A modification of Hermite sampling with a Gaussian multiplier

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Abstract

The Hermite sampling series is used to approximate bandlimited functions. In this paper, we introduce two modifications of Hermite sampling with a Gaussian multiplier to approximate bandlimited and non-bandlimited functions. The convergence rate of those modifications is much higher than the convergence rate of Hermite sampling. Based on complex analysis, we establish some error bounds for approximating different classes of functions by these modifications. Theoretically and numerically, we demonstrate that the approximation by these modifications is highly efficient.

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1. Introduction

For $p \in [1, \infty]$, let $L^p(\mathbb{R})$ denotes the classical space of p-th power Lebesgue integrable functions with the usual norm $\|.\|_p$. Further, we denote by B^p_σ the class of entire functions of exponential type σ, $\sigma > 0$ which belong to $L^p(\mathbb{R})$ when restricted to $\mathbb{R}$; that is $f \in B^p_\sigma$ if and only if, (see [5])

$$|f(z)| \le \|f\|_\infty \mathrm{e}^{\sigma|\Im z|}, \quad z \in \mathbb{C}. \tag{1.1}$$

The following inclusions hold, cf. e.g. [23, p. 24],

$$B^1_\sigma \subseteq B^p_\sigma \subseteq B^q_\sigma \subseteq B^\infty_\sigma, \qquad 1 \le p \le q \le \infty.$$

The spaces B^p_σ are called Bernstein spaces and particularly, the space B^2_σ is called Paley-Wiener space. The function $f \in B^p_\sigma$ for some p is called a bandlimited function with band-limit σ. If $f \in B^p_\sigma$, $1 \le p < \infty$ then we can reconstruct via the following Hermite sampling

$$f(z) = H_\sigma[f](z) := \sum_{n=-\infty}^{\infty} \left\{ f\left(\frac{2n\pi}{\sigma}\right) + \left(z - \frac{2n\pi}{\sigma}\right) f'\left(\frac{2n\pi}{\sigma}\right)\right\} \operatorname{sinc}^2\left((\sigma/2)z - n\pi\right), \quad z \in \mathbb{C}, \tag{1.2}$$

where the sinc function is defined as

$$\operatorname{sinc}\left((\sigma/2)z - n\pi\right) = \begin{cases} \dfrac{\sin((\sigma/2)z - n\pi)}{((\sigma/2)z - n\pi)}, & z \ne \frac{2n\pi}{\sigma}, \\ 1, & z = \frac{2n\pi}{\sigma}. \end{cases} \tag{1.3}$$

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The series (1.2) converges uniformly on any compact subset of $\mathbb{C}$. The Hermite sampling is established by Jagerman and Fogel 1956 in [10]. In 2006 Li and Fang [11] discussed the aliasing error associated with (1.2). The truncation, amplitude and jitter errors appearing in (1.2) are studied by Annaby and Asharabi (2010), [2]. In recent years, the Hermite sampling is used in computing the eigenvalues of Sturm-Liouville problems, see e.g. [3, 22]. The generalized Hermite sampling is derived in different situations by Linden and Abramson (1961), Rawn (1989), Hinsen (1993), Grozev and Rahman (1997) and Shin (2002), see [7, 8, 9, 12, 18, 19]. The series (1.2) has a slow rate of convergence.

In this paper, we introduce two modifications, which improve the rate of convergence, for Hermite sampling by using the Gaussian function

$$G(z) := \mathrm{e}^{-z^2}, \quad z \in \mathbb{C}. \tag{1.4}$$

We call these modifications the Hermite-Gauss sampling formulas. As we see in the sequel of this paper, the Hermite-Gauss formulas have the following advantages:

- They give a slightly better convergence rate and the error bounds are of order $\mathrm{e}^{-\alpha N}$, where α is a positive number and N is the number of terms in the Hermite-Gauss formula.
- They extend to wider classes than B^p_σ like non-bandlimited functions, functions analytic in a strip, functions which do not belong to $L^p(\mathbb{R})$.

The effects of using a Gaussian multiplier with the classical sampling series in $\mathbb{R}$ were studied by Qian (2002), Qian and Creamer (2006), and Qian and Ogawa (2006) in a series of papers, see [15, 16, 17]. Schmeisser and Stenger (2007) and Tanaka, Sugihara and Murota (2008) extended the results of Qian and his co-authors to the complex domain $\mathbb{C}$, see [20, 21]. Some authors called the modification of classical sampling with a Gaussian function by sinc-Gaussian technique and they used it for computing the eigenvalues of Sturm-Liouville problems, see e.g. [1, 4].

The organization of this paper is as follows. In Section 2, we present two modifications of Hermite sampling for the class of entire functions and some particular results. In Section 3, we extend the results of Section 2 to the class of analytic functions. Numerical examples will be addressed in Section 4.

2. Entire functions

Let $E_\sigma(\varphi)$, $\sigma \geq 0$, be the class of entire functions satisfying the following condition

$$|f(z)| \leq \varphi(|\Re z|)\ \mathrm{e}^{\sigma|\Im z|}, \quad z \in \mathbb{C}, \tag{2.1}$$

where φ is a non-decreasing, non-negative function on $[0, \infty)$. Clearly, the class $E_\sigma(\varphi)$ is bigger than the class of entire functions of exponential type σ. For $h \in (0, 2\pi/\sigma]$ and $\alpha := (2\pi - h\sigma)/2$, we define the localization operator $\mathcal{G}_{h,N} : E_\sigma(\varphi) \to E_{L^p(\mathbb{R})}$ as follows

$$\mathcal{G}_{h,N}[f](z) := \sum_{n \in \mathbb{Z}_N(z)} \left\{ \left(1 + \frac{2\alpha(z-nh)^2}{h^2 N}\right) f(nh) + (z-nh)f'(nh) \right\} \operatorname{sinc}^2\!\left(\frac{\pi}{h}z - n\pi\right) \mathrm{e}^{-\frac{\alpha}{N}\left(\frac{z}{h}-n\right)^2}, \tag{2.2}$$

where $\mathbb{Z}_N(z) := \left\{n \in \mathbb{Z} : |[h^{-1}\Re z + 1/2] - n| \leq N\right\}$, $N \in \mathbb{Z}^+$ and $E_{L^p(\mathbb{R})}$ is the class of all entire functions which belong to $L^p(\mathbb{R})$ when restricted to $\mathbb{R}$. Note that the operator maps into $E_{L^p(\mathbb{R})}$ because $\operatorname{sinc}^2(\frac{\pi}{h}z - n\pi)\,\mathrm{e}^{-\frac{\alpha}{N}\left(\frac{z}{h}-n\right)^2} \in E_{L^p(\mathbb{R})}$ for all $n \in \mathbb{Z}_N(z)$. The summation in (2.2) depends on the real part of z. As we will see in this section, the operator $\mathcal{G}_{h,N}$ provides a piecewise analytic approximation for the functions of $E_\sigma(\varphi)$ on each of the vertical strips

$$\left\{ z \in \mathbb{C} : \left(k - \frac{1}{2}\right) h \leq \Re z \leq \left(k + \frac{1}{2}\right) h \right\}, \qquad k \in \mathbb{Z}. \tag{2.3}$$

In the following result, we notice that the Hermite operator H_σ, (1.2), is included in the class of Hermite-Gauss operators $\mathcal{G}_{h,N}$.

