Pointwise Gopengauz Estimates for Interpolation

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Abstract

We derive some new pointwise estimates for the error in simultaneous approximation of a function $f \in C^q[-1,1]$ and its derivatives by a polynomial of interpolation and its respective derivatives. Our estimates incorporate the pointwise modulus of continuity and/or pointwise second modulus of continuity, an improvement in precision over estimates heretofore known. Our new methods of constructing error estimates are applicable to interpolation on an arbitrary set of nodes in $(-1,1)$, augmented by interpolation of some derivatives of the function at ± 1. Application in particular to similarly augmented interpolation on the zeroes of certain Jacobi polynomials yields error estimates which improve upon existing results of ourselves and of others.

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1. Introduction

It is known that to augment the interpolation of a function $f \in C^q[-1,1]$ on some nodes in $(-1,1)$ by the interpolation at ± 1 of some derivatives of f gives better rates of simultaneous convergence of the interpolating polynomial to the function f and of the respective derivatives of the interpolating polynomial to the respective derivatives of f. A tool which has been almost universally used in investigating such phenomena is a theorem of Gopengauz [3], which we will state in the next section. This theorem incorporates a pointwise modulus of continuity in its statement, but the pointwise modulus of continuity has heretofore been lost or sacrificed in the process of obtaining error estimates for simultaneous approximation by interpolation. Here, we obtain error estimates which retain the pointwise modulus of continuity. Interpolation on the zeroes of certain Jacobi polynomials permits a concrete realization of our new methods of estimating error, and we obtain estimates which in particular improve the results obtained in our own previous work [4] and in Theorem 3.3 of Mastroianni [6].

2. Preliminaries and notations

Let $f \in C^q[-1,1]$ be given, where $q \geq 0$. Then for a fixed r such that $\frac{q}{2} < r \leq q+1$ we define a polynomial $H_{n,r}f$ of degree at most $n+2r-1$ which interpolates f on nodes $x_1, \ldots, x_n$ such that $-1 < x_n < \ldots < x_1 < 1$ and interpolates $f^{(0)}, \ldots, f^{(r-1)}$ at ± 1. The polynomial $H_{n,r}f$ may be represented as

$$H_{n,r}f(x) = \sum_{j=1}^{n} f(x_j) \left(\frac{1-x^2}{1-x_j^2} \right)^r l_j(x) \; + \; \sum_{k=0}^{r-1} f^{(k)}(1) h_{1,k}(x) + f^{(k)}(-1) h_{2,k}(x),$$

where

$$l_j(x) = \prod_{\substack{s=1 \\ s \neq j}}^{n} \frac{x - x_s}{x_j - x_s}$$

and $h_{1,k}, h_{2,k}$ are certain polynomials of degree $n+2r-1$.

The approximation properties of $H_{n,r}f$ will be described in terms of weighted Lebesgue sums

$$L_{n,s}(x) = \sum_{j=1}^{n} \left(\frac{1-x^2}{1-x_j^2} \right)^{\frac{s}{2}} |l_j(x)| \, .$$

We remark that $L_{n,0}$ is the ordinary Lebesgue sum of the Lagrange interpolation on the nodes $x_1, \ldots, x_n$. If one wishes to approximate a function in $f \in C^q[-1,1]$ one has of course some flexibility about the choice of r. Clearly, one cannot choose r greater than $q+1$, because one can not interpolate more derivatives than those which exist. On the other hand, the most interesting choices of r for our purposes are also bounded below by $\frac{q}{2}$. For choices of r smaller than this, one has insufficient data to guarantee good simultaneous approximation of $f^{(0)}, \ldots, f^{(q)}$.

The most important tool for our investigations is the Theorem of Gopengauz [3]:

Theorem: *Let $f \in C^q[-1,1]$. Then there exists a polynomial p_n of degree at most n such that for $k = 0, \ldots, q$*

$$|f^{(k)}(x) - p_n^{(k)}(x)| \leq M_q \left(\frac{\sqrt{1-x^2}}{n} \right)^{q-k} \omega \left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} \right).$$

Here $\omega(f,h) = \sup_{|x-y|<h} |f(x) - f(y)|$.

