2)| \le \varphi\left(b(z_1), |\Re z_2|\right)\mathrm{e}^{\sigma|\Im w_1|}\mathrm{e}^{\sigma|\Im z_2|}, \quad w_1 \in R_1. \tag{3.15}$$

Combining (3.13)–(3.15) with (3.12) implies

$$\begin{aligned} |f(z)-\mathcal{G}_{h,N}[f](z)| \le\; & \frac{\varphi\left(b(z_1),|\Re z_2|\right)\mathrm{e}^{\sigma|\Im z_2|}|\sin(\pi h^{-1}z_1)|}{2\pi}\oint_{R_1}\left|\frac{\exp\left(\sigma|\Im w_1| - \frac{\alpha\,(z_1-w_1)^2}{Nh^2}\right)}{(w_1-z_1)\sin(\pi h^{-1}w_1)}\right||\mathrm{d}w_1| \\ & + \frac{\varphi\left(|\Re z_1|, b(z_2)\right)\mathrm{e}^{\sigma|\Im z_1|}|\sin(\pi h^{-1}z_2)|}{2\pi}\oint_{R_2}\left|\frac{\exp\left(\sigma|\Im w_2| - \frac{\alpha\,(z_2-w_2)^2}{Nh^2}\right)}{(w_2-z_2)\sin(\pi h^{-1}w_2)}\right||\mathrm{d}w_2| \\ & + \frac{\varphi\left(b(z_1),b(z_2)\right)}{4\pi^2}\prod_{j=1}^{2}\left|\sin(\pi h^{-1}z_j)\right|\oint_{R_j}\left|\frac{\exp\left(\sigma_j|\Im w_j| - \frac{\alpha\,(z_j-w_j)^2}{Nh^2}\right)}{(w_j-z_j)\sin(\pi h^{-1}w_j)}\right||\mathrm{d}w_j|. \end{aligned} \tag{3.16}$$

The integrals are estimated in [26, p. 203-205], using the residue theorem, as follows:

$$\oint_{R_j}\left|\frac{\exp\left(\sigma|\Im w_j| - \frac{\alpha\,(z_j-w_j)^2}{Nh^2}\right)}{(w_j-z_j)\sin(\pi h^{-1}w_j)}\right||\mathrm{d}w_j| \le 4\pi\theta_N\left(h^{-1}\Im z_j\right)\frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha\, N}}. \tag{3.17}$$

Substituting (3.17) into (3.16), we obtain (3.10). □

In the following corollaries we introduce some special cases of the last theorem. In the first corollary we choose the function f to be in $B_{\sigma,\infty}\left(\mathbb{R}^2\right)$ or $B_{\sigma,2}\left(\mathbb{R}^2\right)$. The second corollary deals with the case when f has exponential growth on one or two axes of $\mathbb{R}^2$.

Corollary 3.4. If $f \in B_{\sigma,\infty}\left(\mathbb{R}^2\right)$, we have

$$|f(z) - \mathcal{G}_{h,N}[f](z)| \le 2\|f\|_\infty \, \mathcal{A}_N(z) \, \frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}}. \tag{3.18}$$

If f is in the Paley-Wiener space $B_{\sigma,2}\left(\mathbb{R}^2\right)$, we have

$$|f(z) - \mathcal{G}_{h,N}[f](z)| \le 2\sigma\|f\|_2 \, \mathcal{A}_N(z) \, \frac{\mathrm{e}^{-\alpha N}}{\pi\sqrt{\pi\alpha N}}.$$

Proof. Since $f \in B_{\sigma,\infty}\left(\mathbb{R}^2\right)$, we have for all $z \in \mathbb{C}^2$

$$|f(z)| \le \|f\|_\infty \prod_{j=1}^{2} \mathrm{e}^{\sigma|\Im z_j|}.$$

If we choose φ as $\|f\|_\infty$, we get (3.18). For $f \in B_{\sigma,2}\left(\mathbb{R}^2\right)$, we have

$$f(x) = \frac{1}{2\pi}\int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \widehat{f}(\omega_1,\omega_2)\mathrm{e}^{\mathrm{i}(x_1\omega_1+x_2\omega_2)}\mathrm{d}\omega_1\mathrm{d}\omega_2,$$

where $\widehat{f}$ is the Fourier transform of f. Using the Cauchy-Schwarz inequality and the Parseval identity yields

$$\|f\|_\infty \le \frac{1}{2\pi}\int_{-\sigma}^{\sigma}\int_{-\sigma}^{\sigma} \left|\widehat{f}(\omega_1,\omega_2)\right| \mathrm{d}\omega_1\mathrm{d}\omega_2 \le \frac{\sigma}{\pi}\,\|\widehat{f}\,\|_2 = \frac{\sigma}{\pi}\,\|f\|_2.$$

The proof is completed by (3.18) and the fact that $B_{\sigma,2}\left(\mathbb{R}^2\right) \subset B_{\sigma,\infty}\left(\mathbb{R}^2\right)$. □

Corollary 3.5. Let f be an entire function satisfying

$$|f(z)| \le M \prod_{j=1}^{2} \mathrm{e}^{\kappa|\Re z_j|+\sigma|\Im z_j|}, \quad z \in \mathbb{C}^2,\ M > 0, \tag{3.19}$$

where σ, κ are non-negative numbers and $\sigma + \kappa \ne 0$. Then we have for $|\Im z_j| < N$

$$|f(z) - \mathcal{G}_{h,N}[f](z)| \le 2M\mathcal{E}_N(z)\frac{\mathrm{e}^{-(\alpha-2h\kappa)N}}{\sqrt{\pi\alpha N}},$$

where $h \in (0, \pi/(\sigma + 4\kappa))$ and

$$\begin{aligned}\mathcal{E}_N(z) := {} & \mathrm{e}^{\kappa(2h+|\Re z_1|+|\Re z_2|)} \sum_{i=1}^{2} \mathrm{e}^{\sigma|\Im z_{3-i}|} \left|\sin(\pi h^{-1} z_i)\right| \theta_N\left(h^{-1}\Im z_i\right) \\ & +2\mathrm{e}^{4\kappa h} \prod_{j=1}^{2} \mathrm{e}^{\kappa|\Re z_j|} \left|\sin(\pi h^{-1} z_j)\right| \theta_N\left(h^{-1}\Im z_j\right).\end{aligned}$$

