A Duality Principle for Trigonometric Wavelets

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Abstract. The aim of this paper is to give a direct proof of the duality principle for decomposition and reconstruction algorithms for nested spaces of trigonometric polynomials. The scaling functions of these spaces are defined as fundamental polynomials of Lagrange interpolation. The decomposition matrix for these scaling functions and wavelets is the transpose of the reconstruction matrix for the dual scaling functions and dual wavelets. Analogously, the original reconstruction matrix is the transpose of the decomposition matrix for the dual functions.

§1. Introduction

The trigonometric polynomials used in this paper were introduced by A. A. Privalov for the construction of orthogonal bases of the space $C_{2\pi}$ [6], whose results were very recently generalized by R. A. Lorentz and A. A. Sahakian [4] using a wavelet approach. The investigation of nested spaces of trigonometric polynomials from a wavelet point of view started with a paper by C. K. Chui and H. N. Mhaskar on trigonometric wavelets [2]. The basic tool in [6] is the construction of a suitable interpolation operator of Lagrange type using de la Vallée Poussin kernels. In [5], the exact structure of the decomposition and reconstruction matrices associated with the nested spaces generated by these Lagrange interpolants was investigated. The use of dual scaling functions and wavelets - as first introduced in [3] - turned out to be a crucial step. The aim of this paper is to prove a duality principle for these nested spaces of trigonometric polynomials. After defining the scaling functions and wavelets and listing their basic properties in Section 2, the corresponding two-scale relations are recalled from [5] in Section 3. In Section 4, the relationship between the inner product matrices of scaling functions and wavelets at a

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level j and the inner product matrix of scaling functions at the next level $j+1$ is described using a convenient matrix notation. Also, the dual scaling functions and wavelets are introduced. The more complicated decomposition relations are again recalled from [5] in Section 5, so that in Section 6 the duality principle can be proven, saying that the decomposition matrix for the original scaling functions and wavelets is the transpose of the reconstruction matrix for the dual scaling functions and dual wavelets. Analogously, the original reconstruction matrix is the transpose of the decomposition matrix for the dual functions.

§2. Definitions

For $\ell \in \mathbb{N}$, the Dirichlet kernel $D_\ell \in T_\ell$ is defined as

$$D_\ell(x) = \frac{1}{2} + \sum_{k=1}^{\ell} \cos kx = \begin{cases} \frac{\sin(\ell+\frac{1}{2})x}{2\sin\frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ \ell + \frac{1}{2} & \text{for } x \in 2\pi\mathbb{Z}, \end{cases}$$

where T_ℓ denotes the linear space of trigonometric polynomials of degree ℓ.

In the following, two different kinds $\phi_{j,0}^D$ and $\phi_{j,0}^F$ of de la Vallée Poussin kernels are used to construct certain interpolatory operators. Therefore, let

$$\begin{aligned} \phi_{j,0}^D(x) &= \frac{1}{3\cdot 2^{2j+1}} \sum_{\ell=2^{j+1}}^{2^{j+2}-1} D_\ell(x) \\ &= \begin{cases} \frac{\sin(3\cdot 2^j x)\sin(2^j x)}{3\cdot 2^{2j+2}\sin^2(\frac{x}{2})} & \text{for } x \notin 2\pi\mathbb{Z}, \\ 1 & \text{for } x \in 2\pi\mathbb{Z} \end{cases} \end{aligned}$$

and

$$\begin{aligned} \phi_{j,0}^F(x) &= \frac{1}{3\cdot 2^{j+1}} \sum_{\ell=3\cdot 2^j-1}^{3\cdot 2^j} D_\ell(x) \\ &= \begin{cases} \frac{\sin(3\cdot 2^j x)\cot\frac{x}{2}}{3\cdot 2^{j+1}} & \text{for } x \notin 2\pi\mathbb{Z}, \\ 1 & \text{for } x \in 2\pi\mathbb{Z}\ . \end{cases} \end{aligned}$$

While the first kernel represents the classical de la Vallée Poussin approach, the second one is only a slight modification of a simple Dirichlet kernel. In the following, the superscripts will be dropped whenever a statement holds for both cases.

