On a build-up polynomial frame for the detection of singularities

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Abstract

Let $f : [-1, 1] \to I\!R$, $x_0 \in (-1, 1)$, $r \geq 0$ be an integer. The point x_0 is called a singularity of f of order r if the derivative $f^{(r)}$ has a jump discontinuity at x_0, but is continuous at every other point of some neighborhood of x_0. In this paper, we propose a sequence of polynomial operators $\{\tau_j\}$ with the following properties. Each τ_j is computed using the values $f(\cos(k\pi/2^j))$, $k = 1, \cdots, 2^j - 1$, and the quantity $\tau_j(f, x)$ is "large" near a singularity, and "small" away from it. Precise quantitative estimates are given.

1 Introduction

Let I be an open real interval, $f : I \to I\!R$, and $r \geq 0$ be an integer. In this paper, a point $x_0 \in I$ will be called a singularity of f of order r if the $(r-1)$-st derivative, $f^{(r-1)}$, is absolutely continuous in a neighborhood of x_0, and $f^{(r)}$ is continuous in this neighborhood, except for a jump discontinuity at x_0. In many applications, for example, image and data compression, prediction of time series, and antenna technology, one needs to determine the location of the singularities of a function of various orders.

The theory of wavelets provides many popular tools for the solution of this problem, provided that the wavelet coefficients of f can be computed. In practical applications,

*This research was supported, in part, by grant no. F-49620-97-1-0211 from the U. S. Air Force Office of Scientific Research.

one computes these coefficients based on a sufficiently large set of values of f, using numerical integration. However, in some applications, the information available about the function consists of the coefficients of the function in some orthogonal polynomial expansion (see, for example, [3, 4, 6, 13]). Since the function is not smooth, it is not economical to compute a good approximation to it using this data, which can then be used to compute the wavelet coefficients. Also, the popular, compactly supported wavelets have the inherent limitation that they cannot simultaneously detect singularities of all orders. For this reason, a study of trigonometric polynomials was initiated in [2], and developed further, both in the context of trigonometric polynomials and orthogonal polynomials, by several authors (e.g., [5, 7, 8, 9, 12, 14, 15, 16, 17]). In [10], we gave a general construction of orthogonal polynomial frames. In the context of Jacobi polynomials, our frames are capable of simultaneously detecting singularities of any orders of the function.

During the conference in Dubna, Spiridonov asked if a similar construction can be given for polynomials orthogonal with respect to a measure supported on finitely many points. He pointed out several applications in quantum physics leading to this problem [18, 19, 20]. Any such construction must necessarily depend upon the values of the function at the points in the support of the measure. A closely related question is to construct such frames which are "built-up" using an increasing number of samples, rather than starting out with the function values at all the points of the support.

In [11], we gave such "build-up" constructions for the detection of singularities in periodic functions. In particular, we constructed a class of trigonometric polynomial frames. A central role in this theory was played by the quadrature formulas, and the aliasing formulas. The current paper is a small step towards answering the question of Spiridonov, in the special case when the Chebyshev polynomials $\{U_j\}_{j=0}^{n-1}$ of second kind are considered as the orthogonal polynomials on the zeros of U_n with respect to the Cotes' numbers as weights. Since the zeros of U_n are contained in those of U_{2n+1}, our constructions are also "build-up" operators.

2 Polynomial frames

For $x \ge 0$, let Π_x denote the class of all polynomials of degree at most x. It is convenient to define $\Pi_{-1} := \{0\}$. The formula

$$U_n(\cos\theta) = \frac{\sin((n+1)\theta)}{\sin\theta}, \qquad n = 0, 1, 2, \cdots \tag{2.1}$$

defines a system of polynomials orthonormalized with respect to the weight function $w(t) := (2/\pi)(1-t^2)^{1/2}$, $t \in [-1, 1]$. Let

$$\theta_{j,m} := \frac{j\pi}{m}, \quad x_{j,m} = \cos(\theta_{j,m}), \quad \lambda_{j,m} := \frac{2\sin^2\theta_{j,m}}{m}, \quad j \in \mathbb{Z},\ m = 2, 3, \cdots, \tag{2.2}$$

and ν_m denote the measure that associates the mass $\lambda_{j,m}$ with $x_{j,m}$, $1 \le j \le m-1$. It is then easy to verify that $\{U_k\}_{k=0}^{m-1}$ is a system of polynomials orthonormalized with respect to ν_m. More generally, we have the quadrature formula

$$\int P(t)d\nu_m(t) = \int P(t)w(t)dt, \qquad P \in \Pi_{2m-3}. \tag{2.3}$$

If $f : [-1,1] \to I\!R$, and fw is integrable, then we may define

$$b_k(f) := \int f(t)U_k(t)w(t)dt, \qquad k = 0, 1, \cdots, \tag{2.4}$$

and its discrete analogue

$$b_{k,m}(f) := \int f(t)U_k(t)d\nu_m(t), \qquad k = 0, 1, \cdots,\ m = 1, 2, \cdots. \tag{2.5}$$