Proof. Setting $\varphi(s,t) = M\mathrm{e}^{\kappa(s+t)}$ for non-negative s and t, we obtain the result as an immediate consequence of Theorem 3.3. □

For $d > 0$, we introduce the bivariate strip

$$\mathcal{S}_d^2 := \left\{z \in \mathbb{C}^2 : |\Im z_j| < d, \quad j = 1, 2\right\}. \tag{3.20}$$

We define $A_d^2(\varphi)$ to be the class of analytic functions $f : \mathcal{S}_d^2 \to \mathbb{C}^2$ which satisfy the condition (3.1) with $\sigma = 0$. For functions from the class $A_d^2(\varphi)$ we study the operator $\mathcal{G}_{\frac{d}{N},N}$ in the special case $h := \frac{d}{N}$ and $\alpha = \pi/2$.

Theorem 3.6. *Let $f \in A_d^2(\varphi)$. Then we have for $z \in \mathcal{S}_{d/4}^2$*

$$\left|f(z) - \mathcal{G}_{\frac{d}{N},N}[f](z)\right| \le \varphi(b(z_1), b(z_2)) \left(2^{3/2} \sum_{j=1}^{2} \left|\sin\left(\frac{\pi N z_j}{d}\right)\right| \vartheta_N\left(\frac{\Im z_j}{d}\right) \frac{\exp\left(-\frac{\pi N}{2}\left(1-\frac{2|\Im z_j|}{d}\right)\right)}{\pi\sqrt{N}} + 2^3 \prod_{j=1}^{2} \left|\sin\left(\frac{\pi N z_j}{d}\right)\right| \vartheta_N\left(\frac{\Im z_j}{d}\right) \frac{\exp\left(-\frac{\pi N}{2}\left(1-\frac{2|\Im z_j|}{d}\right)\right)}{\pi\sqrt{N}}\right), \tag{3.21}$$

where $b(z_j)$ is defined as above (with $h = \frac{d}{N}$) and

$$\vartheta_N(t) = \frac{1}{1-t}\left(\frac{1}{1-\mathrm{e}^{-2\pi N}} + \frac{2\sqrt{2}}{\pi\sqrt{N}(1+t)}\right) = \frac{1}{1-t}\left(1 + O\left(N^{-1/2}\right)\right), \quad \textit{as} \quad N \to \infty.$$

Proof. Define the rectangle R_j which has vertices at $\pm h\left(N+\frac{1}{2}\right) + hN_{z_j/h} + \mathrm{i}d$ and $\pm h\left(N+\frac{1}{2}\right) + hN_{z_j/h} - \mathrm{i}h(d - \Im z_j)$. Since $f \in A_d^2(\varphi)$, we have for all points w on the bivariate rectangle $R_1 \times R_2$

$$|f(w_1, w_2)| \le \varphi(b(z_1), b(z_2)), \tag{3.22}$$

and for z_1 or z_2 fixed we obtain

$$\begin{aligned} |f(z_1, w_2)| &\le \varphi(|\Re z_1|, b(z_2)), \quad w_2 \in R_2, \\ |f(w_1, z_2)| &\le \varphi(b(z_1), |\Re z_2|,), \quad w_1 \in R_1. \end{aligned} \tag{3.23}$$

Letting $C_j := R_j$ in (3.5) and using (3.22) to (3.23) implies

$$\begin{aligned} |f(z) - \mathcal{G}_{\frac{d}{N},N}[f](z)| \le{}& \frac{\varphi\left(b(z_1), |\Re z_2|\right)\left|\sin\left(\frac{\pi N z_1}{d}\right)\right|}{2\pi} \oint_{R_1} \left|\frac{\exp\left(-\frac{\pi N(z_1-w_1)^2}{2d^2}\right)}{(w_1 - z_1)\sin\left(\frac{\pi N w_1}{d}\right)}\right| |\mathrm{d}w_1| \\ &+ \frac{\varphi\left(|\Re z_1|, b(z_2)\right)\left|\sin\left(\frac{\pi N z_2}{d}\right)\right|}{2\pi} \oint_{R_2} \left|\frac{\exp\left(-\frac{\pi N(z_2-w_2)^2}{2d^2}\right)}{(w_2 - z_2)\sin\left(\frac{\pi N w_2}{d}\right)}\right| |\mathrm{d}w_2| \\ &+ \frac{\varphi\left(b(z_1), b(z_2)\right)}{4\pi^2} \prod_{j=1}^{2} \left|\sin\left(\frac{\pi N z_j}{d}\right)\right| \oint_{R_j} \left|\frac{\exp\left(-\frac{\pi N(z_j-w_j)^2}{2d^2}\right)}{(w_j - z_j)\sin\left(\frac{\pi N w_j}{d}\right)}\right| |\mathrm{d}w_j|. \end{aligned} \tag{3.24}$$

These integrals are estimated in [26, p. 209-211], using the residue theorem, as follows:

$$\oint_{\mathcal{R}_j} \left|\frac{\exp\left(\frac{-\pi N(z_j-w_j)^2}{2d^2}\right)}{(w_j - z_j)\sin\left(\frac{\pi N w_j}{d}\right)}\right| |\mathrm{d}w_j| \le 4\sqrt{2}\,\vartheta_N\left(\frac{\Im z_j}{d}\right) \frac{\exp\left(-\frac{\pi N}{2}\left(1-\frac{2|\Im z_j|}{d}\right)\right)}{\sqrt{N}}. \tag{3.25}$$

Combining (3.25) and (3.24) yields (3.21). □

Remark 3.7. The error bound in Theorem 3.6 converges to zero as $N \to \infty$ for every $z \in \mathcal{S}_{d/4}^2$. Since

$$\sinh\left(\frac{\pi N|\Im z_j|}{d}\right) \le \left|\sin\left(\frac{\pi N z_j}{d}\right)\right| \le \cosh\left(\frac{\pi N|\Im z_j|}{d}\right), \tag{3.26}$$

the decisive factor in the error bound (3.21) becomes

$$\exp\left(-\frac{\pi N}{2}\left(1 - \frac{4|\Im z_j|}{d}\right)\right),$$

which guarantees convergence to zero as long as $|\Im z_j| \le d/4$, $j = 1, 2$.