The crucial interpolatory property of $\phi_{j,0}$ is

$$\phi_{j,0}\Big(\frac{k\pi}{3\cdot 2^j}\Big) = \delta_{k,0}, \quad k = 0, 1, \ldots, 3\cdot 2^{j+1}-1.$$

Definition 2.1. *For $j \in \mathbb{N}_0$, the spaces V_j are defined by $V_j = span\{\phi_{j,n} : n = 0, \ldots, 3 \cdot 2^{j+1} - 1\}$, where $\phi_{j,n}(x) = \phi_{j,0}(x - \frac{n\pi}{3 \cdot 2^j})$.*

For notational convenience, let $\phi_{j,n} = \phi_{j,n \bmod 3 \cdot 2^{j+1}}$ for any $n \in \mathbb{Z}$. Note that V_j^D as generated by $\phi_{j,n}^D$ and V_j^F generated by $\phi_{j,n}^F$ are different spaces. Nevertheless, they have the same dimension $3 \cdot 2^{j+1}$ as can be deduced from the corresponding interpolatory property $\phi_{j,n}(\frac{k\pi}{3 \cdot 2^j}) = \delta_{k,n}$, for all $k, n \in \mathbb{Z}$.

Definition 2.2. *For any $j \in \mathbb{N}_0$, the interpolation operator L_j mapping any real-valued 2π-periodic function f into the space V_j is defined as*

$$L_j f(x) = \sum_{n=0}^{3 \cdot 2^{j+1} - 1} f(\frac{n\pi}{3 \cdot 2^j}) \phi_{j,n}(x).$$

The following properties of the operators L_j are well-known ([6],[7]):

(i) $\quad L_j^D f \in T_{2^{j+2}-1}, \quad L_j^F f \in T_{3 \cdot 2^j}$

(ii) $\quad L_j f(\frac{k\pi}{3 \cdot 2^j}) = f(\frac{k\pi}{3 \cdot 2^j}), k \in \mathbb{Z}$

(iii) $\quad L_j^D f = f \quad \text{for all} \quad f \in T_{2^{j+1}} \cup V_j^D,$

$\quad L_j^F f = f \quad \text{for all} \quad f \in T_{3 \cdot 2^j - 1} \cup V_j^F,$

hence

(iv) $\quad T_{2^{j+1}} \subset V_j^D \subset T_{2^{j+2}-1},$

$\quad T_{3 \cdot 2^j - 1} \subset V_j^F \subset T_{3 \cdot 2^j}.$

Moreover, property (iv) implies that

$$V_j \subset V_{j+1} \quad ,$$

i.e., the spaces V_j form a sequence of nested subspaces of L^2, the space of 2π-periodic square integrable functions. Setting $V_{-1} = \{0\}$, it is also clear that

$$L^2 = L^2 - \text{closure of} \quad \bigcup_{j=-1}^{\infty} V_j \quad \text{and} \quad \bigcap_{j=-1}^{\infty} V_j = \{0\} \ .$$

As the next step, the orthogonal complement of V_j relative to V_{j+1}, *i.e.*, the so-called wavelet space W_j needs to be described in more detail.

Definition 2.3. *For $j \in \mathbb{N}_0$, the spaces W_j are defined by $W_j = span\{\psi_{j,n} : n = 0, \ldots, 3 \cdot 2^{j+1} - 1\}$, where*

$$\psi_{j,n}(x) = 2\phi_{j+1,2n+1}(x) - \phi_{j,n}(x - \frac{\pi}{3 \cdot 2^{j+1}}) \in V_{j+1} \quad .$$

The functions $\psi_{j,n}$ also show interpolatory properties, namely for all $k \in \mathbb{Z}$

$$\psi_{j,n}(\frac{(2k+1)\pi}{3\cdot 2^{j+1}}) = \delta_{k,n},$$
$$\psi_{j,n}(\frac{k\pi}{3\cdot 2^{j}}) = -\phi_{j,n}(\frac{(2k-1)\pi}{3\cdot 2^{j+1}}) \ .$$

Therefore, $\dim W_j = 3\cdot 2^{j+1}$.
Let $\langle \cdot,\cdot \rangle$ denote the inner product of two functions f and g in L^2, *i.e.*,

$$\langle f,g\rangle = \frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx.$$

The following fundamental result was established by A. A. Privalov [6]:

Theorem A. *The spaces V_j and W_j are orthogonal, i.e.,*

$$\langle \phi_{j,n}, \psi_{j,k}\rangle = 0 \quad \text{for all} \quad n,k\in \mathbb{Z}.$$

Therefore,

$$V_{j+1} = V_j \oplus W_j,$$

where $\oplus$ denotes orthogonal summation.