Also, the symbol C whenever used without subscript will denote a “generic constant” whose value may not necessarily be the same each time it appears, even if there are two appearances in one line. The quantities on which C may depend or not depend will be listed explicitly if the context leaves the situation unclear.

The Theorem of Gopengauz also holds true if the modulus of continuity ω is replaced by the second modulus,

$\omega_2(f,h) = \sup_{\delta<h}|f(x+\delta) - 2f(x) + f(x-\delta)|$.

This result, which we will also use, is due to Yu Xiang-Ming [8], and Dahlhaus [2] has shown that no higher modulus of smoothness than the second can be used in the Theorem of Gopengauz.

3. Results

Theorem 1. *Let $f \in C^q[-1,1]$. Then for $\frac{q}{2} < r \leq q+1$*

$$|f(x) - H_{n,r}f(x)| \leq M_q \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \cdot$$

$$\cdot (1 + \max\{2L_{n,2r-q-1}(x), L_{n,2r-q}(x)\}).$$

A similar statement holds with ω replaced by ω_2. Specifically:

$$|f(x) - H_{n,r}f(x)| \leq C \left(\frac{\sqrt{1-x^2}}{n}\right)^q \omega_2\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n}\right) \cdot$$

$$\cdot (1 + \max\{4L_{n,2r-q-2}(x), L_{n,2r-q}(x)\}).$$

Furthermore for the derivatives of $H_{n,r}f$ we obtain

Theorem 2. *Let $f \in C^q[-1,1]$. Then for $\frac{q}{2} < r \leq q+1$ and for $k = 0, \ldots, q$ there is a constant C_q depending only upon q such that*

$$|f^{(k)}(x) - H_{n,r}^{(k)}f(x)| \leq C_q \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-k} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \cdot$$

$$\cdot \max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\}.$$

Furthermore, for $0 \leq k < r$ we have

$$|f^{(k)}(x) - H_{n,r}^{(k)}f(x)| \leq e\, C_q \left(\frac{\sqrt{1-x^2}}{n}\right)^{r-k} \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-r} \omega\left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \cdot$$

$$\cdot \max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\}.$$

In the special case $r = q+1$, there is a constant $K_q \leq \max\{4eC_q, 7C_q+7\}$ such that for $k = 0, \ldots, q$ we have

$$|f^{(k)}(x) - H_{n,r}^{(k)} f(x)| \leq K_q \left(\frac{\sqrt{1-x^2}}{n} \right)^{q-k} \omega \left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} \right) \cdot$$

$$\cdot \max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\}.$$

As a consequence of these theorems we can obtain pointwise estimates for the quality of approximation on Jacobi nodes with added interpolation at ± 1 which improve on what has been previously known by including the pointwise modulus of continuity or the pointwise modulus of smoothness:

Theorem 3. *Let $f \in C^q[-1,1]$ and r be given such that $\frac{q}{2} < r \leq q+1$. Then for $2r - q - \frac{5}{2} \leq \alpha, \beta \leq 2r - q - \frac{3}{2}$ we can choose the nodes x_j as the zeros of the ordinary Jacobi polynomial $P_n^{\alpha,\beta}$, and we obtain*

$$\max\{\|L_{n,2r-q-1}(x)\|, \|L_{n,2r-q}(x)\|\} \leq C \log n,$$

whence for these nodes

$$|f(x) - H_{n,r} f(x)| \leq C \left(\frac{\sqrt{1-x^2}}{n} \right)^{q} \omega \left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} \right) \log n,$$

in which the constant C depends upon q, α, β. Again using the nodes generated by $P_n^{(\alpha,\beta)}$ we obtain that

$$\max\{\|L_{n,2r-q-2}(x)\|, \|L_{n,2r-q}(x)\|\} \leq C \log n$$

if and only if $\alpha = \beta = 2r - q - \frac{5}{2}$ so that for the nodes thus determined we obtain

$$|f(x) - H_{n,r} f(x)| \leq C \left(\frac{\sqrt{1-x^2}}{n} \right)^{q} \omega_2 \left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} \right) \log n.$$