We observe that $s_n(f) := \sum_{k=0}^{n-1} b_k(f)U_k$ is the partial sum of the expansion of f in the series of Chebyshev polynomials of second kind, while $L_n(f) := \sum_{k=0}^{n-2} b_{k,n}(f)U_k$ is the polynomial of Lagrange interpolation of f at the data $(x_{j,n}, f(x_{j,n}))$, $j = 1, \cdots, n-1$. In this paper, we are interested in studying the behavior of the operators of the form

$$\tau_{n,m}(G; f, x) := \sum_{k=0}^{n-2} g_{k,m} b_{k,m}(f)U_k(x), \qquad m \geq n,\ n = 3, 4, \cdots, \tag{2.6}$$

as $n \to \infty$, where $G := (g_{k,m})$ is a real (infinite) matrix. (In the notation $\tau_{n,m}$, $n-2$ is the degree of the polynomial, while m is the number of evaluations of f. It is convenient for our later calculations to let m be a multiple of n, but define the factors $g_{k,m}$ in terms of m rather than n.) It will turn out that for a suitable choice of the factors $g_{k,m}$, the numbers $\tau_{n,m}(G; f, x_{j,m})$ will enable us to detect the singularities of f of order at least equal to 2. We observe that the operators $\tau_{n,m}$ are *built-up* from an increasing set of values of f at the discrete points $x_{j,m}$. These operators also help us to define a frame as follows.

We write for $n = 0, 1, 2, \cdots$,

$$V_n := \Pi_{2^n-2}, \qquad W_n := \text{span}\{U_k\ :\ k = 2^n - 1, \cdots, 2^{n+1} - 2\}. \tag{2.7}$$

Let L^2 denote the Hilbert space of all measurable functions $f : [-1,1] \to I\!R$ such that

$$\|f\|_2^2 := \int |f(t)|^2 w(t)dt < \infty, \tag{2.8}$$

with the convention that two functions are identified if they are equal almost everywhere. Then $L^2 = V_1 \oplus \oplus_{n=1}^{\infty} W_n$ and $V_{n+1} = V_n \oplus W_n$, where $\oplus$ denotes the orthogonal direct sum with respect to the inner product of L^2.

Theorem 2.1 *Let $n \geq 1$, $m \geq 2^{n+1}$ be integers, $g_{k,m} \neq 0$ for $2^n - 1 \leq k \leq 2^{n+1} - 2$, and equal to 0 for other values of k. Let*

$$\Psi_{n,m,j}(G; x) := \lambda_{j,m}^{1/2} \sum_{k=2^n-1}^{2^{n+1}-2} g_{k,m}^{-1} U_k(x_{j,m})U_k(x), \qquad j = 1, \cdots, m-1.$$

For any $P \in W_n$, we have

$$P(x) := \sum_{j=1}^{m-1} \lambda_{j,m}^{1/2} \tau_{2^{n+1},m}(G; P, x_{j,m}) \Psi_{n,m,j}(G; x). \tag{2.9}$$

Further, we have the following frame bounds for $P \in W_n$:

$$\begin{aligned} \left\{ \min_{2^n-1 \le k \le 2^{n+1}-2} |g_{k,m}|^2 \right\} \|P\|_2^2 &\le \sum_{j=1}^{m-1} \lambda_{j,m} |\tau_{2^{n+1},m}(G; P, x_{j,m})|^2 \\ &\le \left\{ \max_{2^n-1 \le k \le 2^{n+1}-2} |g_{k,m}|^2 \right\} \|P\|_2^2. \end{aligned} \tag{2.10}$$

Proof. Let $P \in W_n$. In view of (2.3), we see that $b_{k,m}(P) = b_k(P)$ for $0 \le k \le 2^{n+1} - 2$, and hence,

$$\tau_{n,m}(G; P, x) = \sum_{k=2^n-1}^{2^{n+1}-2} g_{k,m} b_k(P) U_k(x). \tag{2.11}$$