§3. Two-scale Relations

As $V_j \subset V_{j+1}$, there have to exist specific coefficients $p_{j,n,s}$ such that $\phi_{j,n} = \sum_s p_{j,n,s}\phi_{j+1,s}$. The following result establishes their precise values.

Theorem 3.1. [5] *For $j\in \mathbb{N}_0$ and $n = 0,1\ldots,3\cdot 2^{j+1}-1$, it holds that*

$$\phi_{j,n}(x) = \phi_{j+1,2n}(x) + \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,n}(\frac{(2s+1)\pi}{3\cdot 2^{j+1}})\phi_{j+1,2s+1}(x).$$

From $W_j \subset V_{j+1}$, it is clear that there also must be coefficients $q_{j,n,s}$ such that $\psi_{j,n} = \sum_s q_{j,n,s}\phi_{j+1,s}$ which are determined using the following.

Theorem 3.2. [5] *For $j\in \mathbb{N}_0$ and $n = 0,1\ldots,3\cdot 2^{j+1}-1$, the two-scale relation for the wavelet functions $\psi_{j,n}$ is*

$$\psi_{j,n}(x) = \phi_{j+1,2n+1}(x) - \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,n}(\frac{(2s-1)\pi}{3\cdot 2^{j+1}})\phi_{j+1,2s}(x).$$

Let $\underline{\phi}_j$ denote the vector $(\phi_{j,0},\phi_{j,1},\ldots,\phi_{j,3\cdot 2^{j+1}-1})^T$ and, analogously, $\underline{\psi}_j$ the vector $(\psi_{j,0},\psi_{j,1},\ldots,\psi_{j,3\cdot 2^{j+1}-1})^T$. Furthermore, we define a reordering for the vector of scaling functions by

$$P_j\underline{\phi}_{j+1} = (\phi_{j+1,0}, \phi_{j+1,2}, \ldots, \phi_{j+1,2m}, \ldots, \phi_{j+1,3\cdot 2^{j+2}-2},$$
$$\phi_{j+1,1}, \phi_{j+1,3}, \ldots, \phi_{j+1,2m+1}, \ldots, \phi_{j+1,3\cdot 2^{j+2}-1})^T,$$

i.e., P_j is chosen to be the suitable permutation matrix for this ordering.

Consequently, the two-scale relation or *reconstruction matrix* C_j has the following form

$$C_j = \begin{pmatrix} I_j & K_j \\ -K_j^T & I_j \end{pmatrix},$$

where I_j is an identity matrix of dimension $3\cdot 2^{j+1}$ and K_j is a knot evaluation matrix:

$$K_j = \left(\phi_{j,r}\left(\frac{(2s+1)\pi}{3\cdot 2^{j+1}}\right)\right)_{r,s=0}^{3\cdot 2^{j+1}-1}.$$

Due to the definition of the functions $\phi_{j,r}$, C_j is a square matrix of dimension $3\cdot 2^{j+2}$ with circulant blocks. Theorems 3.1 and 3.2 can now be expressed as

$$\begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} = C_j P_j \underline{\phi}_{j+1} \quad .$$

As a consequence of Theorem A, both the sets $\{\phi_{j,r}, \psi_{j,r}\}_{r=0}^{3\cdot 2^{j+1}-1}$ and $\{\phi_{j+1,r}\}_{r=0}^{3\cdot 2^{j+2}-1}$ are bases of V_{j+1}. Therefore, the reconstruction matrix C_j is non-singular and its inverse D_j is the decomposition matrix such that

$$P_j\underline{\phi}_{j+1} = D_j \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix}.$$

For a detailed analysis of the entries of D_j, it is necessary to investigate inner products of scaling functions and to introduce dual scaling functions.