Lemma 2.1. *Let $\varphi := \|f\|_\infty$, then we have*

$$\lim_{N\to\infty} \mathcal{G}_{2\pi/\sigma,N} f = H_\sigma f = f, \quad \text{for all } f \in B_\sigma^p \subset E_\sigma(\varphi).$$

Proof. Because of $\alpha = 0$ in (2.2) and taking $N \to \infty$, the result follows immediately. □

Before stating and proving the main result of this section, we would like to summarize some inequalities which will be used in the proof of the result of Sections 2,3, namely

$$\left|\mathrm{e}^{-z^2}\right| \le \mathrm{e}^{-(\Re z)^2} \mathrm{e}^{(\Im z)^2}, \quad \text{for all } z \in \mathbb{C}, \tag{2.4}$$

$$|\sin z| \ge |\sinh(\Im z)| = \frac{\mathrm{e}^{|\Im z|}}{2}\left(1 - \mathrm{e}^{-2|\Im z|}\right), \quad \text{for all } z \in \mathbb{C}, \tag{2.5}$$

$$|\sin \pi(n + 1/2 + i\Im z)| = |\cosh(\pi \Im z)| \ge \frac{\mathrm{e}^{\pi|\Im z|}}{2}, \quad n \in \mathbb{Z},\ z \in \mathbb{C}. \tag{2.6}$$

For the proof of the following result we use the technique of [20].

Theorem 2.2. *For $f \in E_\sigma(\varphi)$ and $|\Im z| < N$ we have*

$$|f(z) - \mathcal{G}_{h,N}[f](z)| \le 2\left|\sin^2(h^{-1}\pi z)\right| \varphi\left(|\Re z| + h(N+1)\right) \beta_N\left(h^{-1}\Im z\right) \frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}}, \tag{2.7}$$

where

$$\begin{aligned}\beta_N(t) &:= \frac{4\mathrm{e}^{\alpha t^2/N}}{\sqrt{\pi\alpha N}\left(1 - (t/N)^2\right)} + \frac{\mathrm{e}^{-2\alpha t}}{(1 - \mathrm{e}^{-2\pi(N+t)})^2} + \frac{\mathrm{e}^{2\alpha t}}{(1 - \mathrm{e}^{-2\pi(N-t)})^2} \\ &= 2\cosh(2\alpha t) + O(N^{-1/2}), \quad \text{as} \quad N \to \infty. \end{aligned} \tag{2.8}$$

Proof. We may assume that $\sigma < 2\pi$ and $h = 1$, since the general case can be deduced from the special case by considering the function $\zeta \to f(h\zeta)$. Set $z := x + \mathrm{i}y$, $x, y \in \mathbb{R}$. Also we may assume that $y \ge 0$ and the general case can be deduced from the special case by considering the function $\zeta \to \overline{f(\overline{\zeta})}$. Denote by $\mathcal{R}$ the rectangle with vertices at $\pm N' + N_z + \mathrm{i}(y \pm N)$, where $N' = N + 1/2$ $N_z := \lfloor \Re z + 1/2 \rfloor$. We consider the kernel function

$$\mathcal{K}(z,\zeta) := \frac{\sin^2(\pi z)}{2\pi \mathrm{i}} . \frac{f(\zeta)\,\mathrm{e}^{-\frac{\alpha}{N}(z-\zeta)^2}}{(\zeta - z)\sin^2(\pi\zeta)}, \quad z \in \mathbb{C},\ z \notin \mathbb{Z}. \tag{2.9}$$

This kernel $\mathcal{K}(z,\zeta)$, as a function of ζ, has a simple pole at $\zeta = z$ and a pole of order 2 at $\zeta = n$, $n \in \mathbb{Z}$. Using the residue theorem we obtain

$$\int_{\mathcal{R}} \mathcal{K}(z,\zeta)\mathrm{d}\zeta = 2\pi\mathrm{i}\,\mathrm{Res}\,(\mathcal{K}; z) + 2\pi\mathrm{i} \sum_{n=N_z-N}^{N_z+N} \mathrm{Res}\,(\mathcal{K}; n), \quad z \notin \mathbb{Z}, \tag{2.10}$$

where $\mathrm{Res}\,(\mathcal{K}; \xi)$ denotes the residue of $\mathcal{K}$ at ξ. The residue at each pole is given by

$$\mathrm{Res}\,(\mathcal{K}; z) = \frac{f(z)}{2\pi\mathrm{i}}, \tag{2.11}$$

and for $n \in \mathbb{Z}$

$$\begin{aligned}\operatorname{Res}(\mathcal{K};n) &= \frac{\sin^2(\pi z)}{2\pi \mathrm{i}} \lim_{\zeta\to n}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left\{\frac{f(\zeta)\,\mathrm{e}^{-\frac{\alpha}{N}(z-\zeta)^2}}{(\zeta-z)}\left(\frac{\zeta-n}{\sin(\pi\zeta)}\right)^2\right\}\\ &= \frac{-1}{2\pi\mathrm{i}}\left\{\left(1+\frac{2\alpha(z-n)^2}{N}\right)f(n)+(z-n)f'(n)\right\}\operatorname{sinc}^2(\pi z-n\pi)\,\mathrm{e}^{-\frac{\alpha}{N}(z-n)^2}. \quad (2.12)\end{aligned}$$

Substituting from (2.11), (2.12) into (2.10), yields

$$\int_{\mathcal{R}}\mathcal{K}(z,\zeta)\mathrm{d}\zeta = f(z)-\mathcal{G}_{1,N}[f](z), \quad z\notin\mathbb{Z}. \quad (2.13)$$

In the case $z=n\in\mathbb{Z}$, we have from (2.2), where $h=1$, that $f(n)=\mathcal{G}_{1,N}[f](n)$. Error bounds for $f(z)$ and its Hermite-Gaussian sampling are now computed by estimation of the integral $\int_{\mathcal{R}}\mathcal{K}(z,\zeta)\mathrm{d}\zeta$. From (2.9) we have

$$\int_{\mathcal{R}}\mathcal{K}(z,\zeta)\mathrm{d}\zeta = \frac{\sin^2(\pi z)}{2\pi\mathrm{i}}\int_{\mathcal{R}}\frac{f(\zeta)\,\mathrm{e}^{-\frac{\alpha}{N}(z-\zeta)^2}}{(\zeta-z)\sin^2(\pi\zeta)}\mathrm{d}\zeta. \quad (2.14)$$

