Orthogonal Algebraic Polynomial Schauder Bases of Optimal Degree

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Abstract

For any fixed $\varepsilon > 0$ we construct an orthonormal Schauder basis $\{p_\mu\}_{\mu=0}^\infty$ for $C[-1,1]$ consisting of algebraic polynomials p_μ with $\deg p_\mu \leq (1+\varepsilon)\mu$.

The orthogonality is with respect to the Chebyshev weight.

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1 Introduction

In [9] A. A. Privalov constructed for each $\varepsilon > 0$ a trigonometric polynomial Schauder basis $\{t_\mu\}_{\mu=1}^\infty$ of $C_{2\pi}$ such that

$$\deg t_\mu \;\leq\; (1+\varepsilon)\frac{\mu}{2}\,, \tag{1}$$

where Schauder basis means that for every $f \in C_{2\pi}$ there exist unique coefficients a_μ such that $f = \sum\limits_{\mu=1}^{\infty} a_\mu t_\mu$. Moreover, he mentioned that one obtains as an easy corollary for each $\varepsilon > 0$ an algebraic polynomial Schauder basis $\{g_\mu\}_{\mu=0}^\infty$ of $C[-1,1]$ such that $\deg g_\mu \leq (1+\varepsilon)\mu$. Conversely, in [8] he had shown that for any polynomial Schauder basis $\{g_\mu\}_{\mu=0}^\infty$ of $C[-1,1]$, there is an $\varepsilon > 0$ such that $\deg g_\mu \geq (1+\varepsilon)\mu$ for sufficiently large μ.

Many authors have given attention to this problem (see e.g. D. Offin and K. Oskolkov [5], A. A. Privalov [10], and P. L. Ul'yanov [14] for further references). In [13] P. L. Ul'yanov raised the question of the minimal growth of the degree of the polynomials also for orthonormal algebraic and trigonometric polynomial Schauder bases of $C[-1,1]$ and $C_{2\pi}$, respectively.

Using wavelet packet constructions on the real line and a periodization technique using the Poisson summation formula, in [4] R. A. Lorentz and A. A. Sahakian gave a final answer for the trigonometric case (see also K. Woźniakowski [15]). I.e., for any $\varepsilon > 0$ they constructed an orthonormal trigonometric Schauder basis $\{t_\mu\}_{\mu=1}^\infty$ of $C_{2\pi}$ which satisfies (1).

However, the question of whether there exists an orthonormal algebraic polynomial Schauder basis remained open. In particular, merely to apply the transformation $x = \cos\theta$ to the trigonometric polynomials t_μ fails to resolve the question, since the usual technique of splitting the trigonometric polynomials into their even and odd parts can destroy orthogonality.

The aim of this paper is to present an orthonormal algebraic polynomial Schauder basis $\{p_\mu\}_{\mu=0}^\infty$ of optimal degree $\deg p_\mu \leq (1+\varepsilon)\mu$. The methods which we will use for construction of the basis are an adaptation of wavelet techniques. Specifically we adapt the concept of wavelet packets, using a generalized translation. Orthogonality is given with respect to the weighted inner product

$$\langle f, g \rangle \;=\; \frac{2}{\pi}\int_{-1}^{1} f(x)g(x)\frac{dx}{\sqrt{1-x^2}} \, .$$

With this weighted inner product, the Chebyshev polynomials $T_n(x) = \cos n \arccos x$ ($n \in I\!N_0$) satisfy the orthogonality relations

$$\langle T_n, T_m \rangle = \begin{cases} 2 & \text{for } n = m = 0, \\ 1 & \text{for } n = m > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Using this, we will give a direct and explicit description of the orthonormal algebraic polynomials p_μ in terms of their respective Chebyshev expansions. Our method of construction is based on the trigonometric de la Vallée Poussin kernels and corresponding shift-invariant spaces (see e.g. Privalov [10], and [7]).

Note that in our construction the Chebyshev polynomials can be replaced by other polynomial systems orthonormal with respect to an arbitrary weight function w. If this is done, it is clear that our methods will always yield at least an orthonormal Schauder basis of the Hilbert space $L^2_w[-1,1]$. It is an open question, for which classes of weights w it is also a Schauder basis in $L^p_w[-1,1]$ for a scope of p to be determined. In particular, in order to show it to be a Schauder basis for $C[-1,1]$, one would need to show that the corresponding Lebesgue constants (see Lemma 3.6) are uniformly bounded.

The organization of the paper is as follows. In Section 2 we define the polynomials p_μ which only depend on the parameter $\varepsilon > 0$, and we state the main result. In order to simplify the proofs we show in Section 3 how the polynomials p_μ fit into a multiresolution and wavelet packet decomposition of the weighted L^2-space. Finally in Section 4 we obtain the orthonormality and the Schauder basis property as a reformulation of results from Section 3.

2 Definition of the Basis

For arbitrary $\varepsilon > 0$ we define a polynomial sequence $\{p_\mu\}_{\mu \in \mathbb{N}_0}$ such that the degree $\deg p_\mu \leq \mu(1+\varepsilon)$. In order to define this sequence, we first choose a natural number η satisfying

$$\begin{aligned} &\eta = 3, && \text{for} \quad \tfrac{1}{2} \le \varepsilon, \\ &\tfrac{3}{2^{\eta}-2} \le \varepsilon < \tfrac{3}{2^{\eta-1}-2}, && \text{for} \quad 0 < \varepsilon < \tfrac{1}{2}. \end{aligned} \tag{3}$$

Observe that, given ε, these conditions uniquely determine η, with $\eta \ge 3$.