§4. Inner Products of Scaling Functions and Wavelets. Dual Functions

In order to investigate dual functions and to be able to give a concise formulation of the duality principle for trigonometric wavelets, in this section we study the inner products of the functions $\phi_{j,k}$ and $\psi_{j,k}$.

For any level $j \in \mathbb{N}_0$, the inner product matrix G_j for scaling functions is defined as

$$G_j = (\langle \phi_{j,k}, \phi_{j,l}\rangle)_{k,l=0}^{3\cdot 2^{j+1}-1}.$$

Our next goal is to express the matrix G_j in terms of the matrix G_{j+1}, *i.e.*, inner products at level j by means of inner products at level $j+1$.

Lemma 4.1. *For $j \in \mathbb{N}_0$, let K_j be the knot evaluation matrix, I_j the identity matrix and P_j the permutation matrix defined in Section 3. It holds that*

$$G_j = (I_j \quad K_j) P_j G_{j+1} P_j^T \begin{pmatrix} I_j \\ K_j^T \end{pmatrix}.$$

Proof: Using the two-scale relation for $\phi_{j,k}$ of Theorem 3.1 yields for $k, l = 0, \ldots, 3 \cdot 2^{j+1} - 1$

$$\begin{aligned}\langle \phi_{j,k}, \phi_{j,l} \rangle = &\langle \phi_{j+1,2k}, \phi_{j+1,2l} \rangle \\ &+ \sum_{r=0}^{3\cdot 2^{j+1}-1} \phi_{j,k}\left(\frac{(2r+1)\pi}{3\cdot 2^{j+1}}\right) \langle \phi_{j+1,2r+1}, \phi_{j+1,2l} \rangle \\ &+ \sum_{s=0}^{3\cdot 2^{j+1}-1} \phi_{j,l}\left(\frac{(2s+1)\pi}{3\cdot 2^{j+1}}\right) \langle \phi_{j+1,2k}, \phi_{j+1,2s+1} \rangle \\ + \sum_{r=0}^{3\cdot 2^{j+1}-1} \sum_{s=0}^{3\cdot 2^{j+1}-1} &\phi_{j,k}\left(\frac{(2r+1)\pi}{3\cdot 2^{j+1}}\right) \phi_{j,l}\left(\frac{(2s+1)\pi}{3\cdot 2^{j+1}}\right) \langle \phi_{j+1,2r+1}, \phi_{j+1,2s+1} \rangle.\end{aligned}$$

Introducing the following square matrices of dimension $3 \cdot 2^{j+1}$, namely, $\Phi_e = (\langle \phi_{j+1,2k}, \phi_{j+1,2l} \rangle)_{k,l=0}^{3\cdot 2^{j+1}-1}$, $\Phi_o = (\langle \phi_{j+1,2k+1}, \phi_{j+1,2l+1} \rangle)_{k,l=0}^{3\cdot 2^{j+1}-1}$ and $\Phi_m = (\langle \phi_{j+1,2k}, \phi_{j+1,2l+1} \rangle)_{k,l=0}^{3\cdot 2^{j+1}-1}$, the relations above can be written in matrix notation as

$$\begin{aligned} G_j &= \Phi_e + \Phi_m K_j^T + K_j \Phi_m^T + K_j \Phi_o K_j^T \\ &= (I_j \quad K_j) \begin{pmatrix} \Phi_e & \Phi_m \\ \Phi_m^T & \Phi_o \end{pmatrix} \begin{pmatrix} I_j \\ K_j^T \end{pmatrix}. \end{aligned}$$

By careful inspection, it is obtained that indeed

$$P_j G_{j+1} P_j^T = \begin{pmatrix} \Phi_e & \Phi_m \\ \Phi_m^T & \Phi_o \end{pmatrix}$$

which concludes the proof of the lemma. ■

A similar statement holds for the inner product matrix of the wavelets

$$H_j = (\langle \psi_{j,k}, \psi_{j,l} \rangle)_{k,l=0}^{3\cdot 2^{j+1}-1}.$$

Lemma 4.2. *For $j \in \mathbb{N}_0$, it holds that*

$$H_j = (-K_j^T \quad I_j) P_j G_{j+1} P_j^T \begin{pmatrix} -K_j \\ I_j \end{pmatrix}.$$