4. Hermite-Gauss sampling

This section is devoted to a modification of the Hermite sampling (1.2) based on samples of a function f and its partial and mixed second partial derivatives. For $h \in (0, 2\pi/\sigma)$ and $\beta := (2\pi - h\sigma)/2$, we study the operator $\mathcal{H}_{h,N} : E^2_\sigma(\varphi) \to E^2 \cap L^p(\mathbb{R}^2)$ for every $1 \le p \le \infty$ given by

$$\begin{aligned}\mathcal{H}_{h,N}[f](z) = & \sum_{k\in\mathbb{Z}^2_N(z)} \left\{ \left(1 + 2\sum_{j=1}^{2} \frac{\beta\,(z_j - k_j h)^2}{Nh^2} + 4\prod_{j=1}^{2} \frac{\beta\,(z_j - k_j h)^2}{Nh^2} \right) f(kh) \right. \\ & + \sum_{i+j=1} (z_{j+1} - k_{j+1}h) \left(1 + \frac{2\beta_{i+1}(z_{i+1} - k_{i+1}h)^2}{N_{i+1}h^2} \right) f^{(i,j)}(kh) \\ & + \left. \left(\prod_{j=1}^{2} (z_j - k_j h) \right) f^{(1,1)}(kh) \right\} \prod_{j=1}^{2} \operatorname{sinc}^2\left(\pi h^{-1} z_j - k_j \pi\right) \exp\left(- \frac{\beta\,(z_j - k_j h)^2}{Nh^2} \right),\end{aligned}$$

where $z \in \mathbb{C}^2$, $k := (k_1, k_2) \in \mathbb{Z}^2_+$ and $\mathbb{Z}^2_N(z)$ is defined in (3.3). For functions from $E^2_\sigma(\varphi)$ the operator $\mathcal{H}_{h,N}$ provides a piecewise analytic approximation on every bivariate strip defined in (3.4). Now we consider the kernel function

$$\mathbf{K}(z, w) := f(w)H(z, w) \prod_{j=1}^{2} \frac{\exp\left(-\frac{\beta\,(z_j - w_j)^2}{Nh^2}\right)}{(w_j - z_j)\sin^2(\pi h^{-1} w_j)},$$

where $w := (w_1, w_2)$, $z := (z_1, z_2)$ and

$$H(z, w) := \sin^2(\pi h^{-1} z_1)\sin^2(\pi h^{-1} w_2) + \sin^2(\pi h^{-1} z_2)\sin^2(\pi h^{-1} w_1) - \sin^2(\pi h^{-1} z_1)\sin^2(\pi h^{-1} z_2).$$

The kernel $\mathbf{K}$ has a singularity of order one at all the points of the set $\{(z_1, \mathbb{C}), (\mathbb{C}, z_2) \ : z_1, z_2 \in \mathbb{C}\}$ and a singularity of order two at all the points of the set $\{(k_1 h, \mathbb{C}), (\mathbb{C}, k_2 h) : \ k_1, k_2 \in \mathbb{Z}\}$.

Lemma 4.1. *Let $f \in E^2_\sigma(\varphi)$. Then we have for all $z \in \mathbb{C}^2$*

$$f(z) - \mathcal{H}_{h,N}[f](z) = \frac{-1}{4\pi^2} \oint_{C_2} \oint_{C_1} \mathbf{K}(z, w)\, \mathrm{d}w_1 \mathrm{d}w_2, \tag{4.1}$$

where C_1 and C_2 are the rectifiable piecewise smooth Jordan curves defined above.

Proof. Applying the classical Cauchy integral formula on w_1-plane and w_2-plane respectively, as we have done in Lemma 3.2, we obtain

$$\frac{-1}{4\pi^2} \oint_{C_2} \oint_{C_1} \mathbf{K}(z, w)\, \mathrm{d}w_1 \mathrm{d}w_2 = \operatorname{Res}(\mathbf{K}; (z_1, z_2)) + \sum_{k\in\mathbb{Z}^2_N(z)} \operatorname{Res}(\mathbf{K}; (k_1 h, k_2 h)). \tag{4.2}$$

The residues at these points are

$$\operatorname{Res}(\mathbf{K}; (z_1, z_2)) = f(z), \tag{4.3}$$

and for $k \in \mathbb{Z}^2_N(z)$

$$\operatorname{Res}(\mathbf{K}; (k_1 h, k_2 h)) = \lim_{w_2 \to k_2 h} \frac{\partial}{\partial w_2} \left\{ \frac{D(w_2)\, \mathrm{e}^{-\frac{\beta\,(z_2 - w_2)^2}{Nh^2}}}{(w_2 - z_2)} \left(\frac{w_2 - k_2 h}{\sin(\pi h^{-1} w_2)} \right)^2 \right\}, \tag{4.4}$$

where $k = (k_1, k_2)$ and

$$D(w_2) := \lim_{w_1 \to k_1 h} \frac{\partial}{\partial w_1} \left\{ \frac{f(w)H(w, z)\, \mathrm{e}^{-\frac{\beta\,(z_1 - w_1)^2}{Nh^2}}}{(w_1 - z_1)} \left(\frac{w_1 - k_1 h}{\sin(\pi h^{-1} w_1)} \right)^2 \right\}.$$

It is easy to check that

$$\sum_{k\in\mathbb{Z}^2_N(z)} \operatorname{Res}\left(\mathbf{K};(k_1h,k_2h)\right) = -\mathcal{H}_{h,N}[f](z). \tag{4.5}$$

Substituting (4.5) and (4.3) into (4.2), we get (4.1). □

The following theorem is devoted to error bounds for $|f(z)-\mathcal{H}_{h,N}[f](z)|$.