Then every index $\mu \in I\!N$, $\mu > 2^{\eta+1}$ determines uniquely the triplet of integers j, λ, s such that

$$\begin{aligned} \mu &= 2^j + (\lambda - 1)2^{j-\eta+1} + s\,, \\ \text{with} \qquad & j \ge \eta + 1\,, \quad 0 \le \lambda \le 2^{\eta-1} - 2\,, \quad 1 \le s \le 2^{j-\eta+1}\,. \end{aligned} \tag{4}$$

For notational convenience, we further introduce two functions,

$$g_1(x) := \begin{cases} \frac{2+x}{\sqrt{2+2(x+1)^2}} & \text{for } -2 \le x < 0, \\ \frac{2-x}{\sqrt{2+2(x-1)^2}} & \text{for } \quad 0 \le x \le 2, \end{cases}$$

and

$$g_0(x) := \begin{cases} 0 & \text{for } -2 \le x < -\frac{3}{2}, \\ \frac{x+\frac{3}{2}}{\sqrt{\frac{1}{2}+2(x+1)^2}} & \text{for } -\frac{3}{2} \le x < -\frac{1}{2}, \\ 1 & \text{for } -\frac{1}{2} \le x < 0, \\ \frac{2-x}{\sqrt{2+2(x-1)^2}} & \text{for } \quad 0 \le x \le 2, \end{cases}$$

the sampling of which will give us polynomial coefficients.

Definition 2.1 *Let $\varepsilon > 0$. Then with η as in (3), the polynomials p_μ are given by*

$$p_0 := \frac{1}{\sqrt{2}}\,,$$

$$
\begin{aligned}
p_k &:= T_k\,, \qquad \textit{for } k = 1, \ldots, 2^{\eta+1} - 2\,, \\
p_{2^{\eta+1}-1} &:= T_{2^{\eta+1}}\,, \\
p_{2^{\eta+1}} &:= \frac{3}{\sqrt{10}} T_{2^{\eta+1}-1} + \frac{1}{\sqrt{10}} T_{2^{\eta+1}+1}\,,
\end{aligned}
$$

and for $\mu > 2^{\eta+1}$ *with* j, λ, s *as in* (4), *by*

$$
p_\mu := 2^{(\eta-j)/2} \sum_{k=-3\cdot 2^{j-\eta}+1}^{2^{j-\eta}-1} g_1\left(1 + \frac{k}{2^{j-\eta}}\right) \sin\left(\frac{k(2s-1)\pi}{2^{j-\eta+2}}\right) T_{2^j+\lambda 2^{j-\eta+1}+k}\,,
$$

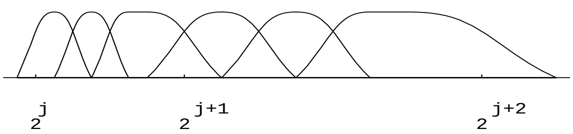
if λ *is odd, and by*

$$
p_\mu := 2^{(\eta-j)/2} \sum_{k=-3\cdot 2^{j-\eta}+1}^{2^{j-\eta}-1} g_\lambda\left(1 + \frac{k}{2^{j-\eta}}\right) \cos\left(\frac{k(2s-1)\pi}{2^{j-\eta+2}}\right) T_{2^j+\lambda 2^{j-\eta+1}+k}\,,
$$

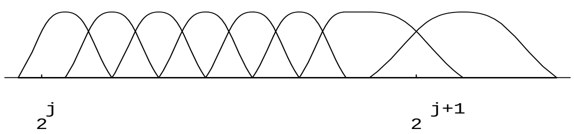
if λ *is even, with* $g_\lambda := g_1$ *for all* $\lambda > 0$.

Observe that each polynomial p_μ, with $\mu > 2^{\eta+1}$, consists of a linear combination of $2^{j-\eta+2} - 1$ succeeding Chebyshev polynomials where the coefficients in this Chebyshev expansion are a product of the function g_λ with a sine or cosine function evaluated at certain equidistant nodes. The special choice of g_λ guarantees the orthogonality of p_μ and p_ν if μ and ν do not correspond to the same j, λ. The cosine and sine factors can be seen as a result of a generalized translation (see e.g. [11, 6]) which, together with g_λ, ensure orthogonality for p_μ and p_ν corresponding to the same j, λ.

In order to illustrate the construction principle, we have drawn functions g_λ for $\eta = 3$ and $\eta = 4$ in the way in which, for different j, λ, the functions g_λ are equidistantly sampled in the Chebyshev expansions of corresponding p_μ.



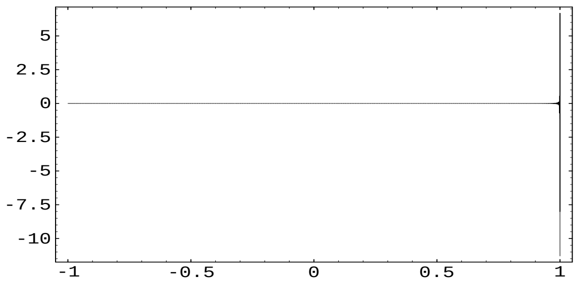
$\eta = 3$



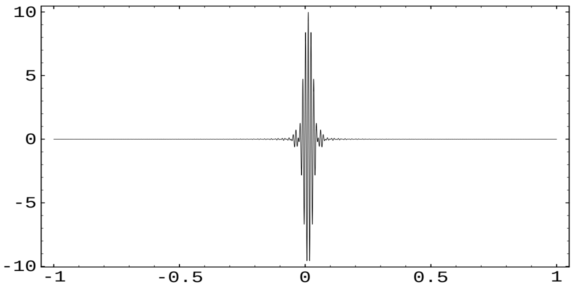
$\eta = 4$

In the next Figure, we give some examples to show the shape of the polynomials p_μ with μ from (4) for different values of η corresponding to different $\varepsilon > 0$. For $\eta = 3$, we take $j = 9$, $\lambda = 1$, and $s = 1$ determining the value of $\mu = 513$, and $s = 64$ giving $\mu = 576$.