Proof: This time we use the two-scale relation for the wavelets given by Theorem 3.2, to obtain that

$$\begin{aligned} H_j &= \Phi_o - K_j^T \Phi_m - \Phi_m^T K_j + K_j^T \Phi_e K_j \\ &= (-K_j^T \quad I_j) \begin{pmatrix} \Phi_e & \Phi_m \\ \Phi_m^T & \Phi_o \end{pmatrix} \begin{pmatrix} -K_j \\ I_j \end{pmatrix}. \end{aligned} \qquad ■$$

In a similar way, the orthogonality relations

$$\langle \phi_{j,k}, \psi_{j,l} \rangle = 0, \quad j,k = 0,\ldots,3 \cdot 2^{j+1} - 1$$

of Theorem A can be written in matrix terms using Theorems 3.1 and 3.2:

$$(I_j \quad K_j) P_j G_{j+1} P_j^T \begin{pmatrix} -K_j \\ I_j \end{pmatrix} = (-K_j^T \quad I_j) P_j G_{j+1} P_j^T \begin{pmatrix} I_j \\ K_j^T \end{pmatrix} = (0).$$

Putting all the above matrix relations together, one obtains

$$\begin{pmatrix} I_j & K_j \\ -K_j^T & I_j \end{pmatrix} P_j G_{j+1} P_j^T \begin{pmatrix} I_j & -K_j \\ K_j^T & I_j \end{pmatrix} = \begin{pmatrix} G_j & 0 \\ 0 & H_j \end{pmatrix},$$

i.e., the following

Proposition 4.1. *For $j \in \mathbb{N}_0$, the two-scale relation for the inner product matrices G_j, H_j and G_{j+1} is given by*

$$\begin{pmatrix} G_j & 0 \\ 0 & H_j \end{pmatrix} = C_j P_j G_{j+1} P_j^T C_j^T.$$

As a next step, it is possible to describe in detail the entries of the inverse matrix of G_j.

Theorem 4.1. [5] *For any $j \in \mathbb{N}_0$, the inverse matrix for the inner product matrix G_j is given by*

$$G_j^{-1} = (\alpha_{j,r,s})_{r,s=0}^{3\cdot 2^{j+1}-1},$$

where the coefficients for $r,s = 0,\ldots,3 \cdot 2^{j+1} - 1$ are given as

$$\alpha_{j,r,s}^F = 3 \cdot 2^{j+1} \delta_{r,s} + (-1)^{r-s} \quad \textit{and}$$

$$\alpha_{j,r,s}^D = 3 \cdot 2^{j+1} \delta_{r,s} + \sum_{\ell=2^{j+1}+1}^{2^{j+2}-1} \left(\frac{2^{2j+1}}{\ell^2 - 3\ell 2^{j+1} + 5 \cdot 2^{2j+1}} - 1 \right) \cos \frac{\ell (r-s)\pi}{3 \cdot 2^j}.$$

In order to find the decomposition sequences for the orthogonal sum $V_{j+1} = V_j \oplus W_j$, it is convenient to make use of the so-called dual scaling functions and wavelets. The concept of duality was introduced in [3] for a multiresolution analysis of $L^2(\mathbb{R})$ and is used here in a form suitably adapted to our purposes.

Definition 4.1. *For any $j \in \mathbb{N}_0$, the functions $\tilde{\phi}_{j,r} \in V_j, r = 0,\ldots,3 \cdot 2^{j+1} - 1$, uniquely determined by the conditions*

$$\langle \tilde{\phi}_{j,r}, \phi_{j,k} \rangle = \delta_{r,k} \quad \textit{for all} \quad r,k = 0,\ldots,3 \cdot 2^{j+1} - 1,$$

are called dual scaling functions (or dual to the functions $\phi_{j,r}$).