Theorem 4. *Let $f \in C^q[-1,1]$ and $r = q+1$. Then for $q - \frac{1}{2} \leq \alpha, \beta \leq q + \frac{1}{2}$ we can choose the nodes x_j as the zeros of the ordinary Jacobi polynomial $P_n^{\alpha,\beta}$, and we obtain for $k = 0, \ldots, q$*

$$|f^{(k)}(x) - H_{n,r}^{(k)} f(x)| \leq C \left(\frac{\sqrt{1-x^2}}{n} \right)^{q-k} \omega \left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} \right) \log n,$$

in which the constant C depends upon q, α, β. Also, using again nodes generated by Jacobi polynomials,the statement

$$|f(x) - H_{n,r}f(x)| \leq C \left(\frac{\sqrt{1-x^2}}{n} \right)^q \omega_2 \left(f^{(q)}, \frac{\sqrt{1-x^2}}{n} \right) \log n$$

holds for $\alpha = \beta = q - \frac{1}{2}$.

4. Proofs

Proof of Theorem 1: Using the theorem of Gopengauz and the projection properties of $H_{n,r}$ we may estimate

$$\begin{aligned} |f(x) - H_{n,r}f(x)| &= |f(x) - p_n(x) - H_{n,r}(f - p_n)(x)| \\ &\leq |f(x) - p_n(x)| + |H_{n,r}(f - p_n)(x)| \\ &\leq M_q \left(\frac{\sqrt{1-x^2}}{n} \right)^q \omega(f^{(q)}, \frac{\sqrt{1-x^2}}{n}) + |H_{n,r}(f - p_n)(x)|. \end{aligned}$$

Because of the fact that $f^{(k)}(\pm 1) - p_n^{(k)}(\pm 1) = 0$ for $k = 0, \ldots, q$ we observe that

$$H_{n,r}(f - p_n)(x) = \sum_{j=1}^{n} (f(x_j) - p_n(x_j)) \left(\frac{1-x^2}{1-x_j^2} \right)^r l_j(x).$$

The remaining terms in the expansion are all zero. Thus in turn we may estimate

$$\begin{aligned} |H_{n,r}(f - p_n)(x)| &\leq \sum_{j=1}^{n} |f(x_j) - p_n(x_j)| \left(\frac{1-x^2}{1-x_j^2} \right)^r |l_j(x)| \\ &\leq \sum_{j=1}^{n} M_q \left(\frac{\sqrt{1-x_j^2}}{n} \right)^q \omega \left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n} \right) \left(\frac{1-x^2}{1-x_j^2} \right)^r |l_j(x)| \\ &\leq M_q \left(\frac{\sqrt{1-x^2}}{n} \right)^q \sum_{j=1}^{n} \omega \left(f^{(q)}, \frac{\sqrt{1-x_j^2}}{n} \right) \left(\frac{1-x^2}{1-x_j^2} \right)^{r-\frac{q}{2}} |l_j(x)|. \end{aligned}$$

Now, given an $x \in [-1, 1]$, the set of indices $\{1, \ldots, n\}$ can be partitioned into two disjoint subsets

$$A(x) = \{j : 1 - x_j^2 < 1 - x^2\}$$

and

$$B(x) = \{j : 1 - x_j^2 \geq 1 - x^2\},$$

and the sum over each may be considered separately. For the set of indices $A(x)$ we obtain, using the monotonicity of $\omega(f^{(q)}, h)$,

$$M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q\sum_{j\in A}\omega\left(f^{(q)},\frac{\sqrt{1-x_j^2}}{n}\right)\left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}}|l_j(x)|$$

$$\le M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q\omega\left(f^{(q)},\frac{\sqrt{1-x^2}}{n}\right)\sum_{j\in A}\left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}}|l_j(x)|$$

$$\le M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q\omega\left(f^{(q)},\frac{\sqrt{1-x^2}}{n}\right)L_{n,2r-q}(x).$$