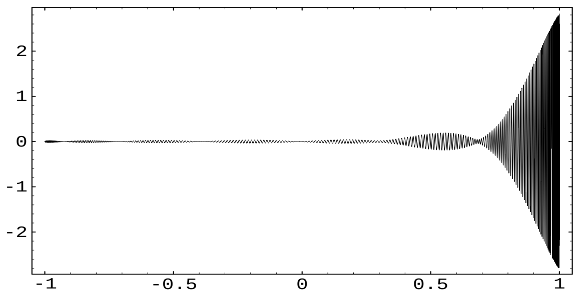
Then for $\eta = 7$, we take again $j = 9$, $\lambda = 1$, with $s = 1$ and $s = 4$, respectively. Different values of η, and hence of ε, yield different localization properties whereas different values of s for the same j, λ yield a kind of generalized translation on $[-1, 1]$.



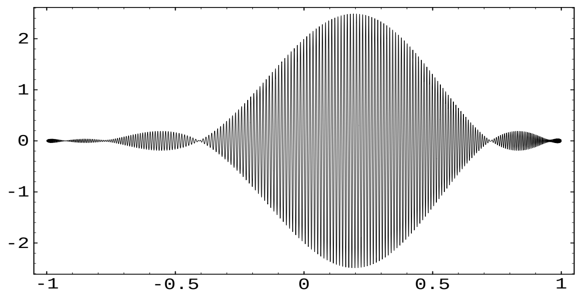
p_{513} for $\eta = 3$



p_{576} for $\eta = 3$



p_{513} for $\eta = 7$



p_{516} for $\eta = 7$

Note that for $\eta = 3$ the polynomials p_{513} and p_{576} are of exact degree 703 and would result from an $\varepsilon \geq \frac{1}{2}$, whereas for $\eta = 7$, i.e., $\frac{3}{126} \leq \varepsilon < \frac{3}{62}$, the polynomials p_{513} and p_{516} are both of exact degree 523.

Now we have the following main result.

Theorem 2.2 *Let $\varepsilon > 0$ be given. Then $\{p_\mu\}_{\mu \in \mathbb{N}_0}$ is an orthonormal polynomial Schauder basis of optimal degree for $C[-1,1]$. I.e., we have for all $\mu, \nu \in \mathbb{N}_0$*

$$\deg\ p_\mu \leq \mu(1+\varepsilon), \qquad (5)$$

$$\langle p_\mu \,,\, p_\nu \rangle = \delta_{\mu,\nu}, \qquad (6)$$

and for all $f \in C[-1,1]$

$$\left\| f - \sum_{s=0}^{\mu} \langle f \,,\ p_s \rangle p_s \right\|_{C[-1,1]} \ \le\ C E_{[\mu(1-\varepsilon)]}(f, C[-1,1])\,. \tag{7}$$

Here $E_n(f, C[-1,1])$ means the best approximation of f in the maximum norm on $[-1,1]$ by algebraic polynomials of degree n.

Note that $C = C(\varepsilon)$ is a positive constant only depending on ε. From the proof of Lemma 3.6 and in accordance with the negative result in [8] we have $C(\varepsilon) \longrightarrow \infty$ if $\varepsilon \to 0$.

PROOF. The inequality (5) can be easily checked for $\mu \le 2^{\eta+1}$. For $\mu > 2^{\eta+1}$, with j, λ, s as in (4), the inequality (5) follows from

$$\begin{aligned} \frac{\deg\, p_\mu}{\mu} \ &=\ \frac{2^j + (2\lambda+1)2^{j-\eta} - 1}{2^j + (2\lambda-2)2^{j-\eta} + s} \\ &\le\ 1 + \frac{3}{2^\eta + (2\lambda - 2)} \ \le\ 1 + \varepsilon\,. \end{aligned}$$

The proof of (6) is straightforward, but we will defer its presentation until Section 4, where we will also prove (7). This is because, in order to show (7), it is helpful to describe the polynomial spaces and orthogonal projections onto them in a more general way (see also [3]). □

Using standard arguments (see B. S. Kashin and A. A. Sahakian [1], Chap. 1, §4, Th. 9), we can conclude that our polynomials are also a Schauder basis for the spaces L^p $(1 \le p < \infty)$ with Chebyshev weight.

Corollary 2.3 *Let $\omega(x) = (1 - x^2)^{-1/2}$ and $1 \le p < \infty$. Then for all $f \in L^p_\omega[-1,1]$ we have*

$$\left\| f - \sum_{s=0}^{\mu} \langle f \,,\ p_s \rangle p_s \right\|_{L^p_\omega[-1,1]} \ \le\ C E_{[\mu(1-\varepsilon)]}(f, L^p_\omega[-1,1])\,,$$

where $E_n(f, L^p_\omega[-1,1])$ means the best approximation of f in the L^p_ω-norm on $[-1,1]$ by algebraic polynomials of degree n.

3 A Wavelet Approach

The results in Section 2 can be obtained by using a wavelet approach on the interval (see also [2, 3, 6, 11]).