The formula (2.9) can now be deduced easily using the quadrature formula again (cf. [10]). Using the quadrature formula, (2.11), and the Parseval identity, we obtain that

$$\sum_{j=1}^{m-1} \lambda_{j,m} |\tau_{2^{n+1},m}(G; P, x_{j,m})|^2 = \sum_{k=2^n-1}^{2^{n+1}-2} g_{k,m}^2 |b_k(P)|^2.$$

The estimates (2.10) are now clear. □

3 Detection of singularities

For integer $r \ge 0$, we write

$$x_+^r := \begin{cases} x^r, & \text{if } x \ge 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.1}$$

If $f : [-1, 1] \to I\!R$, $r \ge 0$ is an integer, and $y \in (-1, 1)$ is a singularity of order r of f, then the function $f - d(\cdot - y)_+^r$ is r times continuously differentiable in a neighborhood of y for a suitable constant d. Hence, if $R \ge 0$ is an integer, and a function f has only finitely many singularities of order at most R, all in $(-1, 1)$, then it can be expressed in the form

$$f(x) = \sum_{k=0}^{R} \sum_{j=1}^{n_k} d_{k,j} (x - y_{k,j})_+^k + F(x), \tag{3.2}$$

where F is R times continuously differentiable on $[-1, 1]$. Hence, the ability of an operator $\tau_{n,m}$ of the form (2.6) can be analyzed in three steps:

1. We need to show that $\tau_{n,m}(G; (\cdot - y)_+^r, x)$ is "large" for x "near" y,
2. If F is R times continuously differentiable on $[-1, 1]$, the values $\tau_{n,m}(G; F, x)$ should be uniformly smaller than the "large" values, and

3. The values $\tau_{n,m}(G;(\cdot-y)_+^r,x)$ should be sufficiently "small" for x "away" from y; smaller than the bounds in the first two steps.

In Theorem 3.1 below, we attempt to address these requirements for a special choice of the factors $g_{k,m}$. In the sequel, we adopt the following conventions on the constants. The symbols $c, c_1, \cdots$ will denote positive constants depending only on the fixed parameters in question, but their value may be different at different occurences, even within the same formula. For a bounded function $f:[-1,1]\to I\!R$, and $x\ge 0$, we write

$$\|f\| := \sup_{t\in[-1,1]} |f(t)|, \quad E_x(f) := \inf_{P\in\Pi_x} \|f-P\|.$$

Theorem 3.1 *Let $q,r\ge 2$ be integers, $\alpha\ge 1$, $g: I\!R\to[0,\infty)$ be a 2π-periodic, even function, such that $g(t)=0$ if $t\in[0,\pi/(2\alpha)]\cup[\pi/\alpha,\pi]$. We assume that the q-th derivative $g^{(q)}$ has bounded variation on $[-\pi,\pi]$, and that $g^{(j)}(\pi)=0$, $j=0,1$. Let*

$$g_{k,m} := g\left(\frac{\pi(k+1)}{m}\right). \tag{3.3}$$

Let $\delta>0$, $y=\cos\phi$, $\delta<\phi<\pi-\delta$. All the constants below will depend upon r, g, and δ, and m_n will denote the smallest integer not less than αn.
(a) *There exists an interval $I=I(g,r)$ such that for $j=0,\cdots,m_n-1$,*

$$n^r|\tau_{n,m_n}(G;(\cdot-y)_+^r,x_{j,m_n})|\ge c, \qquad \text{if } n|\theta_{j,m_n}-\phi|\in I. \tag{3.4}$$

(b) *If $f:[-1,1]\to I\!R$ is a continuous function then*

$$\|\tau_{n,m_n}(G;f)\|\le cE_{n/2}(f). \tag{3.5}$$

In particular, if $R\ge 0$ is an integer, and $F:[-1,1]\to I\!R$ is R times continuously differentiable on $[-1,1]$ then

$$\|\tau_{n,m_n}(G;F)\|\le \epsilon_n n^{-R}, \tag{3.6}$$

where $\epsilon_n\to 0$ as $n\to\infty$.
(c) *For $\theta_{j,m_n}\ne\phi$, we have*

$$n^r(1-x_{j,m_n}^2)^{1/2}|\tau_{n,m_n}(G;(\cdot-y)_+^r,x_{j,m_n})|\le c(n|\theta_{j,m}-\phi|)^{-q-1}. \tag{3.7}$$

We observe that if f is a continuously differentiable function on $[-1,1]$, and has finitely many singularities of different orders up to R, then the representation (3.2) implies that

$$\tau_{n,m_n}(G;f,x)=\sum_{k=2}^{R}\sum_{j=1}^{n_k} d_{k,j}\tau_{n,m_n}(G;(\cdot-y_{k,j})_+^k,x)+\tau_{n,m_n}(G;F,x). \tag{3.8}$$