Theorem 4.2. *Let $f\in E^2_\sigma(\varphi)$. Then we have for all $|\Im z_j|<N$, $j=1,2$*

$$|f(z)-\mathcal{H}_{h,N}[f](z)| \le 2\,\varphi\left(b(z_1),b(z_2)\right)\mathcal{B}_N(z)\,\frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}, \tag{4.6}$$

where

$$\mathcal{B}_N(z) := \sum_{i=1}^{2} \mathrm{e}^{\sigma|\Im z_{3-i}|}\left|\sin^2(\pi h^{-1}z_i)\right|\chi_N\left(h^{-1}\Im z_i\right) + 2\prod_{j=1}^{2}\left|\sin^2(\pi h^{-1}z_j)\right|\chi_N\left(h^{-1}\Im z_j\right),$$

and χ_N is given by

$$\begin{aligned}\chi_N(t) &:= \frac{4\mathrm{e}^{\beta\,t^2/N}}{\sqrt{\pi\beta\,N}\left(1-(t/N)^2\right)} + \frac{\mathrm{e}^{-2\beta\,t}}{(1-\mathrm{e}^{-2\pi(N+t)})^2} + \frac{\mathrm{e}^{2\beta\,t}}{(1-\mathrm{e}^{-2\pi(N-t)})^2}\\ &= 2\cosh(2\beta\;t) + O\left(N^{-1/2}\right), \quad \text{as} \quad N\to\infty.\end{aligned}$$

Proof. Let R_j be the rectangles defined in the proof of Theorem 3.3. Using the Cauchy integral formula in one dimension, the integral in (4.4) may be rewritten (with $C_j := R_j$) as follows:

$$\begin{aligned} f(z)-\mathcal{H}_{h,N}[f](z) &= \frac{\sin^2(\pi h^{-1}z_1)}{2\pi\mathrm{i}}\oint_{R_1}\frac{\exp\left(-\frac{\beta\,(z_1-w_1)^2}{Nh^2}\right)f(w_1,z_2)}{(w_1-z_1)\sin^2(\pi h^{-1}w_1)}\,\mathrm{d}w_1\\ &+ \frac{\sin^2(\pi h^{-1}z_2)}{2\pi\mathrm{i}}\oint_{R_2}\frac{\exp\left(-\frac{\beta\,(z_2-w_2)^2}{Nh^2}\right)f(z_1,w_2)}{(w_2-z_2)\sin^2(\pi h^{-1}w_2)}\,\mathrm{d}w_2\\ &+ \frac{1}{4\pi^2}\prod_{m=1}^{2}\sin^2(\pi h^{-1}z_m)\oint_{R_2}\oint_{R_1} f(w_1,w_2)\prod_{j=1}^{2}\frac{\exp\left(-\frac{\beta\,(z_j-w_j)^2}{Nh^2}\right)}{(w_j-z_j)\sin^2(\pi h^{-1}w_j)}\,\mathrm{d}w_1\mathrm{d}w_2. \end{aligned} \tag{4.7}$$

Since $f\in E^2_\sigma(\varphi)$, we can combine the inequalities (3.13)–(3.15) with (4.7) and obtain

$$\begin{aligned} |f(z)-\mathcal{H}_{h,N}[f](z)| &\le \frac{\varphi\left(b(z_1),|\Re z_2|\right)\mathrm{e}^{\sigma|\Im z_2|}|\sin^2(\pi h^{-1}z_1)|}{2\pi}\oint_{R_1}\left|\frac{\exp\left(\sigma|\Im w_1|-\frac{\beta\,(z_1-w_1)^2}{Nh^2}\right)}{(w_1-z_1)\sin^2(\pi h^{-1}w_1)}\right|\,|\mathrm{d}w_1|\\ &+ \frac{\varphi\left(|\Re z_1|,b(z_2)\right)\mathrm{e}^{\sigma|\Im z_1|}|\sin^2(\pi h^{-1}z_2)|}{2\pi}\oint_{R_2}\left|\frac{\exp\left(\sigma|\Im w_2|-\frac{\beta\,(z_2-w_2)^2}{Nh^2}\right)}{(w_2-z_2)\sin^2(\pi h^{-1}w_2)}\right|\,|\mathrm{d}w_2|\\ &+ \frac{\varphi\left(b(z_1),b(z_2)\right)}{4\pi^2}\prod_{j=1}^{2}\left|\sin^2(\pi h^{-1}z_j)\right|\oint_{R_j}\left|\frac{\exp\left(\sigma_j|\Im w_j|-\frac{\beta\,(z_j-w_j)^2}{Nh^2}\right)}{(w_j-z_j)\sin^2(\pi h^{-1}w_j)}\right|\,|\mathrm{d}w_j|. \end{aligned} \tag{4.8}$$

The integrals in (4.8) are estimated in [4], using the residue theorem, as follows:

$$\oint_{R_j} \left| \frac{\exp\left(\sigma_j |\Im w_j| - \frac{\beta\,(z_j - w_j)^2}{N h^2}\right)}{(w_j - z_j)\sin^2(\pi h^{-1} w_j)} \right| |\mathrm{d}w_j| \le 4\pi \chi_N\left(h^{-1}\Im z_j\right) \frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}. \tag{4.9}$$

Substituting (4.9) into (4.8), we obtain (4.6). □

In the following corollary, we note some special cases of Theorem 4.2. The proofs are similar to the proofs of Corollary 3.4 and Corollary 3.5 and will be omitted.