Definition 3.1 *Let $N, M \in I\!N$ be fixed, with $8M|N$. Then we define*

$$\phi_{N,s}^M \ := \ \frac{1}{2}T_0 + \sum_{k=1}^{N-M} \cos\frac{ks\pi}{N}\, T_k + \sum_{k=N-M+1}^{N+M-1} g_1(\frac{k-N+M}{M}) \cos\frac{ks\pi}{N}\, T_k \tag{8}$$

for $s = 0, \ldots, N$. For $p = 0, ..., \frac{N}{4M} - 1$ and $s = 1, \ldots, 2M$ let

$$\psi_{N,2p,s}^M \ := \ \sum_{k=-2M+1}^{2M-1} g_p(\frac{k}{M}) \cos\left((k-M)\frac{(2s-1)\pi}{4M}\right)\, T_{N+(4p-1)M+k}\,, \tag{9}$$

where $g_p = g_1$ for all $p \geq 1$. Furthermore, for $p = 1, ..., \frac{N}{4M} - 1$ and $s = 1, \ldots, 2M$ let

$$\psi_{N,2p-1,s}^M \ := \ \sum_{k=-2M+1}^{2M-1} g_1(\frac{k}{M}) \sin\left((k-M)\frac{(2s-1)\pi}{4M}\right)\, T_{N+(4p-3)M+k}\,. \tag{10}$$

Remark. The results of Section 3 are written down for arbitrary N, M with $8M|N$. However, to prove our Theorem 2.2 we only have to choose $N = 2^j$ and $M = 2^{j-\eta}$, for arbitrary $j \in I\!N$, $j \geq \eta + 1$ and $\eta \in I\!N$, $\eta \geq 3$ as given by (3). For then in view of (4) the correspondence

$$p_\mu \ = \ \frac{1}{\sqrt{M}}\, \psi_{N,\lambda,s}^M \tag{11}$$

is valid for $\mu > 2^{\eta+1}$. Moreover, we will see that the orthogonal projection $\sum_{s=0}^{2^j} \langle f, p_s\rangle p_s$ can also be expressed in terms of the orthogonal basis $\{\phi_{N,s}^M : \ s = 0, \ldots, N\}$.

First we prove the following orthogonality result.

Lemma 3.2 *Let $N, M \in I\!N$, with $8M|N$. For all $r, s = 0, \ldots, N$ we have*

$$\langle \phi_{N,r}^M \,,\, \phi_{N,s}^M \rangle \ = \ N\delta_{r,s}\frac{1 + \delta_{s,0} + \delta_{s,N}}{2} \tag{12}$$

and for all $\ell = 0, ..., \frac{N}{2M} - 2$ and $r, s = 1, \ldots, 2M$

$$\langle \psi_{N,\ell,r}^M \,,\, \psi_{N,\ell,s}^M \rangle \ = \ M\delta_{r,s}\,. \tag{13}$$

Proof. For the proof, we will use the orthonormality properties (2) of the T_k, and we will also need

$$
\begin{aligned}
g_\ell^2(1+x)+g_\ell^2(1-x) &= 1 \qquad \text{for} \quad x\in[0,1],\ \ell=0,1\,,\\
g_\ell^2(-1+x)+g_\ell^2(-1-x) &= 1 \qquad \text{for} \quad x\in[0,1],\ \ell=0,1\,.
\end{aligned}
$$

To show (12), we note that

$$
\begin{aligned}
\langle \phi^M_{N,r}\ ,\ \phi^M_{N,s}\rangle &= \frac{1}{2}+\sum_{k=1}^{N-M}\cos\frac{kr\pi}{N}\cos\frac{ks\pi}{N}+\\
&\qquad +\sum_{k=-M+1}^{M-1} g_1^2\Big(\frac{M-k}{M}\Big)\cos\frac{(N-k)r\pi}{N}\cos\frac{(N-k)s\pi}{N}\\
&= \frac{1+(-1)^{r-s}}{2}+\sum_{k=1}^{N-1}\cos\frac{kr\pi}{N}\cos\frac{ks\pi}{N}\\
&= \frac{2+(-1)^{r-s}+(-1)^{r+s}}{4}+\frac{1}{2}\sum_{k=1}^{N-1}\left(\cos\frac{k(r-s)\pi}{N}+\cos\frac{k(r+s)\pi}{N}\right)\\
&= N\delta_{r,s}\frac{1+\delta_{s,0}+\delta_{s,N}}{2}\,,
\end{aligned}
$$

where we used that

$$
\frac{1}{2}+\frac{(-1)^r}{2}+\sum_{k=1}^{N-1}\cos\frac{kr\pi}{N} \;=\; N\delta_{r,0\,\mathrm{mod}\,2N}\,.
$$

To prove (13) for $\ell=2p$, $p=0,\ldots,\frac{N}{4M}-1$, we write

$$
\langle \psi^M_{N,2p,r}\ ,\ \psi^M_{N,2p,s}\rangle = \sum_{k=-2M+1}^{2M-1} g_p^2\Big(\frac{k}{M}\Big)\cos\frac{(M-k)(2r-1)\pi}{4M}\cos\frac{(M-k)(2s-1)\pi}{4M}.
$$