Denote by $I_H^{\pm}$ the contributions to the integral coming from the two horizontal parts of $\mathcal{R}$, where $+$ and $-$ refer to the upper and the lower line segment, respectively. Similarly, denote by $I_V^{\pm}$ the contributions coming from the two vertical parts of $\mathcal{R}$, where $+$ and $-$ refer to the right and the left line segment, respectively. Therefore

$$\int_{\mathcal{R}}\frac{f(\zeta)\,\mathrm{e}^{-\frac{\alpha}{N}(z-\zeta)^2}}{(\zeta-z)\sin^2(\pi\zeta)}\mathrm{d}\zeta = I_H^- + I_V^+ + I_H^+ + I_V^-, \quad (2.15)$$

where

$$I_V^{\pm} := \pm\mathrm{i}\int_{-N+y}^{N+y}\frac{f(\pm N'+N_z+\mathrm{i}t)\,\mathrm{e}^{-\frac{\alpha}{N}\left(z\mp N'-N_z-\mathrm{i}t\right)^2}}{(\pm N'+N_z+\mathrm{i}t-z)\sin^2\pi(\pm N'+N_z+\mathrm{i}t)}\mathrm{d}t, \quad (2.16)$$

$$I_H^{\pm} := \mp\int_{-N'+N_z}^{N'+N_z}\frac{f(t+\mathrm{i}(y\pm N))\,\mathrm{e}^{-\frac{\alpha}{N}(x-t\mp\mathrm{i}N)^2}}{(t-x\mp\mathrm{i}N)\sin^2\pi(t+\mathrm{i}(y\pm N))}\mathrm{d}t. \quad (2.17)$$

From (2.1), we have for any point $z\in\mathcal{R}$

$$|f(z)| \le \varphi\left(N'+|N_z|\right)\,\mathrm{e}^{\sigma|y|} \le \varphi\left(|x|+N+1\right)\,\mathrm{e}^{\sigma|y|}. \quad (2.18)$$

The following inequality holds for $z\in\mathbb{C}$

$$|\pm N'+N_z-\Re z| \ge N. \quad (2.19)$$

Using (2.18), (2.19) and the inequalities (2.4)–(2.6), we get

$$\begin{aligned}\left|I_V^{\pm}\right| &\le \frac{4\varphi\left(|x|+N+1|\right)\,\mathrm{e}^{-\alpha N}}{N}\int_{-N+y}^{N+y}\mathrm{e}^{-(2\pi-\sigma)|t|+\frac{\alpha}{N}(y-t)^2}\mathrm{d}t\\ &= \frac{4\varphi\left(|x|+N+1|\right)\,\mathrm{e}^{-\alpha N}}{N}\int_{-N}^{N}\mathrm{e}^{-2\alpha|u+y|+\frac{\alpha}{N}u^2}\mathrm{d}u. \quad (2.20)\end{aligned}$$

Substituting from the estimation, [20, p. 205],

$$\int_{-N}^{N}\mathrm{e}^{-2\alpha|u+y|+\frac{\alpha}{N}u^2}\mathrm{d}u \le \frac{2\,\mathrm{e}^{\alpha y^2/N}}{\alpha\left(1-(y/N)^2\right)} \quad (2.21)$$

into (2.20), implies

$$\left|I_V^{\pm}\right| \le \frac{8\varphi\left(|x|+N+1|\right)\,\mathrm{e}^{\alpha y^2/N}\,\mathrm{e}^{-\alpha N}}{\alpha N\left(1-(y/N)^2\right)}. \tag{2.22}$$

Again we use the inequalities (2.4)–(2.6) and (2.18) to estimate the integral in (2.17)

$$\begin{aligned}\left|I_H^{\pm}\right| &\le \frac{4\varphi\left(|x|+N+1|\right)\mathrm{e}^{(\sigma-2\pi)|y\pm N|}\,\mathrm{e}^{\alpha N}}{N(1-\mathrm{e}^{-2\pi|y\pm N|})^2}\int_{-N'+N_z}^{N'+N_z}\mathrm{e}^{-\frac{\alpha}{N}(x-t)^2}\mathrm{d}t\\ &\le \frac{4\varphi\left(|x|+N+1|\right)\,\mathrm{e}^{-\alpha N}\,\mathrm{e}^{\mp 2\alpha y}}{N(1-\mathrm{e}^{-2\pi|y\pm N|})^2}\int_{-\infty}^{\infty}\mathrm{e}^{-\frac{\alpha}{N}u^2}\mathrm{d}u\\ &= \frac{4\sqrt{\pi}\varphi\left(|x|+N+1|\right)\,\mathrm{e}^{-\alpha N}\,\mathrm{e}^{\mp 2\alpha y}}{\sqrt{\alpha N}(1-\mathrm{e}^{-2\pi|y\pm N|})^2}.\end{aligned} \tag{2.23}$$

Therefore,

$$\left|I_H^{+}\right|+\left|I_H^{-}\right| \le 4\pi\varphi\left(|x|+N+1|\right)\frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}}\left(\frac{\mathrm{e}^{-2\alpha y}}{(1-\mathrm{e}^{-2\pi(N+y)})^2}+\frac{\mathrm{e}^{2\alpha y}}{(1-\mathrm{e}^{-2\pi(N-y)})^2}\right). \tag{2.24}$$

Combining (2.24), (2.22), (2.15) and (2.14), yields

$$\left|\int_{\mathcal{R}}\mathcal{K}(z,\zeta)\mathrm{d}\zeta\right| \le 4\pi\left|\sin^2(\pi z)\right|\varphi\left(|\Re z|+(N+1)\right)\beta_N\left(\Im z\right)\frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}}, \tag{2.25}$$

where $\beta_N(y)$ is defined in (2.8). Substituting from (2.25) into (2.13), we get (2.2) with $h=1$. □

In the following corollaries, we state and prove some interesting consequences of Theorem 2.1.

Corollary 2.3. Let $f \in B_\sigma^\infty$. Then, in the notation of Theorem 2.2, we have

$$|f(z)-\mathcal{G}_{h,N}[f](z)| \le 2\left|\sin^2(h^{-1}\pi z)\right|\|f\|_\infty\beta_N\left(h^{-1}\Im z\right)\frac{\mathrm{e}^{-\alpha N}}{\sqrt{\pi\alpha N}}, \quad |\Im z|<N. \tag{2.26}$$

Proof. Since $f \in B_\sigma^\infty$, we have (1.1). Replacing (2.1) by (1.1) in Theorem 2.1, yields (2.26). □

Corollary 2.4. For $1<p\le 2$ and $1/p+1/q=1$ let $f\in B_\sigma^p$. Then, with the notation of Theorem 2.2, we have

$$|f(z)-\mathcal{G}_{h,N}[f](z)| \le \sqrt{2}(2\sigma)^{1/q}\left|\sin^2(h^{-1}\pi z)\right|\|f\|_q\beta_N\left(h^{-1}\Im z\right)\frac{\mathrm{e}^{-\alpha N}}{\pi\sqrt{\alpha N}}, \quad |\Im z|<N. \tag{2.27}$$