Note that the dual scaling functions $\tilde{\phi}_{j,r}$ lie in the same space V_j as the original scaling functions $\phi_{j,k}$. Consequently, the dual functions can be written as linear combinations of these scaling functions. In fact, for any $j \in \mathbb{N}_0$,

$$\tilde{\phi}_{j,r} = \sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,r,s}\phi_{j,s}.$$

Thus, the duality conditions lead to a linear system of equations for each dual function $\tilde{\phi}_{j,r}$, $r = 0, \ldots, 3 \cdot 2^{j+1} - 1$, namely,

$$\sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,r,s}\langle \phi_{j,k}, \phi_{j,s}\rangle = \delta_{r,k}, \quad k = 0, \ldots, 3 \cdot 2^{j+1} - 1,$$

with G_j as its coefficient matrix and different right hand sides for the different dual functions.

Definition 4.2. *For $j \in \mathbb{N}_0$, the functions $\tilde{\psi}_{j,r} \in W_j$, $r = 0, \ldots, 3 \cdot 2^{j+1} - 1$, uniquely determined by the conditions*

$$\langle \tilde{\psi}_{j,r}, \psi_{j,k}\rangle = \delta_{r,k} \quad \text{for all} \quad r, k = 0, \ldots, 3 \cdot 2^{j+1} - 1,$$

are called dual wavelets (or dual to the functions $\psi_{j,r}$).

As for the scaling functions, the dual wavelets $\tilde{\psi}_{j,r}$ lie in the same space W_j as the original wavelets $\psi_{j,k}$. Therefore, also the dual wavelets can be written as linear combinations of the original ones, *i.e.*, for $r = 0, \ldots, 3 \cdot 2^{j+1} - 1$, $\tilde{\psi}_{j,r} = \sum_{s=0}^{3\cdot 2^{j+1}-1} \beta_{j,r,s}\psi_{j,s}$, where the $\beta_{j,r,s}$ are the entries of the matrix H_j^{-1}.

Remark. The dual scaling functions are again translates of a single function just like the original ones, as the inner product matrices are circulant, *i.e.*,

$$\begin{aligned}\tilde{\phi}_{j,r}(x) &= \sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,r,s}\phi_{j,0}(x - \frac{s\pi}{3\cdot 2^j}) = \sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,0,s-r}\phi_{j,0}(x - \frac{s\pi}{3\cdot 2^j}) \\ &= \sum_{s=0}^{3\cdot 2^{j+1}-1} \alpha_{j,0,s}\phi_{j,0}(x - \frac{r\pi}{3\cdot 2^j} - \frac{s\pi}{3\cdot 2^j}) = \tilde{\phi}_{j,0}(x - \frac{r\pi}{3\cdot 2^j})\end{aligned}$$

and, analogously, for the dual wavelets

$$\tilde{\psi}_{j,r}(x) = \tilde{\psi}_{j,0}(x - \frac{r\pi}{3\cdot 2^j}), \quad r = 0, \ldots, 3 \cdot 2^{j+1} - 1.$$

§5. Decomposition Sequences

The other pair of important sequences are the so-called decomposition sequences. As $V_{j+1} = V_j \oplus W_j$ for $j \in \mathbb{N}_0$, any $\phi_{j+1,n} \in V_{j+1}$ can be written as a linear combination of the basis functions of V_j and W_j, *i.e.*, $\phi_{j,k}$ and $\psi_{j,k}$. In the following, we will distinguish two cases.

Theorem 5.1. [5] *For any $j \in \mathbb{N}_0$ and $m = 0, \ldots, 3 \cdot 2^{j+1} - 1$, it holds that*

$$\phi_{j+1,2m}(x) = \sum_{k=0}^{3\cdot 2^{j+1}-1} \left(a_{j,m,k}\phi_{j,k}(x) + b_{j,m,k}\psi_{j,k}(x)\right),$$

where for $n = 0, \ldots, 3 \cdot 2^{j+1} - 1$, the decomposition coefficients $a_{j,m,n}$ and $b_{j,m,n}$ are given by

$$\begin{aligned} a_{j,m,n} &= \frac{\alpha_{j,m,n}}{3 \cdot 2^{j+2}}, \\ b_{j,m,n} &= -\sum_{s=0}^{3\cdot 2^{j+1}-1} \frac{\alpha_{j,m,s}}{3 \cdot 2^{j+2}} \phi_{j,s}\left(\frac{(2n+1)\pi}{3 \cdot 2^{j+1}}\right) \\ &= -\frac{\tilde{\phi}_{j,m}\left(\frac{(2n+1)\pi}{3\cdot 2^{j+1}}\right)}{3 \cdot 2^{j+2}}, \end{aligned}$$

with the terms $\alpha_{j,m,n}$ being the coefficients of the dual scaling functions as described in Theorem 4.1.