Splitting this sum at the values $k=-M,0$ and M, shifting indices and using some standard trigonometric identities, we rewrite this as

$$
\begin{aligned}
&\langle \psi^M_{N,2p,r}\ ,\ \psi^M_{N,2p,s}\rangle\\
&= \cos\frac{(2r-1)\pi}{4}\cos\frac{(2s-1)\pi}{4}+\frac{1}{2}\Big(1+\cos\frac{(2r-1)\pi}{2}\cos\frac{(2s-1)\pi}{2}\Big)\\
&\quad+\sum_{k=1}^{M-1}\left(g_p^2\Big(\frac{k+M}{M}\Big)+g_p^2\Big(\frac{-k+M}{M}\Big)\right)\cos\frac{k(2r-1)\pi}{4M}\cos\frac{k(2s-1)\pi}{4M}\\
&\quad+\sum_{k=1}^{M-1}\left(g_p^2\Big(\frac{k-M}{M}\Big)+g_p^2\Big(\frac{-k-M}{M}\Big)\right)\cos\frac{(2M-k)(2r-1)\pi}{4M}\cos\frac{(2M-k)(2s-1)\pi}{4M}\\
&= \frac{1}{2}\left(1+\frac{(-1)^{r-s}+(-1)^{r+s-1}}{2}+\sum_{k=1}^{2M-1}\left(\cos\frac{k(r-s)\pi}{2M}+\cos\frac{k(r+s-1)\pi}{2M}\right)\right)\\
&= M\delta_{r-s,0\,\mathrm{mod}\,4M}+M\delta_{r+s-1,0\,\mathrm{mod}4M}\,.
\end{aligned}
$$

Since $1 \le r,s \le 2M$, the second term always vanishes. Showing (13) for odd ℓ, we obtain from the definition (10) that

$$\langle \psi^M_{N,\ell,r}, \psi^M_{N,\ell,s} \rangle = (-1)^{r+s} \sum_{k=-2M+1}^{2M-1} g_1^2(\frac{k}{M}) \cos \frac{(M-k)(2r-1)\pi}{4M} \cos \frac{(M-k)(2s-1)\pi}{4M}.$$

The rest of the proof is identical to the previous case, and we thus obtain

$$\langle \psi^M_{N,\ell,r} \ , \ \psi^M_{N,\ell,s} \rangle \ = \ M\delta_{r-s,0 \bmod 4M} - M\delta_{r+s-1,0 \bmod 4M}.$$

□

Now, we introduce polynomial spaces that come from a multiresolution approach (see e.g. [2, 11]) and a splitting of the corresponding wavelet spaces.

Definition 3.3 *For $N, M \in \mathbb{N}$, with $8M|N$, we define spaces*

$$\begin{aligned} V_N^M \ &:= \ \text{span} \ (\{T_k : k = 0, \ldots, N-M;\} \cup \\ &\cup \{\frac{M+k}{2M} T_{N-k} + \frac{M-k}{2M} T_{N+k} : k = 0, \ldots, M-1\}) \ . \end{aligned}$$

Moreover, let

$$\begin{aligned} W^M_{N,\ell} \ := \ \text{span} \ (\{&\frac{M-k}{2M} T_{N+2(\ell-1)M-k} - \frac{M+k}{2M} T_{N+2(\ell-1)M+k} : k = 1, \ldots, M\} \cup \\ \cup \{&\frac{M+k}{2M} T_{N+2\ell M-k} + \frac{M-k}{2M} T_{N+2\ell M+k} : k = 0, \ldots, M-1\}) \end{aligned}$$

for $\ell = 1, ..., \frac{N}{2M} - 2$, and

$$\begin{aligned} W^{2M}_{2N,0} \ := \ \text{span} \ (\{&\frac{M-k}{2M} T_{2N-4M-k} - \frac{M+k}{2M} T_{2N-4M+k} : k = 1, \ldots, M\} \cup \\ \cup \ &\{T_k : k = 2N-3M+1, \ldots, 2N-2M\} \cup \\ \cup \ &\{\frac{2M+k}{4M} T_{2N-k} + \frac{2M-k}{4M} T_{2N+k} : k = 0, \ldots, 2M-1\}) \ . \end{aligned}$$

Easily one can check the following

Lemma 3.4 *The set Π_{N-M} of polynomials up to degree $N-M$ is a subset of V_N^M. The dimensions of the spaces in Definition 3.3 are*

$$\begin{aligned} \dim V_N^M &= N+1 \\ \dim W_{N,\ell}^M &= 2M \quad \textit{for} \quad \ell = 0, ..., \frac{N}{2M} - 2 \,. \end{aligned}$$

Furthermore, we have for $\lambda = 1, \ldots, \frac{N}{2M} - 2$ the orthogonal splittings

$$V_{N+2\lambda M}^M = V_N^M \oplus \bigoplus_{\ell=1}^{\lambda} W_{N,\ell}^M \,,$$

and in particular

$$V_{2N}^{2M} = V_N^M \oplus \left(\bigoplus_{\ell=1}^{\frac{N}{2M}-2} W_{N,\ell}^M \right) \oplus W_{2N,0}^{2M} \,.$$

Now, we can show that the functions defined in Definition 3.1 are a basis of the corresponding spaces which we introduced in Definition 3.3.

Theorem 3.5 *For $N, M \in I\!N$, with $8M|N$, we have*

$$V_N^M = \text{span} \, \{\phi_{N,s}^M : s = 0, \ldots, N\}$$

and

$$W_{N,\ell}^M = \text{span} \, \{\psi_{N,\ell,s}^M : s = 1, \ldots, 2M\}$$

for $\ell = 0, ..., \frac{N}{2M} - 2$.