Proof. For $f\in B_\sigma^p$ the well known Paley-Wiener theorem [14] ensures

$$f(t)=\frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}\widehat{f}(\omega)\mathrm{e}^{\mathrm{i}t\omega}\mathrm{d}\omega, \tag{2.28}$$

where $\widehat{f}$ is the Fourier transform of f. Using Hölder's inequality and the Hausdorff-Young identity, [6, p. 177], yields

$$\|f\|_\infty \le \frac{1}{\sqrt{2\pi}}\int_{-\sigma}^{\sigma}|\widehat{f}(\omega)|\mathrm{d}\omega \le \frac{(2\sigma)^{1/q}}{\sqrt{2\pi}}\|\widehat{f}\|_p \le \frac{(2\sigma)^{1/q}}{\sqrt{2\pi}}\|f\|_q. \tag{2.29}$$

From (2.29), (2.26) and $B_\sigma^p\subset B_\sigma^\infty$, we finally obtain (2.27). □

Corollary 2.5. Let f be an entire function satisfying

$$|f(z)| \le M\,\mathrm{e}^{\kappa|\Re z|+\sigma|\Im z|}, \quad M>0, \tag{2.30}$$

where σ, κ are non-negative numbers and let $h \in (0, 2\pi/(\sigma+2\kappa))$. Then, in the notation of Theorem 2.2, we have

$$|f(z)-\mathcal{G}_{h,N}[f](z)| \le 2\left|\sin^2(h^{-1}\pi z)\right| M\mathrm{e}^{\kappa(|\Re z|+h)}\beta_N\left(h^{-1}\Im z\right)\frac{\mathrm{e}^{-(\alpha-h\kappa)N}}{\sqrt{\pi\alpha N}}, \quad |\Im z|<N. \tag{2.31}$$

Proof. Setting $\phi(x) = M\,\mathrm{e}^{\kappa|\Re z|}$ in Theorem 2.2 implies immediately (2.31) after we restrict h to be in the interval $(0, 2\pi/(\sigma+2\kappa))$. □

At the end of this section, we define another operator also modified for the Hermite sampling formula. For $0 < h < \pi/\sigma$ and $\delta := (\pi - h\sigma)/2$, let the operator $\mathcal{C}_{h,N} : B^\infty_\sigma \to E_{L^p(\mathbb{R})}$ be given by

$$\mathcal{C}_{h,N}[f](z) := \sum_{n\in\mathbb{Z}_N(z)} \{f(nh)+(z-nh)f'(nh)\}\operatorname{sinc}\left(\pi h^{-1}z-n\pi\right)\mathrm{e}^{-\frac{\delta}{N}\left(\frac{z}{h}-n\right)^2}. \tag{2.32}$$

The domain of $\mathcal{C}_{h,N}$ is B^∞_σ which is bigger than the domain of the operator H_σ but smaller than that of $\mathcal{G}_{h,N}$.

Theorem 2.6. *Let $f \in B^\infty_\sigma$. Then, we have*

$$|f(z)-\mathcal{C}_{h,N}[f](z)| \le \frac{\|f\|_\infty\left|\sin(h^{-1}\pi z)\right|\chi_N\left(h^{-1}\Im z\right)}{\sqrt{\pi\alpha N}}(\sigma N+1)\,\mathrm{e}^{-\delta N}, \tag{2.33}$$

where $|\Im z| < N$ and

$$\begin{aligned}\chi_N(t) &:= 2\cosh(2\delta t)+\frac{4\,\mathrm{e}^{\delta t^2/N}}{\sqrt{\pi\delta N}\left(1-(t/N)^2\right)}+\frac{\mathrm{e}^{2\delta t}}{\mathrm{e}^{2\pi(N-t)}-1}+\frac{\mathrm{e}^{-2\delta t}}{\mathrm{e}^{2\pi(N+t)}-1}\\ &= 2\cosh(2\delta t)+O\left(N^{-1/2}\right), \quad \text{as} \quad N\to\infty.\end{aligned} \tag{2.34}$$

Proof. We consider the kernel function

$$\mathcal{K}(z,\zeta) := \frac{\sin(\pi z)}{2\pi\mathrm{i}}.\frac{\{f(\zeta)+(z-\zeta)f'(\zeta)\}\,\mathrm{e}^{-\frac{\delta}{N}(z-\zeta)^2}}{(\zeta-z)\sin(\pi\zeta)}, \quad z\in\mathbb{C}\backslash\{\mathbb{Z}\}. \tag{2.35}$$

This kernel $\mathcal{K}(z,\zeta)$, as a function of ζ, has simple poles at $\zeta = z$ and $\zeta \in \mathbb{Z}$. Since $f \in B^\infty_\sigma$ we have $f' \in B^\infty_\sigma$ and by using Bernstein's inequality, cf. [13, p. 115], we obtain

$$|f'(z)| \le \sigma\|f\|_\infty \mathrm{e}^{\sigma|\Im z|}. \tag{2.36}$$

The proof is completed if we follow the same arguments as in the proof of Theorem 2.2 with the kernel (2.35) and the inequalities (1.1) and (2.36). □

Remark 2.7. In the following table, we compare between the rate of convergence of sinc-Gaussian and Hermite-Gauss formulas and we get that the rate of convergence of Hermite-Gauss formula $\mathcal{G}_{h,N}$ is the best

Formula	Rate of convergence	Class of functions
Sinc-Gaussian	$\mathrm{e}^{-(\pi-\sigma h)\frac{N}{2}}/\sqrt{N}$	for all $f\in E_\sigma(\varphi)$
Hermite-Gauss, $\mathcal{G}_{h,N}$	$\mathrm{e}^{-(2\pi-\sigma h)\frac{N}{2}}/\sqrt{N}$	for all $f\in E_\sigma(\varphi)$
Hermite-Gauss, $\mathcal{C}_{h,N}$	$\sqrt{N}\mathrm{e}^{-(\pi-\sigma h)\frac{N}{2}}$	for all $f\in B^\infty_\sigma$

The rate of convergence of sinc-Gaussian formula (modification of classical sampling with Gaussian Multiplier) was studied in [20, Theorem 2.1].

Remark 2.8. From last table, we can see that the rate of convergence of Hermite-Gauss formulas depends on the parameter h and its goes to be more speed when h decreases without any additional cost.

3. Functions analytic in a strip

For $d > 0$ a horizontal strip $\mathcal{S}_d$ is defined as

$$\mathcal{S}_d := \{z \in \mathbb{C} : |\Im z| < d\} . \tag{3.1}$$

Furthermore, we define $A_d(\varphi)$ to be the class of analytic functions $f : \mathcal{S}_d \to \mathbb{C}$ which satisfy the condition (2.1) with $\sigma = 0$. For functions from the class $A_d(\varphi)$ we study the operator $\mathcal{G}_{\frac{d}{N},N}$ in the special case $h := \frac{d}{N}$ and $\alpha = \pi$.