Corollary 4.3. If $f \in B_{\sigma,\infty}\left(\mathbb{R}^2\right)$, we have

$$|f(z) - \mathcal{H}_{h,N}[f](z)| \le 2\|f\|_\infty \mathcal{B}_N(z) \frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}. \tag{4.10}$$

If f is in the Paley-Wiener space $B_{\sigma,2}\left(\mathbb{R}^2\right)$, we have

$$|f(z) - \mathcal{H}_{h,N}[f](z)| \le 2\sigma\|f\|_2 \mathcal{B}_N(z) \frac{\mathrm{e}^{-\beta N}}{\pi\sqrt{\pi\beta N}}.$$

Corollary 4.4. Let f be an entire function satisfying (3.19). Then we have for $|\Im z_j| < N$

$$|f(z) - \mathcal{H}_{h,N}[f](z)| \le 2M\mathcal{F}_N(z) \frac{\mathrm{e}^{-(\beta - 2h\kappa)N}}{\sqrt{\pi\beta N}},$$

where $h \in (0, 2\pi/(\sigma + 4\kappa))$ and

$$\begin{aligned}\mathcal{F}_N(z) := \mathrm{e}^{\kappa(2h + |\Re z_1| + |\Re z_2|)} \sum_{i=1}^{2} \mathrm{e}^{\sigma|\Im z_{3-i}|} \left|\sin^2(\pi h^{-1} z_i)\right| \chi_N\left(h^{-1}\Im z_i\right) \\ + \mathrm{e}^{4\kappa h} \prod_{j=1}^{2} \mathrm{e}^{\kappa|\Re z_j|} \left|\sin^2(\pi h^{-1} z_j)\right| \chi_N\left(h^{-1}\Im z_j\right).\end{aligned}$$

For functions from the class $A_d^2(\varphi)$ we study the operator $\mathcal{H}_{h,N}$ in the special case $h := \frac{d}{N}$ and $\beta = \pi$. For this particular case we denote the operator by $\mathcal{H}_{\frac{d}{N},N}$. In the following theorem we establish a bound for the error $\left|f(z) - \mathcal{H}_{\frac{d}{N},N}[f](z)\right|$.

Theorem 4.5. *Let $f \in A_d^2(\varphi)$. Then we have for $z \in \mathcal{S}_{d/4}^2$*

$$\begin{aligned}\left|f(z) - \mathcal{H}_{\frac{d}{N},N}[f](z)\right| \le\ & 2^{3/2}\varphi\left(b(z_1), b(z_2)\right) \sum_{j=1}^{2} \left|\sin^2\left(\frac{\pi N z_j}{d}\right)\right| \psi_N\left(\frac{\Im z_j}{d}\right) \frac{\mathrm{e}^{-\pi N\left(1 - \frac{2|\Im z_j|}{d}\right)}}{\pi\sqrt{N}} \\ & + 2^3\varphi(b(z_1), b(z_2)) \prod_{j=1}^{2} \left|\sin^2\left(\frac{\pi N z_j}{d}\right)\right| \psi_N\left(\frac{\Im z_j}{d}\right) \frac{\mathrm{e}^{-\pi N\left(1 - \frac{2|\Im z_j|}{d}\right)}}{\pi\sqrt{N}},\end{aligned} \tag{4.11}$$

where $b(z_j)$ is defined above (with $h := \frac{d}{N}$) and

$$\psi_N(t) := \frac{1}{1-t}\left(\frac{1}{\left(1 - \mathrm{e}^{-2\pi N}\right)^2} + \frac{2}{\pi\sqrt{N}(1+t)}\right) = \frac{1}{1-t}\left(1 + O\left(N^{-1/2}\right)\right), \quad \textit{as} \quad N \to \infty.$$

Proof. Let R_j be the rectangle that is defined in the proof of Theorem 3.6. Since $f \in A_d^2(\varphi)$, we use the inequalities (3.22) and (3.23). Letting $C_j := R_j$ in (4.1) and using (3.22) to (3.23) implies

$$\begin{aligned}|f(z) - \mathcal{H}_{\frac{d}{N},N}[f](z)| \leq & \frac{\varphi\left(b(z_1), |\Re z_2|\right) \left|\sin^2\left(\frac{\pi N z_1}{d}\right)\right|}{2\pi} \oint_{R_1} \left| \frac{\exp\left(-\frac{\pi N (z_1 - w_1)^2}{d^2}\right)}{(w_1 - z_1)\sin^2\left(\frac{\pi N w_1}{d}\right)} \right| |\mathrm{d}w_1| \\ & + \frac{\varphi\left(|\Re z_1|, b(z_2)\right) \left|\sin^2\left(\frac{\pi N z_2}{d}\right)\right|}{2\pi} \oint_{R_2} \left| \frac{\exp\left(-\frac{\pi N (z_2 - w_2)^2}{d^2}\right)}{(w_2 - z_2)\sin^2\left(\frac{\pi N w_2}{d}\right)} \right| |\mathrm{d}w_2| \\ & + \frac{\varphi\left(b(z_1), b(z_2)\right)}{4\pi^2} \prod_{j=1}^{2} \left|\sin^2\left(\frac{\pi N z_j}{d}\right)\right| \oint_{R_j} \left| \frac{\exp\left(-\frac{\pi N (z_j - w_j)^2}{d^2}\right)}{(w_j - z_j)\sin^2\left(\frac{\pi N w_j}{d}\right)} \right| |\mathrm{d}w_j|. \end{aligned} \tag{4.12}$$

These integrals are estimated in [4], using the residue theorem, as follows:

$$\oint_{\mathcal{R}_j} \left| \frac{\exp\left(\frac{-\pi N (z_j - w_j)^2}{d^2}\right)}{(w_j - z_j)\sin^2\left(\frac{\pi N w_j}{d}\right)} \right| |\mathrm{d}w_j| \leq 4\sqrt{2}\, \psi_N\left(\frac{\Im z_j}{d}\right) \frac{\exp\left(-\pi N\left(1 - \frac{2|\Im z_j|}{d}\right)\right)}{\sqrt{N}}. \tag{4.13}$$