In view of the part (b), the term $\tau_{n,m_n}(G;F,x)$ is $o(n^{-R})$ for all $x\in[-1,1]$. Let g be chosen as in the above theorem with $q\ge R+1$, and x be one of the points $\{x_{\ell,m_n}\}$. If x is near some $y_{k,j}$ as indicated by the part (a), then $|\tau_{n,m_n}(G;(\cdot-y_{k,j})_+^k,x)|\ge cn^{-k}$, and the part (c) shows that every other term in the summation above is $o(n^{-R})$. Thus, the operators τ_{n,m_n} are capable of detecting singularities of arbitrarily high order. A few related numerical experiments are described in [10]. We remark that the bounds in Theorem 3.1 are valid only for the points $\{x_{j,m_n}\}$, while the corresponding bounds in [10] are valid for all $x = \cos\theta$, $\theta \in [\delta, \pi - \delta]$. On the other hand, our estimates are valid uniformly for all x_{j,m_n}'s. Also, the bound (3.5) is better than the corresponding estimate in [10].

We will prove the part (b) of the above theorem first. The proof relies upon the boundedness of the $(C, 2)$-means of the Chebyshev series. Let

$$K_\ell(x,t) := \sum_{j=0}^{\ell} \frac{(\ell-j+1)(\ell-j+2)}{(\ell+1)(\ell+2)} U_j(x)U_j(t) \tag{3.9}$$

denote the kernel of these means. For $f : [-1, 1] \to I\!R$ for which fw is integrable, we write

$$\sigma_\ell^{[2]}(f,x) := \int f(t)K_\ell(x,t)w(t)dt, \qquad \sigma_{\ell,m}^{[2]}(f,x) := \int f(t)K_\ell(x,t)d\nu_m(t). \tag{3.10}$$

Let $C[-1, 1]$ denote the class of all continuous functions from $[-1, 1]$ to $I\!R$. It is well known that the operators $\sigma_\ell^{[2]}$ are uniformly bounded as operators on $C[-1, 1]$ (cf. [21], Chapter 9, Section 9.7(1)). In order to translate this result into the boundedness of the operators $\sigma_{\ell,m}^{[2]}$, we prove the following lemma.

Lemma 3.1 *Let $n \ge 0$, $m \ge n + 1$ be integers. For any $P \in \Pi_n$, we have*

$$\int |P(x)|d\nu_m(x) = \sum_{j=1}^{m-1} \lambda_{j,m}|P(x_{j,m})| \le c \int |P(x)|w(x)dx. \tag{3.11}$$

Proof. We may assume that $n \ge 1$. Let $T(t) := \sin^2 tP(\cos t)$. Then T is an even, trigonometric polynomial of degree at most $n + 2$. It is easy to verify that

$$\sum_{j=1}^{m-1} \lambda_{j,m}|P(x_{j,m})| = \frac{1}{m}\sum_{j=1}^{2m} |T(\theta_{j,m})|.$$

Since $2m \ge (1 + 1/4)(n + 2)$, the Marcinkiewicz-Zygmund inequality ([22], Theorem X(7.5), and the remark thereafter), implies that

$$\sum_{j=1}^{m} \lambda_{j,m}|P(x_{j,m})| \le c\int_{-\pi}^{\pi} |T(t)|dt = c\int |P(x)|w(x)dx.$$

$\square$

Proof of Theorem 3.1(b). It is well known (cf. [21], Chapter 9, Section 9.7(1)) that $\|\sigma_\ell^{[2]}(f)\| \le c\|f\|$ for all $f \in C[-1, 1]$, or equivalently,

$$\sup_{x\in[-1,1]} \int |K_\ell(x,t)|w(t)dt \le c.$$

Since $K_\ell(x,\cdot) \in \Pi_\ell$, Lemma 3.1 shows that for all $m \ge \ell+1$,

$$\sup_{x\in[-1,1]} \int |K_\ell(x,t)| d\nu_m(t) \le c.$$

In turn, this implies that

$$\|\sigma^{[2]}_{\ell,m}(f)\| \le c\|f\|, \qquad f \in C[-1,1],\ m \ge \ell+1,\ \ell = 0,1,\cdots. \tag{3.12}$$

Now, we let $s^{[2]}_{\ell,m}(f) = (1/2)(\ell+1)(\ell+2)\sigma^{[2]}_{\ell,m}(f)$. A simple calculation shows that

$$b_{\ell,m}(f)U_\ell(x) = s^{[2]}_{\ell,m}(f,x) - 3s^{[2]}_{\ell-1,m}(f,x) + 3s^{[2]}_{\ell-2,m}(f,x) - s^{[2]}_{\ell-3,m}(f,x), \quad \ell \ge 3,\ m \ge \ell+1.$$