Theorem 3.1. *For $f \in A_d(\varphi)$ and $z \in \mathcal{S}_{d/4}$ we have*

$$\left|f(z) - \mathcal{G}_{d/N,N}[f](z)\right| \le 2\left|\sin^2\left(\frac{N\pi z}{d}\right)\right| \varphi\left(|\Re z| + d + \frac{d}{N}\right)\gamma_N\left(\Im z/d\right)\frac{\mathrm{e}^{-\pi N\left(1-\frac{2|\Im z|}{d}\right)}}{\pi\sqrt{N}}, \tag{3.2}$$

where

$$\gamma_N(t) := \frac{1}{1-t}\left(\frac{1}{(1-\mathrm{e}^{-2\pi N})^2} + \frac{2}{\pi\sqrt{N}(1+t)}\right) = \frac{1}{1-t}\left(1 + O\left(N^{-1/2}\right)\right), \quad \textit{as} \quad N \to \infty.$$

Proof. It is enough to prove the theorem for an arbitrary but fixed $N \in \mathbb{N}$. For doing this, we may assume that $d = N$, and so $h = 1$. The general case can be deduced by considering the function $\zeta \to f(h\zeta)$. As explained in the proof of Theorem 2.2 we may assume that $y := \Im z \ge 0$. After these specializations, we denote by $\mathcal{R}$ the positively oriented rectangle with vertices at $\pm N' + N_z + \mathrm{i}N$ and $\pm N' + N_z - \mathrm{i}(N-y)$ where $N' := N + \frac{1}{2}$. Using again the kernel (2.6) and the residue theorem, we obtain

$$f(z) - \mathcal{G}_{1,N}[f](z) = \int_{\mathcal{R}} \mathcal{K}(z,\zeta)\mathrm{d}\zeta = \frac{\sin^2(\pi z)}{2\pi\mathrm{i}}\int_{\mathcal{R}} \frac{f(\zeta)\,\mathrm{e}^{-\frac{\pi}{N}(z-\zeta)^2}}{(\zeta - z)\sin^2(\pi\zeta)}\mathrm{d}\zeta. \tag{3.3}$$

We use the notations $I_H^{\pm}$ and $I_V^{\pm}$ as explained in Theorem 2.2. Therefore,

$$\int_{\mathcal{R}} \frac{f(\zeta)\,\mathrm{e}^{-\frac{\pi}{N}(z-\zeta)^2}}{(\zeta - z)\sin^2(\pi\zeta)}d\zeta = I_H^- + I_V^+ + I_H^+ + I_V^-, \tag{3.4}$$

where

$$I_V^{\pm} := \pm\mathrm{i}\int_{-N+y}^{N} \frac{f(\pm N' + N_z + \mathrm{i}t)\,\mathrm{e}^{\frac{-\pi}{N}\left(z \mp N' - N_z - \mathrm{i}t\right)^2}}{(\pm N' + N_z + \mathrm{i}t - z)\sin^2(\pi(\pm N' + N_z + \mathrm{i}t))}\mathrm{d}t, \tag{3.5}$$

$$I_H^+ := -\int_{-N'+N_z}^{N'+N_z} \frac{f(t + \mathrm{i}N)\,\mathrm{e}^{\frac{-\pi}{N}(z-t-\mathrm{i}N)^2}}{(t - z + \mathrm{i}N)\sin^2\pi(t + \mathrm{i}N)}\mathrm{d}t, \tag{3.6}$$

$$I_H^- := \int_{-N'+N_z}^{N'+N_z} \frac{f(t + \mathrm{i}(y-N))\,\mathrm{e}^{\frac{-\pi}{N}(x-t+\mathrm{i}N)^2}}{(t - x - \mathrm{i}N)\sin^2\pi(t - \mathrm{i}(N-y))}\mathrm{d}t. \tag{3.7}$$

Since $f \in A_d(\varphi)$, for every point $\zeta \in \mathcal{R}$ it holds

$$|f(\zeta)| \le \varphi\left(N' + |N_z|\right) \le \varphi\left(|x| + N + 1\right). \tag{3.8}$$

Using the inequalities (2.4)–(2.19) and (3.8), we find

$$|I_V^{\pm}| \le \frac{4\phi\left(|x| + N + 1\right)}{N}\,\mathrm{e}^{-\pi N}\int_{-N+y}^{N} \mathrm{e}^{\frac{-\pi}{N}(y-t)^2 - 2\pi|t|}\mathrm{d}t. \tag{3.9}$$

The last integral is estimated in [20, p. 211] as follows

$$\int_{-N+y}^{N} \mathrm{e}^{\frac{-\pi}{N}(y-t)^2 - 2\pi|t|}\mathrm{d}t \le \frac{2\mathrm{e}^{\frac{\pi y^2}{N}}}{\pi\left(1 - (y/N)^2\right)} \le \frac{2\mathrm{e}^{2\pi y}}{\pi\left(1 - (y/N)^2\right)} \tag{3.10}$$

and we obtain

$$|I_V^{\pm}| \le \frac{8\varphi\left(|x|+N+1\right)}{\pi N}\,\frac{\mathrm{e}^{-\pi(N-2y)}}{(1-(y/N)^2)}. \tag{3.11}$$

Again using the inequalities (2.4)–(2.19) and (3.8), we easily derive

$$|I_H^{+}| \le \frac{4\varphi\left(|x|+N+1\right)}{N-y}\,\frac{\mathrm{e}^{\frac{\pi}{N}(N-y)^2-2\pi N}}{(1-\mathrm{e}^{-2\pi N})^2}\sqrt{N}, \tag{3.12}$$

$$|I_H^{-}| \le \frac{4\varphi\left(|x|+N+1\right)}{N}\,\frac{\mathrm{e}^{-\pi(N-2y)}}{(1-\mathrm{e}^{-2\pi(N-y)})^2}\sqrt{N}. \tag{3.13}$$

Since $0 \le y < N$, it is easy to verify that

$$\frac{\pi}{N}\left(N-y\right)^2 - 2\pi N \le -\pi(N-2y), \tag{3.14}$$

$$(N-y)\left(1-\mathrm{e}^{-2\pi N}\right)^2 \le N\left(1-\mathrm{e}^{-2\pi(N-y)}\right)^2. \tag{3.15}$$

Hence, (3.11) and (3.12) can be simplified to

$$|I_H^{\pm}| \le \frac{4\varphi\left(|x|+N+1\right)}{N-y}\,\frac{\sqrt{N}\,\mathrm{e}^{-\pi(N-2y)}}{(1-\mathrm{e}^{-2\pi N}))^2}. \tag{3.16}$$

The proof is completed by combining (3.16), (3.11), (3.4) and (3.3) with $h = 1$. □

Remark 3.2. The rate of convergence of sinc-Gaussian formula for analytic functions in a strip $\mathcal{S}_d$ is of order $\mathrm{e}^{-\frac{\pi N}{2}\left(1-\frac{2|\Im z|}{d}\right)}/\sqrt{N}$, [20, Theorem 3.1]. In our Theorem 3.1, the rate of convergence of Hermite-Gauss formula $\mathcal{G}_{\frac{d}{N},N}$ is of order $\mathrm{e}^{-\pi N\left(1-\frac{2|\Im z|}{d}\right)}/\sqrt{N}$ which is better than the rate of sinc-Gaussian.

Let $\mathbf{B}_d$ be the class of all analytic functions f in $\mathcal{S}_d$ such that f, f' are bounded in $\mathcal{S}_d$. In this class, we establish a special case of the operator $\mathcal{C}_{\frac{d}{N},N}$ for $h := \frac{d}{N}$ and $\delta := \pi/2$.