Combining (4.13) and (4.12) yields (4.11). Similarly, as we explained in Remark 3.7, the error bound converges to zero as $N \to \infty$ for each $z \in \mathcal{S}_{d/4}^2$. □

5. Numerical examples

In this section we discuss four examples. The first example is devoted to the comparison between the classical and Hermite sampling and their modifications; $\mathcal{G}_{h,N}$ and $\mathcal{H}_{h,N}$. In Examples 2-4 we approximate the function f at the points $(x_u, x_v) := \left(\left(u - \frac{1}{2}\right)h, \left(v - \frac{1}{2}\right)h\right)$ where $u, v \in \mathbb{Z}$ and we summarize the results in some tables. Furthermore, we illustrate the absolute and relative errors by figures. As predicted by the error estimates, the precision increases when N is fixed but h decreases without any additional cost except that the function is approximated on a smaller domain.

Example 5.1. Consider the function $f(x, y) = \operatorname{sinc}^{(1)}(x)\ \operatorname{sinc}^{(1)}(y) \in B_{1,1}\left(\mathbb{R}^2\right)$. In Table 2 we approximate f using the classical and Hermite sampling and their modifications. The operator $\mathcal{T}_{\sigma,N}$

(x_u, x_v)	Classical sampling and its modification		Hermite sampling and its modification	
	$\lvert f(x) - T_{\sigma,N}[f](x)\rvert$	$\lvert f(x) - \mathcal{G}_{h,N}[f](x)\rvert$	$\lvert f(x) - \mathcal{T}_{\sigma,N}[f](x)\rvert$	$\lvert f(x) - \mathcal{H}_{h,N}[f](x)\rvert$
(x_1, x_1)	2.92024×10^{-3}	3.94251×10^{-8}	2.49081×10^{-2}	8.72566×10^{-15}
(x_1, x_3)	6.61393×10^{-3}	8.65694×10^{-8}	4.68794×10^{-2}	1.87073×10^{-14}
(x_3, x_1)	6.61393×10^{-3}	8.65694×10^{-8}	4.68794×10^{-2}	1.87073×10^{-14}
(x_3, x_3)	1.48818×10^{-2}	1.84838×10^{-7}	8.82321×10^{-2}	3.87468×10^{-14}

Table 2: Error approximating f at (x_u, x_v) for $N = 10$, $h = 1$.

does not always yield a better approximation than $T_{\sigma,N}$ although $\mathcal{T}_{\sigma,N}$ involves three times as many samples, see the last table.

Example 5.2. Consider the function

$$f(z) = \sin(z_1 + z_2), \qquad z = (z_1, z_2) \in \mathbb{C}^2.$$

Note that $f \in B_{1,\infty}\left(\mathbb{R}^2\right)$. Then we apply Corollaries 3.4 and 4.3. In this example, we choose $h = 1$ and $h = 0.2$. Here, the bounds in (3.18) and (4.10) can be written as the following uniform bounds on $\mathbb{R}^2$

$$|f(x) - \mathcal{G}_{h,N}[f](x)| \le 2\,\theta_N(0)\,(2 + \theta_N(0))\ \frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}},$$

$$|f(x) - \mathcal{H}_{h,N}[f](x)| \le 2\,\chi_N(0)\,(2 + \chi_N(0))\ \frac{\mathrm{e}^{-\beta N}}{\sqrt{\pi\beta N}}.$$

In this example, the function φ is constant which means that the error bounds are quite realistic, see Tables 3, 4.

(x_u, x_v)	Absolute error for $\mathcal{G}_{h,N}$		Absolute error for $\mathcal{H}_{h,N}$	
	$\lvert f(x) - \mathcal{G}_{h,N}[f](x)\rvert$	uniform bound	$\lvert f(x) - \mathcal{H}_{h,N}[f](x)\rvert$	uniform bound
(x_1, x_1)	6.22823×10^{-6}		1.22002×10^{-12}	
(x_1, x_3)	6.14371×10^{-7}		1.83603×10^{-13}	
(x_1, x_5)	6.73956×10^{-6}		1.37312×10^{-12}	
(x_3, x_1)	6.14371×10^{-7}		1.83603×10^{-13}	
(x_3, x_3)	6.73956×10^{-6}	5.37297×10^{-5}	1.37312×10^{-12}	1.24146×10^{-11}
(x_3, x_5)	4.99493×10^{-6}		9.59455×10^{-13}	
(x_5, x_1)	6.73956×10^{-6}		1.37312×10^{-13}	
(x_5, x_3)	4.99493×10^{-6}		9.59455×10^{-13}	
(x_5, x_5)	2.58232×10^{-6}		5.74873×10^{-13}	

Table 3: Approximation of sin at (x_u, x_v) for $N = 10$, $h = 1$.

(x_u, x_v)	Absolute error for $\mathcal{G}_{h,N}$		Absolute error for $\mathcal{H}_{h,N}$	
	$\lvert f(x) - \mathcal{G}_{h,N}[f](x)\rvert$	uniform bound	$\lvert f(x) - \mathcal{H}_{h,N}[f](x)\rvert$	uniform bound
(x_1, x_1)	1.83357×10^{-8}		4.57967×10^{-15}	
(x_1, x_3)	6.14185×10^{-8}		1.37668×10^{-14}	
(x_1, x_5)	9.48047×10^{-8}		2.06501×10^{-14}	
(x_3, x_1)	6.14185×10^{-8}		1.37668×10^{-14}	
(x_3, x_3)	9.48047×10^{-8}	7.8157×10^{-7}	2.06501×10^{-14}	2.07487×10^{-13}
(x_3, x_5)	1.13223×10^{-7}		2.42029×10^{-14}	
(x_5, x_1)	9.48047×10^{-8}		2.06501×10^{-14}	
(x_5, x_3)	1.13223×10^{-7}		2.42029×10^{-14}	
(x_5, x_5)	1.13766×10^{-7}		2.43139×10^{-14}	

Table 4: Approximation of sin at (x_u, x_v) for $N = 10$, $h = 0.2$.