For an even more detailed description of the entries $a_{j,m,n}$ and $b_{j,m,n}$, see Corollary 5.1 in [5].

For the basis functions in V_{j+1} with odd indices the result is

Theorem 5.2. [5] *For any $j \in \mathbb{N}_0$ and $m = 0, \ldots, 3 \cdot 2^{j+1} - 1$, it holds that*

$$\phi_{j+1,2m+1}(x) = \sum_{k=0}^{3\cdot 2^{j+1}-1} \left(\tilde{a}_{j,m,k}\phi_{j,k}(x) + \tilde{b}_{j,m,k}\psi_{j,k}(x)\right),$$

where for $n = 0, \ldots, 3 \cdot 2^{j+1} - 1$, the decomposition coefficients $\tilde{a}_{j,m,n}$ and $\tilde{b}_{j,m,n}$ are given as

$$\begin{aligned} \tilde{a}_{j,m,n} &= -b_{j,n,m}, \\ \tilde{b}_{j,m,n} &= a_{j,m,n}, \end{aligned}$$

and the terms $a_{j,m,n}$ and $b_{j,m,n}$ are defined in Theorem 5.1.

By slightly modifying the symmetric and circulant duality matrix of Section 4, *i.e.*,

$$\tilde{G}_j^{-1} = \frac{1}{3 \cdot 2^{j+2}} G_j^{-1} = \left(\frac{\alpha_{j,r,s}}{3 \cdot 2^{j+2}}\right)_{r,s=0}^{3\cdot 2^{j+1}-1},$$

one obtains by using the notation of Section 3 that the decomposition relations of Theorems 5.1 and 5.2 can be expressed as

$$P_j \underline{\phi}_{j+1} = D_j \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix},$$

using the *decomposition matrix* with circulant blocks

$$D_j = C_j^{-1} = \begin{pmatrix} \tilde{G}_j^{-1} & -\tilde{G}_j^{-1} K_j \\ K_j^T \tilde{G}_j^{-1} & \tilde{G}_j^{-1} \end{pmatrix} \quad .$$

As $V_{j+1} = V_j \oplus W_j$, a function $f_{j+1} \in V_{j+1}$ can be written uniquely as

$$f_{j+1} = f_j + g_j, \quad \text{with} \quad f_j \in V_j \quad \text{and} \quad g_j \in W_j \quad .$$

Using the basis functions of these spaces, one obtains

$$f_{j+1}(x) = \sum_{s=0}^{3\cdot 2^{j+2}-1} c_s^{j+1} \phi_{j+1,s}(x),$$

$$f_j(x) = \sum_{s=0}^{3\cdot 2^{j+2}-1} c_s^j \phi_{j,s}(x) \quad \text{and} \quad g_j(x) = \sum_{s=0}^{3\cdot 2^{j+2}-1} d_s^j \psi_{j,s}(x) \quad .$$

Denoting coefficient vectors by $\underline{c}_j^T = (c_0^j, c_1^j, \ldots, c_{3\cdot 2^{j+1}-1}^j)$ and also $\underline{d}_j^T = (d_0^j, d_1^j, \ldots, d_{3\cdot 2^{j+1}-1}^j)$, respectively, one obtains

$$f_{j+1} = \underline{c}_{j+1}^T \underline{\phi}_{j+1}, \quad f_j = \underline{c}_j^T \underline{\phi}_j \quad \text{and} \quad g_j = \underline{d}_j^T \underline{\psi}_j \quad .$$