Theorem 3.3. *Let $f \in \mathbf{B}_d$. Then we have*

$$\left|f(z) - \mathcal{C}_{\frac{d}{N},N}[f](z)\right| \le 2\sqrt{2}\left|\sin\left(\frac{N\pi z}{d}\right)\right| \eta_N\left(\Im z/d\right)\left(\mu+\mu' N\right)\frac{\mathrm{e}^{-\frac{\pi N}{2}\left(1-\frac{2|\Im z|}{d}\right)}}{\pi\sqrt{N}}, \quad z \in \mathcal{S}_{d/4}, \tag{3.17}$$

where $\mu := \sup_{z\in\mathcal{S}_d} |f(z)|$, $\mu' := \sup_{z\in\mathcal{S}_d} |f'(z)|$ *and*

$$\eta_N(t) := \frac{1}{1-t}\left(\frac{1}{1-\mathrm{e}^{-2\pi N}} + \frac{2\sqrt{2}}{\pi\sqrt{N}(1+t)}\right) = \frac{1}{1-t}\left(1+O\left(N^{-1/2}\right)\right), \quad as \quad N \to \infty.$$

Proof. Using the kernel (2.35) and applying the same technique as in the proof of Theorem 3.1, we are finished. □

4. Examples and comparison

This section contains four examples. In the first example, we compare between the Hermite sampling H_σ, sinc-Gaussian and Hermite-Gauss formulas. We choose the functions of Examples 2-4 from the spaces B_σ^∞, $E_\sigma(\varphi)$ and $A_d(\varphi)$ respectively. We approximate the function f in Example 2 by the operators $\mathcal{G}_{h,N}$ and $\mathcal{C}_{h,N}$ for some h and N and the function of Example 4 by $\mathcal{G}_{\frac{d}{N},N}$ and $\mathcal{C}_{\frac{d}{N},N}$. We can see in the examples, as predicted in Remark 2.7, that the rate of convergence of Hermite-Gauss formula $\mathcal{G}_{h,N}$ is the best. We denote the point $x_{h,j} := \left(j-\frac{1}{2}\right)h$ where $j \in \mathbb{Z}$.

j	Hermite	Hermite-Gauss $\mathcal{C}_{h,N}$	Hermite-Gauss $\mathcal{G}_{h,N}$
1	1.18042×10^{-3}	2.50780×10^{-6}	6.25056×10^{-14}
2	8.96183×10^{-3}	3.18360×10^{-7}	1.52101×10^{-14}
3	1.73756×10^{-2}	2.04345×10^{-6}	4.22162×10^{-14}
4	1.86895×10^{-2}	2.67170×10^{-6}	6.56558×10^{-14}
5	1.16930×10^{-2}	1.13415×10^{-6}	3.82749×10^{-14}
6	2.81562×10^{-3}	1.19174×10^{-6}	1.61260×10^{-14}

Table 1: Error approximation of sinc at $x_{h,j}$ for $N = 10$, $h = 1$.

$h = 1$	Sinc-Gaussian		Hermite-Gauss $\mathcal{G}_{h,N}$	
j	absolute error	bound	absolute error	bound
1	2.96515×10^{-7}	9.40567×10^{-6}	6.25056×10^{-14}	1.80492×10^{-12}
3	1.91518×10^{-7}	9.40567×10^{-6}	4.22162×10^{-14}	1.80492×10^{-12}
5	1.90556×10^{-7}	9.40567×10^{-6}	3.82749×10^{-14}	1.80492×10^{-12}
7	2.43627×10^{-7}	9.40567×10^{-6}	5.42968×10^{-14}	1.80492×10^{-12}
9	1.04706×10^{-7}	9.40567×10^{-6}	8.68750×10^{-15}	1.80492×10^{-12}

Table 2: Approximation of sinc at $x_{h,j}$ for $N = 10$.

Example 4.1. Let $f(z) = \text{sinc}\,(z)$ which is in the space B_1^2. In Table 1, we compare between the approximations of f at the point $x_{h,j}$ using Hermite sampling H_σ and the modifications of it, $\mathcal{G}_{h,N}$ and $\mathcal{C}_{h,N}$.
In the following figures, we show the graphs of the error on the interval $[0, 10]$ for $N = 10$ and $h = 1$.

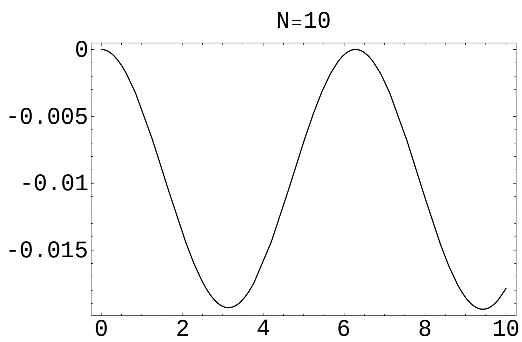

Figure 1: Error of Hermite H_2.

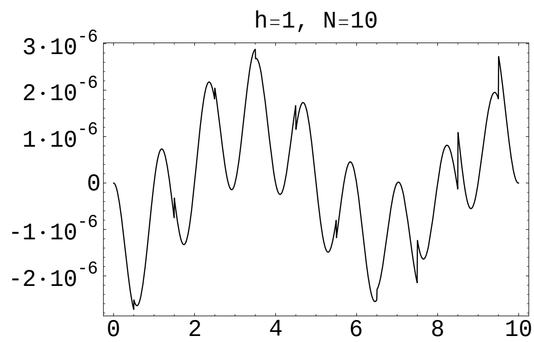

Figure 2: Error of Hermite-Gauss $\mathcal{C}_{h,N}$.

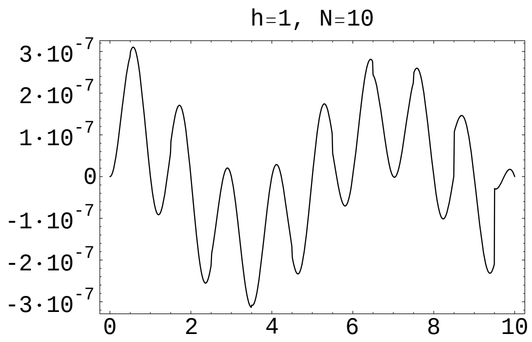

Figure 3: Error of sinc-Gaussian.

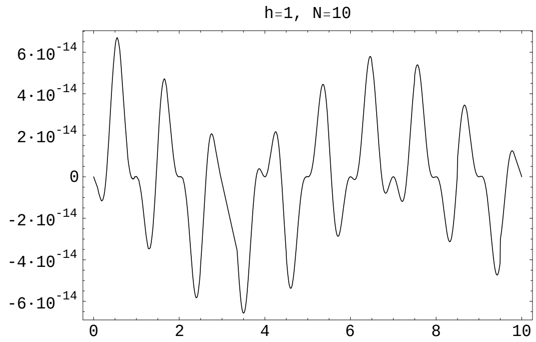

Figure 4: Error of Hermite-Gauss $\mathcal{G}_{h,N}$.