Figures 1, 2 show the graphs of the error of classical Gauss and Hermite-Gauss formulas on the region $[0, 5] \times [0, 5]$ for $N = 15$ and $h = 1$.

Example 5.3. In this example, we approximate the function

$$f(z) = \cosh(z_1 + z_2), \qquad z = (z_1, z_2) \in \mathbb{C}^2,$$

which satisfies the inequality $|f(x)| \le \mathrm{e}^{|x_1|}\mathrm{e}^{|x_2|}$ on $\mathbb{R}^2$. Here we apply Corollaries 3.5 and 4.4 with $\sigma = 0$, $\kappa = 1$, $M = 1$ and $h = 0.5$. Note that f has exponential growth on $\mathbb{R}^2$ and the samples are exponentially increasing in the two axes x_1, x_2. Consequently, the absolute errors increase with u, v,

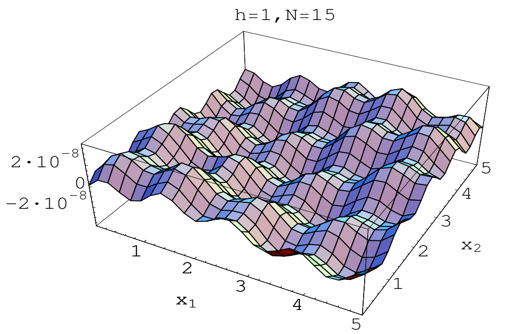

Figure 1: $\sin(x) - \mathcal{G}_{h,N}[\sin](x)$.

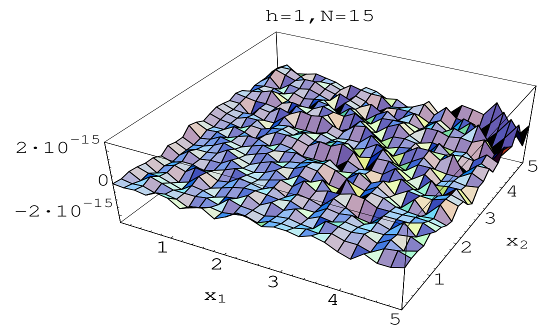

Figure 2: $\sin(x) - \mathcal{H}_{h,N}[\sin](x)$.

(x_u, x_v)	Relative error for $\mathcal{G}_{h,N}$		Relative error for $\mathcal{H}_{h,N}$	
	$\lvert f(x) - \mathcal{G}_{h,N}[f](x)/f(x)\rvert$	bound	$\lvert f(x) - \mathcal{H}_{h,N}[f](x)/f(x)\rvert$	bound
(x_1, x_1)	1.29563×10^{-7}	6.86784×10^{-5}	1.29963×10^{-14}	1.29593×10^{-11}
(x_1, x_5)	3.82956×10^{-8}	9.33205×10^{-5}	7.38665×10^{-15}	1.76082×10^{-11}
(x_1, x_9)	3.60093×10^{-8}	9.39782×10^{-5}	7.57675×10^{-15}	1.77246×10^{-11}
(x_5, x_1)	3.82956×10^{-8}	9.33150×10^{-5}	7.38665×10^{-15}	1.76082×10^{-11}
(x_5, x_5)	3.60093×10^{-8}	9.39377×10^{-5}	7.57675×10^{-15}	1.77246×10^{-11}
(x_5, x_9)	3.59672×10^{-8}	9.39896×10^{-5}	7.52052×10^{-15}	1.77268×10^{-11}
(x_9, x_1)	3.60093×10^{-8}	9.39322×10^{-5}	7.73459×10^{-15}	1.77246×10^{-11}
(x_9, x_5)	3.59672×10^{-8}	9.39490×10^{-5}	7.69144×10^{-15}	1.77268×10^{-11}
(x_9, x_9)	1.80174×10^{-8}	9.39898×10^{-5}	8.14235×10^{-15}	1.77268×10^{-11}

Table 5: Approximation of cosh at (x_u, x_v) for $N = 10$, $h = 0.5$.

but the relative errors are nearly constant for fixed N. In Table 5, we show the numerical results with the relative errors and the graphs of the relative errors are given in Figures 3, 4.

Example 5.4. The function

$$f(z) = \frac{1}{(z_1^2 + 4)\,(z_2^2 + 4)}, \qquad z = (z_1, z_2) \in \mathbb{C}^2,$$

is analytic in the bivariate strip $\mathcal{S}_2^2$. Then we apply Theorems 3.6 and 4.5, where $\varphi = 1$, $d = 2$. Note that f decreases on $\mathbb{R}^2$ and the sample values are decreasing in the two axes x_1, x_2. Therefore, the absolute errors decrease with u, v. First, we approximate f at real points (x_u, x_v) where $u = v = 1, 5, 9$ using the two operators $\mathcal{G}_{h,N}$ and $\mathcal{H}_{h,N}$. The results are shown in Table 6 and the graphs of the relative error are shown in Figures 7, 8 for $N = 10$. In the second test, we compare the quality of the method at the imaginary points $(\mathrm{i}y_1, \mathrm{i}y_2)$ with the absolute errors; see Table 7. As predicted by the theory, the bound converges when $|y_j| < 1/2$, $j = 1, 2$.

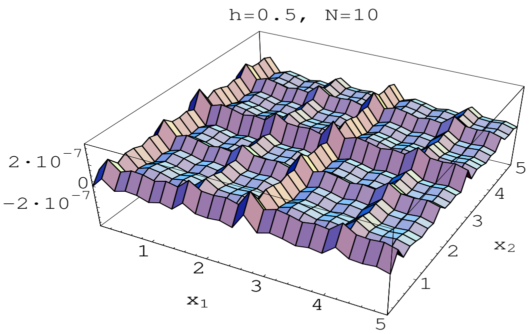

Figure 3: $(\cosh - \mathcal{G}_{h,N}[\cosh]) \,/\, \cosh$.