Hence, if $n \ge c$ is sufficiently large so that $g_{j,m_n} = 0$ for $j = 0,1,2$, and $m := m_n$, we have

$$\begin{aligned} \tau_{n,m}(G;f,x) &= \sum_{\ell=0}^{n-2} g_{\ell,m} b_{\ell,m}(f) U_\ell(x) \\ &= \sum_{\ell=0}^{n-2} g_{\ell,m} s^{[2]}_{\ell,m}(f,x) - 3\sum_{\ell=0}^{n-3} g_{\ell+1,m} s^{[2]}_{\ell,m}(f,x) + 3\sum_{\ell=0}^{n-4} g_{\ell+2,m} s^{[2]}_{\ell,m}(f,x) \\ &\qquad - \sum_{\ell=0}^{n-5} g_{\ell+3,m} s^{[2]}_{\ell,m}(f,x). \end{aligned} \tag{3.13}$$

Since $g(\pi) = g'(\pi) = 0$ and g'' is of bounded variation, we conclude that $|g(\pi(n-j)/m)| \le cm^{-2}$ for $j = 2,3,4$, and

$$\sum_{\ell=0}^{n-2} (\ell+1)(\ell+2) \left| g\left(\frac{\pi\ell}{m}\right) - 3g\left(\frac{\pi(\ell+1)}{m}\right) + 3g\left(\frac{\pi(\ell+2)}{m}\right) - g\left(\frac{\pi(\ell+3)}{m}\right) \right| \le c.$$

Along with (3.12) and (3.13), this shows that $\|\tau_{n,m}(G;f)\| \le c\|f\|$ for all $f \in C[-1,1]$. Since $\tau_{n,m}(G;P) = 0$ if $P \in \Pi_{m/(2\alpha)} \supseteq \Pi_{n/2}$, this implies (3.5). The estimate (3.6) follows from the Jackson theorem in polynomial approximation theory. $\square$

In order to prove the remaining parts of Theorem 3.1, we will obtain an asymptotic formula for $\tau_{n,m_n}(G; (\cdot - y)^r_+, x_{j,m_n})$. A main step towards this goal is the following lemma.

Lemma 3.2 *Let $f : [-1,1] \to I\!R$ be continuous, and $\sum |b_j(f)| < \infty$. Then, with the notations as in Theorem 3.1,*

$$\tau_{n,m_n}(G; f, x_{j,m_n}) = \sum_{k=0}^{\infty} g_{k,m_n} b_k(f) U_k(x_{j,m_n}). \tag{3.14}$$

Proof. In this proof, we will rearrange different infinite series several times. These rearrangements and interchanges in the order of summation are all justified in view of our assumption that $\sum |b_j(f)| < \infty$. This assumption also ensures that $\sum b_j(f)U_j$ converges uniformly and absolutely to f on compact intervals of $(-1,1)$; in particular, at each of the points $\{x_{j,m_n}\}$. We note that both n and m_n are fixed in this proof, and write m instead of m_n. For $k = 0, \cdots, m-2$, we have

$$b_{k,m}(f) = \sum_{j=1}^{m-1} \lambda_{j,m} f(x_{j,m}) U_k(x_{j,m}) = \sum_{\ell=0}^{\infty} b_\ell(f) \sum_{j=1}^{m-1} \lambda_{j,m} U_\ell(x_{j,m}) U_k(x_{j,m}). \tag{3.15}$$

Using the fact that $\nu\theta_{j,m} = j\theta_{\nu,m}$, we calculate that

$$\begin{aligned}
\sum_{j=1}^{m-1} \lambda_{j,m} U_\ell(x_{j,m}) U_k(x_{j,m}) &= \frac{2}{m} \sum_{j=1}^{m-1} \sin((\ell+1)\theta_{j,m}) \sin((k+1)\theta_{j,m}) \\
&= \frac{1}{m} \sum_{j=1}^{m-1} (\cos((\ell-k)\theta_{j,m}) - \cos((\ell+k+2)\theta_{j,m})) \\
&= \begin{cases} 1, & \text{if } \ell - k = 2m\nu,\ \nu \in \mathbb{Z}, \\ -1, & \text{if } \ell + k + 2 = 2m\nu,\ \nu \in \mathbb{Z}, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned} \tag{3.16}$$