If on the other hand $j\in B(x)$ we obtain

$$\frac{n}{\sqrt{1-x_j^2}}\omega\left(f^{(q)},\frac{\sqrt{1-x_j^2}}{n}\right)\le\frac{2n}{\sqrt{1-x^2}}\omega\left(f^{(q)},\frac{\sqrt{1-x^2}}{n}\right)$$

using a standard concavity argument for $\omega(f^{(q)},h)$ (see e.g. Lorentz [5] p.44-45). Therefore

$$M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q\sum_{j\in B}\omega\left(f^{(q)},\frac{\sqrt{1-x_j^2}}{n}\right)\left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q}{2}}|l_j(x)|$$

$$\le 2M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q\omega\left(f^{(q)},\frac{\sqrt{1-x^2}}{n}\right)\sum_{j\in B}\left(\frac{1-x^2}{1-x_j^2}\right)^{r-\frac{q+1}{2}}|l_j(x)|$$

$$\le 2M_q\left(\frac{\sqrt{1-x^2}}{n}\right)^q\omega\left(f^{(q)},\frac{\sqrt{1-x^2}}{n}\right)L_{n,2r-q-1}(x).$$

If ω is replaced by ω_2, a similar argument suffices. We omit the details. The requisite concavity property of ω_2 may also be seen in Lorenz [5], p.47. Hence the proof is completed.

Proof of Theorem 2:

Our first conclusion follows immediately from Theorem 1 and the first theorem of Telyakovskii [7]. Turning to the second conclusion in Theorem 2, we note first that

$$(f^{(k)}-H_{n,r}^{(k)}f)(\pm 1)=0\ \text{ for }\ k=0,\dots,r-1.$$

The first conclusion having been demonstrated, the second conclusion and also the third, in which r takes on the particular value of $q+1$, follow now from the theorem of Balázs and Kilgore [1]. The estimates of the constants in the second and third conclusions here also follow from the proof in Balázs and Kilgore [1].

Proof of Theorem 3:

A lemma from Kilgore and Prestin [4], based upon well known properties of the Jacobi polynomials, states that

$$\|L_{n,s}\| \leq C \log n \qquad \text{if and only if} \qquad s - \frac{5}{2} \leq \alpha, \beta \leq s - \frac{1}{2},$$

when the nodes of interpolation are chosen at the zeroes of the Jacobi polynomial $P_n^{(\alpha,\beta)}$. If

$$2r - q - \frac{5}{2} \leq \alpha, \beta \leq 2r - q - \frac{3}{2}$$

then the lemma guarantees that $\|L_{n,s}\| \leq C \log n$ for $s = 2r - q$ and also for $s = 2r - q - 1$, and the first assertion of Theorem 3 follows from Theorem 1. The second assertion follows similarly from Theorem 1 and the Lemma. The Lemma must hold both for $s = 2r - q$ and for $s = 2r - q - 2$, implying that $\alpha = \beta = 2r - q - \frac{5}{2}$ is the unique choice of α and β which gives $\log n$ (instead of some positive power of n) in the second assertion of Theorem 4.

Proof of Theorem 4:

The first part of Theorem 4 follows from Theorems 1, 2, and 3 in a straightforward manner. We note that in particular if $r = q + 1$, then

$$2r - q - \frac{5}{2} = q - \frac{1}{2},$$

and

$$2r - q - \frac{3}{2} = q + \frac{1}{2}.$$

Also, the second part of the theorem follows immediately from Theorem 3. This completes the proof of Theorem 4.

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