Proof. In view of Lemma 3.2 and Lemma 3.4 we only have to prove

$$\begin{aligned} \phi_{N,s}^M \in V_N^M & \qquad \text{for all } \; s = 0, \ldots, N, \\ \psi_{N,\ell,s}^M \in W_{N,\ell}^M & \qquad \text{for all } \; s = 1, \ldots, 2M, \; \ell = 0, \ldots, \frac{N}{2M} - 2 \,. \end{aligned}$$

To show that $\phi_{N,s}^M \in V_N^M$ it is sufficient to prove that every term in its expansion (8) lies in this space V_N^M as given in Definition 3.3. This is trivially clear for the terms with Chebyshev polynomials up to degree $N-M$. The linear combination of Chebyshev polynomials of degree greater than $N-M$ can be expressed as

$$\begin{aligned}\sum_{k=N-M+1}^{N+M-1} g_1(\frac{k-N+M}{M})\cos\frac{ks\pi}{N}T_k &= g_1(1)\cos s\pi\; T_N + \\ &+\sum_{k=1}^{M-1}\left(g_1(\frac{M-k}{M})\cos\frac{(N-k)s\pi}{N}T_{N-k} + g_1(\frac{M+k}{M})\cos\frac{(N+k)s\pi}{N}T_{N+k}\right) \\ &= \frac{(-1)^s}{\sqrt{2}}\,T_N + \sum_{k=1}^{M-1}\frac{2(-1)^s\cos\frac{ks\pi}{N}}{\sqrt{2+2(\frac{k}{M})^2}}\left(\frac{M+k}{2M}T_{N-k} + \frac{M-k}{2M}T_{N+k}\right)\;.\end{aligned}$$

The proof that $\psi^M_{N,\ell,s}\in W^M_{N,\ell}$ for all $s=1,\dots,2M,\ \ell=1,\dots,\frac{N}{2M}-2$ differs slightly if ℓ is even or if ℓ is odd. We consider first the case that $\ell=2p$, obtaining from the definition (9) that

$$\psi^M_{N,2p,s} \;=\; \sum_{k=-2M+1}^{2M-1} g_1(\frac{k}{M})\cos\left((M-k)\frac{(2s-1)\pi}{4M}\right)T_{N+(4p-1)M+k}\,.$$

Again, splitting the sum at the values $k=-M,0$ and M and shifting indices, we rewrite

$$\begin{aligned}&\psi^M_{N,2p,s}\\ &= \sum_{k=1}^{M-1}\cos k\frac{(2s-1)\pi}{4M}\left(g_1(\frac{-k-M}{M})T_{N+(4p-1)M-M-k} - g_1(\frac{k-M}{M})T_{N+(4p-1)M-M+k}\right) + \\ &+\;\cos\frac{(2s-1)\pi}{4}g_1(0)T_{N+(4p-1)M} + \\ &+\sum_{k=1}^{M-1}\cos k\frac{(2s-1)\pi}{4M}\left(g_1(\frac{M-k}{M})T_{N+(4p-1)M+M-k} + g_1(\frac{M+k}{M})T_{N+(4p-1)M+M+k}\right).\end{aligned}$$

Now, using the definition of g_1, this can again be rewritten as

$$\begin{aligned}\psi^M_{N,2p,s} = 2\sum_{k=1}^{M-1}&\frac{\cos k\frac{(2s-1)\pi}{4M}}{\sqrt{2+2(\frac{k}{M})^2}}\left(\frac{M-k}{2M}T_{N+(4p-2)M-k} - \frac{M+k}{2M}T_{N+(4p-2)M+k}\right) + \\ &+\cos\frac{(2s-1)\pi}{4}T_{N+(4p-1)M} + \\ &+2\sum_{k=1}^{M-1}\frac{\cos k\frac{(2s-1)\pi}{4M}}{\sqrt{2+2(\frac{k}{M})^2}}\left(\frac{M+k}{2M}T_{N+4pM-k} + \frac{M-k}{2M}T_{N+4pM+k}\right)\;.\end{aligned}$$

To consider the case that $\ell=2p-1$, it suffices to see that (10) reduces to

$$\psi^M_{N,2p-1,s} \;=\; (-1)^s\sum_{k=-2M+1}^{2M-1} g_1(\frac{k}{M})\cos\left((M-k)\frac{(2s-1)\pi}{4M}\right)T_{N+(4p-3)M+k}\,.$$

The rest of the proof is identical to that for the even case.

It remains to show that $\psi^{2M}_{2N,0,s} \in W^{2M}_{2N,0}$. From the Definition 3.1 we get

$$\psi^{2M}_{2N,0,s} \;=\; \sum_{k=-4M+1}^{4M-1} g_0(\frac{k}{2M}) \cos\left((2M-k)\frac{(2s-1)\pi}{8M}\right) T_{2N-2M+k}\,.$$

As in the preceding two arguments, the sum can be split apart at $k = -2M, 0$ and $2M$, and rearranged, giving

$$\begin{aligned}
&\psi^{2M}_{2N,0,s} \\
&= \sum_{k=1}^{2M-1} \cos\frac{(4M+k)(2s-1)\pi}{8M}\left(g_0(\frac{-k-2M}{2M})T_{2N-4M-k} - g_0(\frac{k-2M}{2M})T_{2N-4M+k}\right) + \\
&\quad + \cos\frac{(2s-1)\pi}{4} g_0(0)T_{2N-2M} + \\
&\quad + \sum_{k=1}^{2M-1} \cos k\frac{(2s-1)\pi}{4M}\left(g_0(\frac{2M-k}{2M})T_{2N-k} + g_0(\frac{2M+k}{2M})T_{2N+k}\right)\,.
\end{aligned}$$

Now, appealing to the definition of g_0, we see, as in the two preceding cases, that $\psi^{2M}_{2N,0,s}$ is in fact a linear combination of the functions whose span defines $W^{2M}_{2N,0}$, which concludes the proof. □

In order to deal with (7), we estimate certain L^1-norms and their discrete analogues for the functions $\phi^M_{N,s}$ and $\psi^M_{N,\ell,s}$.