(We observe that for $0 \le k \le m-2$, only one of the numbers $\ell - k$ and $\ell + k + 2$ is divisible by $2m$.) Therefore, (3.15) leads to the aliasing formula:

$$b_{k,m}(f) = \sum_{\nu=0}^{\infty} (b_{2m\nu+k}(f) - b_{2m\nu+2m-k-2}(f)). \tag{3.17}$$

Now, since $g(t) = 0$ if $t \in [\pi/a, \pi]$, $g_{k,m} = 0$ for $n-1 \le k \le m-1$, and we have

$$\tau_{n,m}(G; f, x) = \sum_{k=0}^{m-1} g_{k,m} b_{k,m}(f) U_k(x) = \sum_{\nu=0}^{\infty} \sum_{k=0}^{m-1} g_{k,m} U_k(x) (b_{2m\nu+k}(f) - b_{2m\nu+2m-k-2}(f)).$$

Since g is even and 2π-periodic, $g_{k,m} = g_{2m\nu+k} = g_{2m\nu+2m-k-2}$. It is easy to verify that $U_k(x_{j,m}) = U_{2m\nu+k}(x_{j,m}) = -U_{2m\nu+2m-k-2}(x_{j,m})$, and that $U_{2m\nu+m-1}(x_{j,m})$ $= U_{2m\nu+2m-1}(x_{j,m}) = 0$. Therefore,

$$\begin{aligned}
\tau_{n,m}(G; f, x_{j,m}) &= \sum_{\nu=0}^{\infty} \sum_{k=0}^{m-1} \Big(g_{2m\nu+k} b_{2m\nu+k}(f) U_{2m\nu+k}(x_{j,m}) \\
&\qquad + g_{2m\nu+2m-k-2} b_{2m\nu+2m-k-2} U_{2m\nu+2m-k-2}(x_{j,m}) \Big) \\
&= \sum_{\nu=0}^{\infty} \sum_{k=0}^{2m-1} g_{2m\nu+k} b_{2m\nu+k}(f) U_{2m\nu+k}(x_{j,m}) = \sum_{k=0}^{\infty} g_{k,m} b_k(f) U_k(x_{j,m}).
\end{aligned}$$

This completes the proof.

□

Our next lemma is essentially in [11].

Lemma 3.3 *Let g be as in Theorem 3.1, and $\delta > 0$. Then for integer $\ell \ge 3$, and $\psi \in [0, 2\pi(1-\delta)]$, we have*

$$\sum_{k=0}^{\infty} \frac{g_{k,m}}{(k+1)^\ell} e^{i(k+1)\psi} = \left(\frac{\pi}{m}\right)^{\ell-1} \left\{ \int_{\pi/(2\alpha)}^{\infty} \frac{g(t)}{t^\ell} \exp\left(\frac{im\psi t}{\pi}\right) dt + \mathcal{O}(m^{-q-1}) \right\}, \tag{3.18}$$

where the constant involved in the $\mathcal{O}$–term depends upon g, q, ℓ, and δ.

Proof. Let $h(t) := g(t)t^{-\ell}$ if $t \geq 0$ and $h(t) := 0$ if $t < 0$. Then h and $th(t)$ are both integrable and have a bounded variation on $I\!\!R$. Hence, the Fourier transform of h is also integrable, and of bounded variation on $I\!\!R$. Consequently, (cf. [1], Proposition 5.1.29) we obtain that

$$\sum_{k\in\mathbb{Z}} h\left(\frac{k\pi}{m}\right) e^{ik\psi} = \frac{m}{\pi}\sum_{k\in\mathbb{Z}}\int_{I\!\!R} h(t)\exp\left(\frac{im}{\pi}(\psi+2k\pi)t\right) dt;$$

i.e.,

$$\sum_{k=0}^{\infty}\frac{g_{k,m}}{(k+1)^{\ell}}e^{i(k+1)\psi} = \left(\frac{\pi}{m}\right)^{\ell-1}\sum_{k\in\mathbb{Z}}\int_{I\!\!R} h(t)\exp\left(\frac{im}{\pi}(\psi+2k\pi)t\right) dt. \tag{3.19}$$

Since $h^{(q)}$ has bounded variation on $I\!\!R$, an integration by parts shows that for $k \in \mathbb{Z}$, $k \neq 0$,

$$\int_{I\!\!R} h(t)\exp\left(\frac{im}{\pi}(\psi+2k\pi)t\right) dt = \mathcal{O}\Big((m|2k\pi+\psi|)^{-q-1}\Big) = \mathcal{O}\Big((m(|k|-1+\delta))^{-q-1}\Big).$$