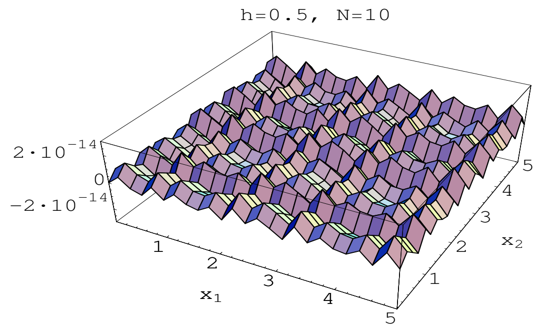

Figure 4: $(\cosh - \mathcal{H}_{h,N}[\cosh]) \,/\, \cosh$.

(x_u, x_v)	Relative error for $\mathcal{G}_{\frac{d}{N},N}$		Relative error for $\mathcal{H}_{\frac{d}{N},N}$	
	$f(x) - \mathcal{G}_{\frac{d}{N},N}[f](x)/f(x)$	bound	$f(x) - \mathcal{H}_{\frac{d}{N},N}[f](x)/f(x)$	bound
(x_1, x_1)	3.49444×10^{-7}	1.77270×10^{-6}	7.05924×10^{-16}	9.57340×10^{-14}
(x_1, x_5)	1.87428×10^{-7}	2.12636×10^{-6}	2.50581×10^{-15}	4.95201×10^{-14}
(x_1, x_9)	1.89442×10^{-7}	3.04586×10^{-6}	2.07430×10^{-15}	5.11876×10^{-14}
(x_5, x_1)	1.87428×10^{-7}	2.12636×10^{-6}	2.50494×10^{-15}	4.97878×10^{-14}
(x_5, x_5)	2.54113×10^{-8}	2.55057×10^{-6}	2.56392×10^{-15}	3.21078×10^{-14}
(x_5, x_9)	2.74261×10^{-8}	3.65352×10^{-6}	1.35655×10^{-15}	5.05915×10^{-14}
(x_9, x_1)	1.89442×10^{-7}	3.04586×10^{-6}	2.07343×10^{-15}	5.11876×10^{-14}
(x_9, x_5)	2.74261×10^{-8}	3.65352×10^{-6}	1.34961×10^{-15}	4.71420×10^{-14}
(x_9, x_9)	1.47165×10^{-8}	5.23342×10^{-6}	3.81639×10^{-16}	3.29404×10^{-14}

Table 6: Approximation of f at real points (x_u, x_v), $N = 10$.

(y_1, y_2)	Absolute error for $\mathcal{G}_{\frac{d}{N},N}$		Absolute error for $\mathcal{H}_{\frac{d}{N},N}$	
	$\lvert f(x) - \mathcal{G}_{h,N}[f](x)\rvert$	bound	$\lvert f(x) - \mathcal{H}_{h,N}[f](x)\rvert$	bound
$(0.1, 0.1)$	9.47352×10^{-8}	1.27109×10^{-6}	3.00218×10^{-13}	8.78625×10^{-13}
$(0.1, 0.2)$	1.06283×10^{-6}	1.66734×10^{-5}	6.86841×10^{-11}	2.68706×10^{-10}
$(0.1, 0.3)$	2.01921×10^{-5}	3.90778×10^{-4}	3.01737×10^{-8}	1.51623×10^{-7}
$(0.2, 0.1)$	1.06283×10^{-6}	1.66734×10^{-5}	6.86841×10^{-11}	2.68706×10^{-10}
$(0.2, 0.2)$	2.04561×10^{-6}	3.20757×10^{-5}	1.38094×10^{-10}	5.36533×10^{-10}
$(0.2, 0.3)$	2.13325×10^{-5}	4.06180×10^{-4}	3.04724×10^{-8}	1.51891×10^{-7}
$(0.3, 0.1)$	2.01921×10^{-5}	3.90778×10^{-4}	3.01737×10^{-8}	1.51623×10^{-7}
$(0.3, 0.2)$	2.13325×10^{-5}	4.06180×10^{-4}	3.04724×10^{-8}	1.51891×10^{-7}
$(0.3, 0.3)$	4.11182×10^{-5}	7.80285×10^{-4}	6.15829×10^{-8}	3.03246×10^{-7}

Table 7: Approximation of f at imaginary points $(\mathrm{i}y_1, \mathrm{i}y_2)$, $N = 10$.

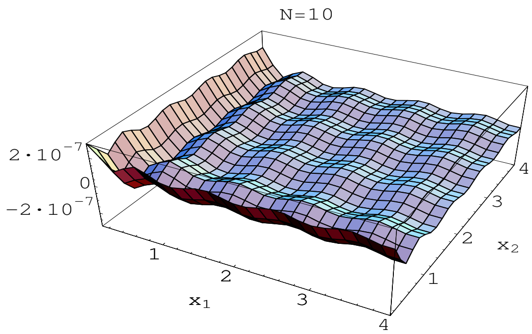

Figure 5: $(f(x) - \mathcal{G}_{\frac{2}{N},N}[f](x))/f(x)$.

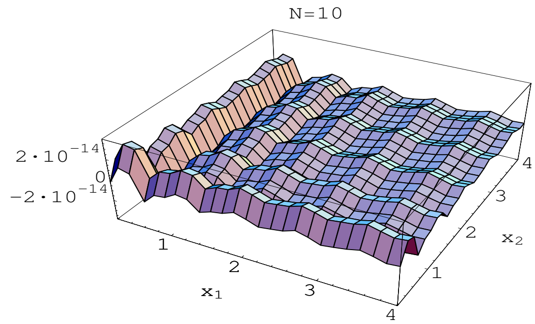

Figure 6: $(f(x) - \mathcal{H}_{\frac{2}{N},N}[f](x))/f(x)$.

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