As $\underline{c}_{j+1}^T \underline{\phi}_{j+1} = (P_j \underline{c}_{j+1})^T P_j \underline{\phi}_{j+1}$, the matrix form of the decomposition relation yields

$$f_{j+1} = (P_j \underline{c}_{j+1})^T P_j \underline{\phi}_{j+1} = (P_j \underline{c}_{j+1})^T D_j \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} \quad .$$

On the other hand, it holds that

$$f_j + g_j = \begin{pmatrix} \underline{c}_j^T & \underline{d}_j^T \end{pmatrix} \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} \quad .$$

Comparing coefficients leads to

$$\begin{pmatrix} \underline{c}_j^T & \underline{d}_j^T \end{pmatrix} = (P_j \underline{c}_{j+1})^T D_j,$$

and taking the transpose finally gives the matrix form of one step of the *decomposition algorithm*

$$\begin{pmatrix} \underline{c}_j \\ \underline{d}_j \end{pmatrix} = D_j^T P_j \underline{c}_{j+1} \quad .$$

Multiplying by the inverse $(D_j^T)^{-1} = C_j^T$ yields the matrix representation of one step of the *reconstruction algorithm*

$$P_j \underline{c}_{j+1} = C_j^T \begin{pmatrix} \underline{c}_j \\ \underline{d}_j \end{pmatrix} \quad .$$

§6. The Duality Principle

The duality principle in the sense that will be used here was first introduced by C. K. Chui and J. Z. Wang in [3] for spline wavelets on the real axis and described in a much more general form in the monograph [1]. In short, it says that the decomposition sequences for the original scaling functions and wavelets become the reconstruction sequences for the dual scaling functions and wavelets, as well as the reconstruction sequences for the original functions are the decomposition sequences for the duals.

In this context of trigonometric scaling functions and wavelets, the duality principle can be formulated as follows.

Theorem 6.1. *(The duality principle). For any $j \in \mathbb{N}_0$, the decomposition matrix for the dual functions is C_j^T and the reconstruction matrix for the duals is D_j^T, i.e., for the original scaling functions and wavelets it holds that*

$$\begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} = C_j P_j \underline{\phi}_{j+1} \qquad (reconstruction),$$

$$P_j \underline{\phi}_{j+1} = D_j \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} \qquad (decomposition),$$

while for the dual functions

$$\begin{pmatrix} \underline{\tilde{\phi}}_j \\ \underline{\tilde{\psi}}_j \end{pmatrix} = D_j^T P_j \underline{\tilde{\phi}}_{j+1} \qquad (reconstruction),$$

$$P_j \underline{\tilde{\phi}}_{j+1} = C_j^T \begin{pmatrix} \underline{\tilde{\phi}}_j \\ \underline{\tilde{\psi}}_j \end{pmatrix} \qquad (decomposition).$$

Proof: Using $D_j = C_j^{-1}$ and $P_j^{-1} = P_j^T$, Proposition 4.1 can be rewritten as

$$G_{j+1} = P_j^T D_j \begin{pmatrix} G_j & 0 \\ 0 & H_j \end{pmatrix} D_j^T P_j.$$

Therefore,

$$G_{j+1}^{-1} = P_j^T C_j^T \begin{pmatrix} G_j^{-1} & 0 \\ 0 & H_j^{-1} \end{pmatrix} C_j P_j.$$

Consequently, using the reconstruction relation for $P_j \underline{\phi}_{j+1}$,

$$\begin{aligned} P_j \underline{\tilde{\phi}}_{j+1} &= P_j G_{j+1}^{-1} \underline{\phi}_{j+1} \\ &= C_j^T \begin{pmatrix} G_j^{-1} & 0 \\ 0 & H_j^{-1} \end{pmatrix} C_j P_j \underline{\phi}_{j+1} \\ &= C_j^T \begin{pmatrix} G_j^{-1} & 0 \\ 0 & H_j^{-1} \end{pmatrix} \begin{pmatrix} \underline{\phi}_j \\ \underline{\psi}_j \end{pmatrix} \\ &= C_j^T \begin{pmatrix} \underline{\tilde{\phi}}_j \\ \underline{\tilde{\psi}}_j \end{pmatrix} \end{aligned}$$

and the reconstruction relation for the duals follows by a simple matrix inversion. ■

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