The equation (3.18) now follows easily from (3.19). □

Finally, we obtain the asymptotics for $\tau_{n,m_n}(G;(\cdot-y)_+^r, x_{j,m_n})$, where we use the following notations:

$$\begin{aligned}
\frac{(k+1)(k-\nu)!}{(k+r+2)!} &=: \frac{1}{(k+1)^{r+\nu+1}}\sum_{\ell=0}^{\infty} d_{\ell,\nu}(k+1)^{-\ell}, \quad d_{0,\nu} := 1,\\
A_\nu(\phi) &:= \binom{\nu+r+1}{\nu}\frac{(-1)^\nu(r+1)\cdots(r+2-\nu)}{(2\sin\phi)^\nu},\\
B_\nu(\phi) &:= \nu\phi + \frac{\nu+r+2}{2}\pi.
\end{aligned} \tag{3.20}$$

(It is understood that the empty product in $A_0(\phi)$ is 1, so that $A_0(\phi) := 1$. Similarly, an empty sum is defined to be 0.)

Lemma 3.4 *With the assumptions as in Theorem 3.1, and $m = m_n$,*

$$\begin{aligned}
\tau_{n,m}(G;(\cdot-y)_+^r, x_{j,m}) &= \frac{r!\sin^{r+1}\phi}{\pi\sin\theta_{j,m}}\left(\frac{\pi}{m}\right)^r\sum_{\nu=0}^{r+1}\sum_{\ell=0}^{q-\nu}\left(\frac{\pi}{m}\right)^{\nu+\ell} d_{\ell,\nu}A_\nu(\phi)\\
&\quad\times\int_{\pi/(2\alpha)}^{\infty}\frac{g(t)}{t^{r+1+\nu+\ell}}\sin\left(\frac{m(\theta_{j,m}-\phi)t}{\pi}+B_\nu(\phi)\right) dt + \mathcal{O}(m^{-r-q-1})\\
&= \frac{r!\sin^{r+1}\phi}{\pi\sin\theta_{j,m}}\left(\frac{\pi}{m}\right)^r\int_{\pi/(2\alpha)}^{\infty}\frac{g(t)}{t^{r+1}}\sin\left(\frac{m(\theta_{j,m}-\phi)t}{\pi}+\frac{r+2}{2}\pi\right) dt + \mathcal{O}(m^{-r-1}).
\end{aligned} \tag{3.21}$$

Proof. We recall ([21], §4.7) that the ultraspherical polynomial $P_k^{(\lambda)}$ is defined by

$$(1-x^2)^{\lambda-1/2}P_k^{(\lambda)}(x) = \frac{(-2)^k}{k!}\frac{\Gamma(k+\lambda)\Gamma(k+2\lambda)}{\Gamma(\lambda)\Gamma(2k+2\lambda)}\left(\frac{d}{dx}\right)^k(1-x^2)^{k+\lambda-1/2}, \tag{3.22}$$

and $U_k(x) = P_k^{(1)}(x)$. From (3.22), and [21, (8.4.3)], we obtain after some computation that

$$\begin{aligned}
\beta_{k,r}(y) &:= b_k((\cdot - y)_+^r) = \frac{2^{r+2} r!(r+1)!}{\pi} \frac{(k-r-1)!(k+1)}{(k+r+2)!} \sin^{2r+3}\phi P_{k-r-1}^{(r+2)}(\cos\phi) \\
&= \frac{2^{r+2} r!(r+1)!}{\pi} \frac{(k-r-1)!(k+1)}{(k+r+2)!} \sin^{2r+3}\phi \\
&\quad \times 2 \binom{k}{k-r-1} \sum_{\nu=0}^{r+1} \binom{\nu+r+1}{\nu} \frac{(-1)^\nu (r+1)\cdots(r+2-\nu)}{(2\sin\phi)^{\nu+r+2}} \\
&\quad \times \frac{\cos\{(k+1-\nu)\phi - (\nu+r+2)\pi/2\}}{k\cdots(k-\nu+1)} \\
&= \frac{2r!\sin^{r+1}\phi}{\pi} \sum_{\nu=0}^{r+1} A_\nu(\phi) \frac{(k-\nu)!(k+1)}{(k+r+2)!} \cos((k+1)\phi - B_\nu(\phi)) \\
&= \frac{2r!\sin^{r+1}\phi}{\pi} \sum_{\nu=0}^{r+1} \sum_{\ell=0}^{q-\nu} d_{\ell,\nu} A_\nu(\phi) \frac{\cos((k+1)\phi - B_\nu(\phi))}{(k+1)^{r+1+\nu+\ell}} + \mathcal{O}\left(\frac{1}{(k+1)^{r+q+2}}\right).
\end{aligned} \tag{3.23}$$