Lemma 3.6 *For all $N, M \in I\!N$, with $8M|N$, and for all $\ell = 0, ..., \frac{N}{2M} - 2$, we have*

$$\max_{0\le s\le N} \langle 1\;,\;|\phi^M_{N,s}|\rangle \;\le\; C\;, \tag{14}$$

$$\max_{1\le s\le 2M} \langle 1\;,\;|\psi^M_{N,\ell,s}|\rangle \;\le\; C\;, \tag{15}$$

$$\left\| \frac{2}{N}\sum_{s=0}^{N} \frac{|\phi^M_{N,s}|}{1+\delta_{s,0}+\delta_{s,N}} \right\|_{C[-1,1]} \;\le\; C\;, \tag{16}$$

and

$$\left\| \frac{1}{M}\sum_{s=1}^{2M} |\psi^M_{N,\ell,s}| \right\|_{C[-1,1]} \;\le\; C\;, \tag{17}$$

where the constants C depends only on the quotient N/M.

Proof. The proof is based essentially on the following inequality for trigonometric polynomials t_n of degree n (see Timan [12], Chap. 4.9.1.(3)). For all $m \in I\!N$ it holds

$$\int_0^{2\pi} |t_n(\theta)| d\theta \;\le\; \sup_\xi \frac{2\pi}{m} \sum_{r=0}^{m-1} |t_n(\xi - \frac{2r\pi}{m})| \;\le\; (1 + \frac{2n\pi}{m}) \int_0^{2\pi} |t_n(\theta)| d\theta \,. \tag{18}$$

Notice that we can replace the supremum over all real ξ by the maximum over all $0 \le \xi \le 2\pi/m$ because the sum in (18) is $2\pi/m$-periodic.

At first we prove (14). By the standard transformation $x = \cos\theta$ we can write

$$\langle 1 \,,\, |\phi_{N,s}^M| \rangle \;=\; \frac{1}{\pi} \int_0^\pi \left| \sum_{k=-2N}^{2N-1} g^\star(\frac{k\pi}{2N}) \cos 2s \frac{k\pi}{2N} \cos k\theta \right| d\theta \,,$$

where the function $g^\star$ is defined by

$$g^\star(x) \;:=\; \begin{cases} 1 & \text{for} \quad |x| \le \frac{(N-M)\pi}{2N} \,, \\ \frac{(N+M)\pi - 2Nx}{\sqrt{2(\pi M)^2 + 2N^2(2x-\pi)^2}} & \text{for} \quad \frac{(N-M)\pi}{2N} < |x| \le \frac{(N+M)\pi}{2N} \,, \\ 0 & \text{for} \quad \frac{(N+M)\pi}{2N} < |x| \le \pi \,. \end{cases}$$

By (18) we conclude

$$\langle 1 \,,\, |\phi_{N,s}^M| \rangle \;\le\; \frac{C}{N} \max_{0 \le \xi \le 1} \sum_{r=-2N}^{2N-1} \left| \sum_{k=-2N}^{2N-1} g^\star(\frac{k\pi}{2N}) \cos 2s \frac{k\pi}{2N} \cos k(\frac{\xi\pi}{2N} + \frac{r\pi}{2N}) \right| \,. \tag{19}$$

Rewriting

$$\begin{aligned} & \cos 2s \frac{k\pi}{2N} \cos k(\frac{\xi\pi}{2N} + \frac{r\pi}{2N}) \\ &= \frac{1}{2} \cos \xi \frac{k\pi}{2N} \left(\cos(2s+r)\frac{k\pi}{2N} + \cos(2s-r)\frac{k\pi}{2N} \right) + \\ &\quad + \frac{1}{2} \sin \xi \frac{k\pi}{2N} \left(\sin(2s-r)\frac{k\pi}{2N} - \sin(2s+r)\frac{k\pi}{2N} \right) \,, \end{aligned}$$

we see that the inner sum in (19) can be understood as a combination of four discrete Fourier coefficients of $g^\star(x)\cos\xi x$ and $g^\star(x)\sin\xi x$. These functions, given for $x \in [-\pi, \pi]$ with fixed $\xi \in [0,1]$, are zero at $\pm\pi$, hence we can continue them 2π-periodically. The first derivatives with respect to x of $g^\star(x)\cos\xi x$ and $g^\star(x)\sin\xi x$ are of bounded variation and possess four jumps in $[-\pi, \pi]$. For the cosine Fourier coefficients a_ν of $g^\star(x)\cos\xi x$, $\nu = |2s \pm r|$, we have (see Zygmund [16], Chap. 2 and 10)

$$|a_\nu| \;=\; \left|\frac{1}{\pi}\int_{-\pi}^{\pi} g^\star(x)\cos\xi x \cos\nu x\,dx\right| \;\le\; C(\nu+1)^{-2}$$

and by aliasing it follows that

$$\begin{aligned}
&\left|\frac{1}{N}\sum_{k=-2N}^{2N-1} g^\star(\frac{k\pi}{2N})\cos\xi\frac{k\pi}{2N}\cos\nu\frac{k\pi}{2N}\right| \qquad (20)\\
&= \left|\frac{1}{2N}\sum_{k=-2N}^{2N-1}\sum_{m=-\infty}^{\infty} a_{|m|}\;\cos m\frac{k\pi}{2N}\cos\nu\frac{k\pi}{2N}\right|\\
&= \left|\sum_{\ell=-\infty}^{\infty} a_{|4N\ell+\nu|} + a_{|4N\ell-\nu|}\right|\\
&\le\; C\max\Big\{(\nu+1)^{-2},\;\; (|4N-\nu|+1)^{-2}\Big\}\,.
\end{aligned}$$

The same is true for the sine Fourier coefficients for $g^\star(x)\sin\xi x$. Hence,

$$\langle 1\;,\;|\phi^M_{N,s}|\rangle \;\le\; C\sum_{r=-2N}^{2N-1}\max\Big\{(|2s\pm r|+1)^{-2},\;\;(|4N-(2s\pm r)|+1)^{-2}\Big\} \;\le\; C'$$

which proves (14). The proof of (15) follows the same lines.