Example 4.2. Let $f(z) = \cos(2z)$ which is a B_2^∞-function. We approximate $f(z)$ at points $x_{h,j}$ using $\mathcal{G}_{h,N}$ and $\mathcal{C}_{h,N}$. In Tables 4,5 and Figures 5,6,7 we illustrate the results.

$h = 1$	$N = 8$		$N = 10$		$N = 12$	
j	absolute error	bound	absolute error	bound	absolute error	bound
1	4.9583×10^{-9}	2.65×10^{-8}	6.6429×10^{-11}	3.18×10^{-10}	8.1546×10^{-13}	3.93×10^{-12}
3	2.8650×10^{-9}	2.65×10^{-8}	3.1158×10^{-11}	3.18×10^{-10}	4.5086×10^{-13}	3.93×10^{-12}
5	8.7037×10^{-9}	2.65×10^{-8}	1.0716×10^{-10}	3.18×10^{-10}	1.4047×10^{-12}	3.93×10^{-12}
7	8.5133×10^{-9}	2.65×10^{-8}	1.0893×10^{-10}	3.18×10^{-10}	1.3863×10^{-12}	3.93×10^{-12}
9	2.4255×10^{-9}	2.65×10^{-8}	3.5246×10^{-11}	3.18×10^{-10}	4.0756×10^{-13}	3.93×10^{-12}

Table 3: Approximation of $\cos(2z)$ at $x_{h,j}$ by $\mathcal{G}_{1,N}$.

$h = \frac{1}{2}$	$N = 8$		$N = 10$		$N = 12$	
j	absolute error	bound	absolute error	bound	absolute error	bound
1	2.0718×10^{-4}	1.49×10^{-3}	3.1751×10^{-5}	1.90×10^{-4}	4.1735×10^{-6}	2.39×10^{-5}
5	3.7708×10^{-5}	1.49×10^{-3}	6.8240×10^{-6}	1.90×10^{-4}	1.2449×10^{-6}	2.39×10^{-5}
9	1.5788×10^{-4}	1.49×10^{-3}	2.2830×10^{-5}	1.90×10^{-4}	2.5460×10^{-6}	2.39×10^{-5}
13	2.4411×10^{-4}	1.49×10^{-3}	3.6670×10^{-5}	1.90×10^{-4}	4.5732×10^{-6}	2.39×10^{-5}
17	1.6124×10^{-4}	1.49×10^{-3}	2.5108×10^{-5}	1.90×10^{-4}	3.4326×10^{-6}	2.39×10^{-5}

Table 4: Approximation of $\cos(2z)$ at $x_{h,j}$ by $\mathcal{C}_{\frac{1}{2},N}$.

In the following table, we see the accuracy of the approximate $\cos(2z)$ at the complex points $z_{h,j} := x_{h,j} + \mathrm{i}h$ using the operator $\mathcal{G}_{1,N}$.

$h=1$	$N=8$		$N=10$		$N=12$	
$z_{h,j}$	absolute error	bound	absolute error	bound	absolute error	bound
0.5+i	4.6789×10^{-5}	1.58×10^{-4}	5.8110×10^{-7}	2.39×10^{-6}	7.3427×10^{-9}	3.57×10^{-8}
3.5+i	4.6793×10^{-5}	1.58×10^{-4}	5.8126×10^{-7}	2.39×10^{-6}	7.3385×10^{-9}	3.57×10^{-8}
5.5+i	4.6835×10^{-5}	1.58×10^{-4}	5.8114×10^{-7}	2.39×10^{-6}	7.3411×10^{-9}	3.57×10^{-8}
7.5+i	4.6819×10^{-5}	1.58×10^{-4}	5.8102×10^{-7}	2.39×10^{-6}	7.3446×10^{-9}	3.57×10^{-8}
9.5+i	4.9695×10^{-5}	1.58×10^{-4}	5.6404×10^{-7}	2.39×10^{-6}	7.2426×10^{-9}	3.57×10^{-8}

Table 5: Approximation of $\cos(2z)$ at $z_{h,j}$ by $\mathcal{G}_{1,N}$.

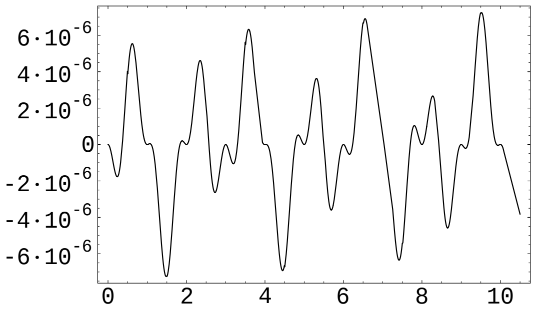

Figure 5: $\cos(z) - \mathcal{G}_{1,5}[\cos](z)$.

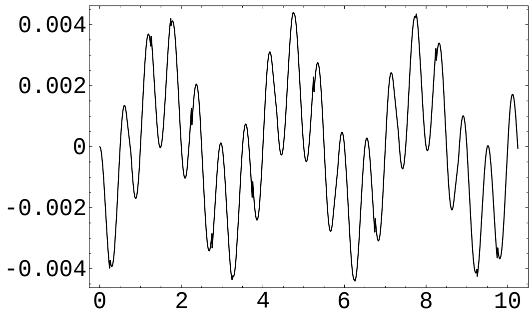

Figure 6: $\cos(z) - \mathcal{C}_{\frac{1}{2},5}[\cos](z)$.

Example 4.3. Let $f(z) = \cosh(z)$. Since $|\cosh(z)| = |\cos(\mathrm{i}z)| \le \mathrm{e}^{|\Re z|}$, then $f \in E_0\left(\mathrm{e}^{|\Re z|}\right)$ and we apply Corollary 2.5 with $\kappa = 1$, $M = 1$ and $\sigma = 0$. Note that the samples $f(nh)$, $f'(nh)$ of this example have an exponential growth for $n \to \infty$. That means $\cosh(z)$ can't be approximated by $\mathcal{C}_{h,N}$.

$h=\frac{1}{2}$	$N=8$		$N=10$		$N=12$	
j	absolute error	bound	absolute error	bound	absolute error	bound
1	2.8670×10^{-12}	7.75×10^{-10}	7.7716×10^{-15}	3.45×10^{-12}	2.2205×10^{-16}	1.58×10^{-14}
5	1.7740×10^{-11}	5.73×10^{-9}	1.9540×10^{-14}	2.55×10^{-11}	1.7764×10^{-15}	1.16×10^{-13}
9	1.3060×10^{-10}	4.23×10^{-8}	1.2790×10^{-13}	1.88×10^{-10}	7.1054×10^{-15}	8.60×10^{-13}
13	9.6452×10^{-10}	3.13×10^{-7}	1.5916×10^{-12}	1.39×10^{-9}	4.5475×10^{-13}	6.36×10^{-12}
17	7.1266×10^{-9}	2.31×10^{-6}	1.1141×10^{-11}	1.03×10^{-8}	3.4106×10^{-12}	4.70×10^{-11}

Table 6: Approximation of $\cosh(z)$ at $x_{h,j}$ by $\mathcal{G}_{\frac{1}{2},N}$.