Consequently, using Lemma 3.2 and Lemma 3.3,

$$\begin{aligned}
&\tau_{n,m}(G; (\cdot - y)_+^r, x_{j,m}) = \sum_{k=0}^{\infty} g_{k,m} \beta_{k,r}(y) U_k(x_{j,m}) \\
&= \frac{r!\sin^{r+1}\phi}{\pi \sin\theta_{j,m}} \sum_{\nu=0}^{r+1} \sum_{\ell=0}^{q-\nu} d_{\ell,\nu} A_\nu(\phi) \sum_{k=0}^{\infty} \frac{g_{k,m}}{(k+1)^{r+1+\nu+\ell}} \\
&\quad \times \{\sin((k+1)(\theta_{j,m} - \phi) + B_\nu(\phi)) + \sin((k+1)(\theta_{j,m} + \phi) - B_\nu(\phi))\} \\
&\qquad + \mathcal{O}\left(\sum_{k=0}^{\infty} \frac{g_{k,m}}{(k+1)^{r+q+2}}\right) \\
&= \frac{r!\sin^{r+1}\phi}{\pi \sin\theta_{j,m}} \sum_{\nu=0}^{r+1} \sum_{\ell=0}^{q-\nu} d_{\ell,\nu} A_\nu(\phi) \sum_{k=0}^{\infty} \frac{g_{k,m}}{(k+1)^{r+1+\nu+\ell}} \\
&\quad \times \{\sin((k+1)(\theta_{j,m} - \phi) + B_\nu(\phi)) + \sin((k+1)(\theta_{j,m} + \phi) - B_\nu(\phi))\} \\
&\qquad + \mathcal{O}\left(\frac{1}{m^{r+q+1}}\right).
\end{aligned} \tag{3.24}$$

Since $r + 1 \geq 3$, we may use Lemma 3.3 to obtain

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{g_{k,m}}{(k+1)^{r+1+\nu+\ell}} \sin((k+1)(\theta_{j,m} - \phi) + B_\nu(\phi)) \\
&= \left(\frac{\pi}{m}\right)^{r+\nu+\ell} \left\{ \int_{\pi/(2\alpha)}^{\infty} \frac{g(t)}{t^{\ell+r+\nu+1}} \sin\left(\frac{m(\theta_{j,m} - \phi)t}{\pi} + B_\nu(\phi)\right) dt + \mathcal{O}(m^{-q-1}) \right\},
\end{aligned} \tag{3.25}$$

and similarly,

$$\sum_{k=0}^{\infty} \frac{g_{k,m}}{(k+1)^{r+1+\nu+\ell}} \sin((k+1)(\theta_{j,m} + \phi) - B_\nu(\phi))$$

$$= \left(\frac{\pi}{m}\right)^{r+\nu+\ell} \left\{ \int_{\pi/(2\alpha)}^{\infty} \frac{g(t)}{t^{\ell+r+\nu+1}} \sin\left(\frac{m(\theta_{j,m}+\phi)t}{\pi} - B_\nu(\phi) \right) dt + \mathcal{O}(m^{-q-1}) \right\}.$$

Using integration by parts we get

$$\int_{\pi/(2a)}^{\infty} \frac{g(t)}{t^{\ell+r+\nu+1}} \sin\left(\frac{mt}{\pi}(\theta_{j,m}+\phi) - B_\nu(\phi) \right) dt = \mathcal{O}(m^{-q-1}).$$

Along with (3.24) and (3.25), this implies (3.21). □

Proof of Theorem 3.1(a) and (c): Part (c) is obtained easily from (3.21) using integration by parts. The function

$$z \mapsto \int_{\pi/(2a)}^{\infty} \frac{g(t)}{t^{r+1}} \sin(zt - \frac{r+2}{2}\pi) dt$$

is a nonzero entire function. Therefore, there exists an interval I on which it has no zeros. This implies the part (a). □

4 Conclusions

We have considered the problem of detecting the location of the jump discontinuities of different derivatives of a function. We have constructed a sequence of operators, which can be evaluated using an increasing number of samples of the function. Our operators provide a frame consisting of Chebyshev polynomials of second kind, which can be thought of as orthogonal polynomials on a discrete set. The values of the frame coefficients are large near the singularities, and small away from them. We have given precise quantitative estimates, but at this time, are unable to detect the discontinuities in the function itself and its first derivative. Unlike the classical, compactly supported wavelets, our constructions can detect the singularities of an arbitrarily high order, at least equal to 2.

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