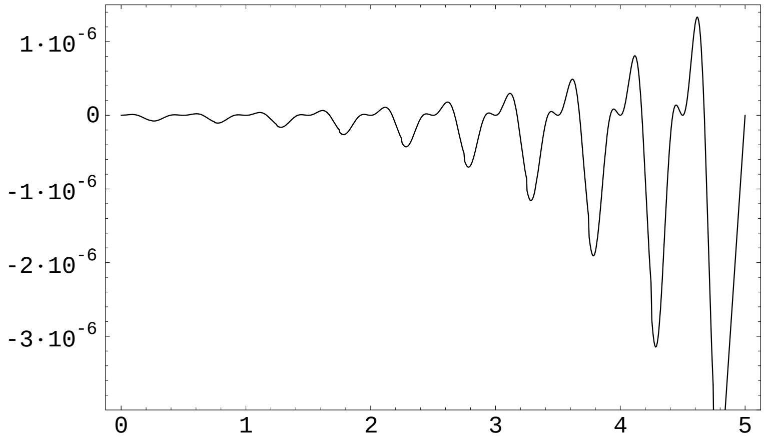

Figure 7: $\cosh(z) - \mathcal{G}_{\frac{1}{2},5}[\cosh](z)$

Example 4.4. Consider the function $f(z) = \dfrac{1}{z^2+2}$ which is analytic in the strip $\mathcal{S}_d$ with $d < \sqrt{2}$. Here we apply Theorem 3.1 and Theorem 3.2 with $d = 1$.

	$N=5$		$N=7$		$N=9$	
z	absolute error	bound	absolute error	bound	absolute error	bound
0.15	8.9350×10^{-9}	2.76×10^{-8}	2.9426×10^{-12}	2.06×10^{-12}	5.8065×10^{-14}	1.07×10^{-13}
1.15	9.3956×10^{-9}	2.76×10^{-8}	8.8979×10^{-13}	2.06×10^{-12}	9.3259×10^{-15}	1.07×10^{-13}
2.15	3.7196×10^{-9}	2.76×10^{-8}	4.3074×10^{-13}	2.06×10^{-12}	6.3838×10^{-16}	1.07×10^{-13}
3.15	1.7151×10^{-9}	2.76×10^{-8}	2.5113×10^{-13}	2.06×10^{-12}	1.4155×10^{-15}	1.07×10^{-13}
4.15	8.9562×10^{-10}	2.76×10^{-8}	1.6086×10^{-13}	2.06×10^{-12}	1.3600×10^{-15}	1.07×10^{-13}

Table 7: Approximation of $f(z)$ by $\mathcal{G}_{\frac{1}{N},N}$.

	$N=5$		$N=7$		$N=9$	
z	absolute error	bound	absolute error	bound	absolute error	bound
0.15	3.2904×10^{-5}	1.89×10^{-4}	3.0495×10^{-7}	1.72×10^{-6}	3.6828×10^{-9}	4.11×10^{-7}
1.15	3.1943×10^{-5}	1.89×10^{-4}	5.3292×10^{-8}	1.72×10^{-6}	7.1783×10^{-8}	4.11×10^{-7}
2.15	1.5727×10^{-5}	1.89×10^{-4}	3.7314×10^{-9}	1.72×10^{-6}	2.8828×10^{-8}	4.11×10^{-7}
3.15	7.8896×10^{-6}	1.89×10^{-4}	5.2174×10^{-9}	1.72×10^{-6}	1.5348×10^{-8}	4.11×10^{-7}
4.15	4.5321×10^{-6}	1.89×10^{-4}	4.8163×10^{-9}	1.72×10^{-6}	9.3114×10^{-9}	4.11×10^{-7}

Table 8: Approximation of $f(z)$ by $\mathcal{C}_{\frac{1}{N},N}$.

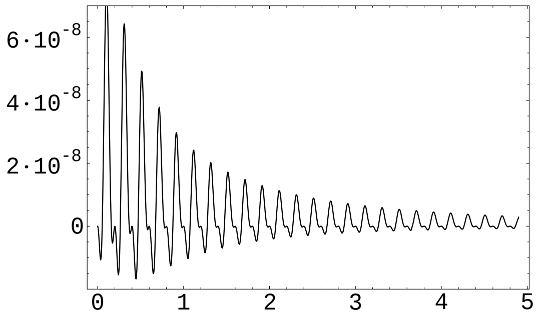

Figure 8: $f(z) - \mathcal{G}_{\frac{1}{5},5}[f](z)$.

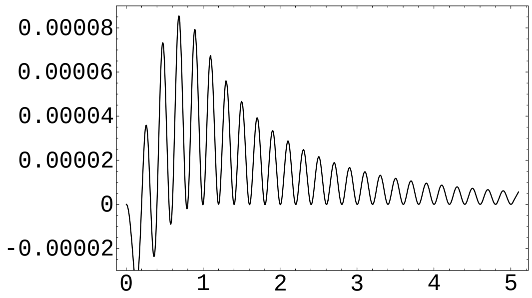

Figure 9: $f(z) - \mathcal{C}_{\frac{1}{5},5}[f](z)$.

Note. The convergence of Hermite-Gauss approximations $\mathcal{G}_{h,N}$ and $\mathcal{C}_{h,N}$ cannot be achieved when α and δ are negative numbers. In other words, the convergence of $\mathcal{G}_{h,N}$ fails for $h > 2\pi/\sigma$ and the convergence of $\mathcal{C}_{h,N}$ fails for $h > \pi/\sigma$. The following figures show this fact.

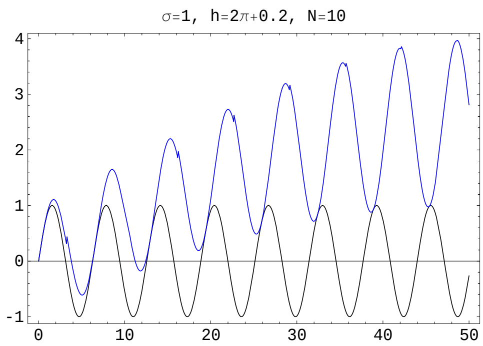

Figure 10: $\sin(z)$, $\mathcal{G}_{h,N}[\sin](z)$.

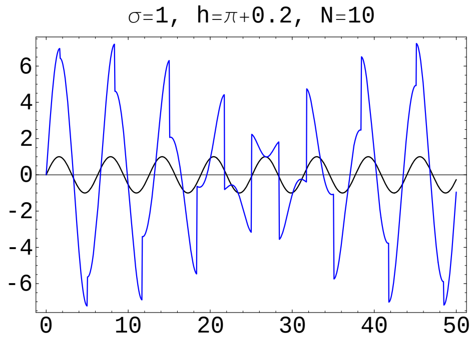

Figure 11: $\sin(z)$, $\mathcal{C}_{h,N}[\sin](z)$.

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