Similar ideas can be applied to show (16) and (17). Let us restrict ourselves to (17) for even $\ell = 2p$. We obtain from (9) that

$$\begin{aligned}
\left\|\frac{1}{M}\sum_{s=1}^{2M}|\psi^M_{N,2p,s}|\,\right\|_{C[-1,1]} &= \max_{0\le\xi\le\pi}\frac{1}{M}\sum_{s=1}^{2M}\left|\sum_{k=-2M+1}^{2M-1} g_p(\frac{k}{M})\times\right.\\
&\quad\left.\times\cos(N+(4p-1)M+k)\frac{(2s-1)\pi}{4M}\cos(N+(4p-1)M+k)\xi\right|\\
&= \max_{0\le\xi\le 2M}\frac{1}{M}\sum_{s=1}^{2M}\left|\sum_{k=-2M}^{2M-1} g_p(\frac{k}{M})\;\Bigg(\cos(N+(4p-1)M+k)\frac{(s+\xi-\frac12)\pi}{2M}+\right.\\
&\quad\left.+\cos(N+(4p-1)M+k)\frac{(s-\xi-\frac12)\pi}{2M}\Bigg)\right|\\
&\le \max_{0\le\xi\le 2M}\frac{1}{2M}\sum_{s=-2M}^{2M-1}\left|\sum_{k=-2M}^{2M-1} g_p(\frac{k}{M})\;\Bigg(\cos(N+(4p-1)M+k)\frac{(s+\xi+\frac12)\pi}{2M}\Bigg)\right|\\
&= \max_{0\le\xi\le 1}\frac{1}{2M}\sum_{s=-2M}^{2M-1}\left|\sum_{m=-3M}^{M-1} g^c(\frac{m\pi}{2M})\cos s\frac{m\pi}{2M} - g^s(\frac{m\pi}{2M})\sin s\frac{m\pi}{2M}\right|,
\end{aligned}$$

where

$$
\begin{aligned}
g^c(\theta) &= g_p(1+\frac{2}{\pi}\theta)\ \cos\Big((\xi+\frac{1}{2})\theta+2\pi\xi(\frac{N}{4M}+p)\Big)\,, \\
g^s(\theta) &= g_p(1+\frac{2}{\pi}\theta)\ \sin\Big((\xi+\frac{1}{2})\theta+2\pi\xi(\frac{N}{4M}+p)\Big)\,.
\end{aligned}
$$

Since the derivatives of g^c and g^s are again of bounded variation, we obtain similar estimates as for (20) which yield (17).

Note that the total variation of the derivative of $g^\star$ in $[0, 2\pi]$ depends on N/M whereas the total variation of the derivative of g^c and g^s in $[0, 2\pi]$ are bounded by an absolute constant. □

4 Continuation of the Proof of Theorem 2.2

It remains to prove (6) and (7).

To show the orthonormality of the polynomials let us note that

$$\text{span}\{p_s : s = 0, \dots, 2^j\} \; = \; V_{2^j}^{2^{j-\eta}}\,. \tag{21}$$

For $j = \eta + 1$ this is clear by definition, and for $j > \eta + 1$ it follows from (11), Lemma 3.4 and Theorem 3.5. Furthermore by (11) and Theorem 3.5, for $j \geq \eta+1,\ \lambda = 0, \dots, 2^{\eta-1}-2$ we have

$$\text{span}\{p_{2^j+(\lambda-1)2^{j-\eta+1}+s} : s = 1, \dots, 2^{j-\eta+1}\} \; = \; W_{2^j,\lambda}^{2^{j-\eta}}\,. \tag{22}$$

Thus, $\langle p_\mu, p_\nu\rangle = \delta_{\mu,\nu}$ follows directly from the results in Section 3. In particular, if p_μ and p_ν are from different spaces $W_{2^j,\lambda}^{2^{j-\eta}}$ or $V_{2^{\eta+1}}^2$, then the orthogonality follows from Lemma 3.4. For p_μ and p_ν from the same space $W_{2^j,\lambda}^{2^{j-\eta}}$, the orthonormality is a consequence of Lemma 3.2. For $p_\mu, p_\nu \in V_{2^{\eta+1}}^2$ we only refer to (2).

Last but not least we have to deal with the approximation result for the partial sums of the orthogonal projection. In order to prove (7), by the usual triangle inequality it is sufficient to prove a reproduction property for polynomials and the uniform boundedness of the norm of the orthogonal projection operator. Since they are of special interest we state these results as an extra theorem.

Theorem 4.1 *Let $\varepsilon > 0$ be given. Then, for arbitrary $\mu \in \mathbb{N}_0$ we have for all q with $\deg q \le \mu(1-\varepsilon)$ that*

$$\sum_{s=0}^{\mu} \langle q\ ,\ p_s \rangle p_s \ = \ q\,. \tag{23}$$

Furthermore, for all $f \in C[-1,1]$

$$\left\| \sum_{s=0}^{\mu} \langle f\ ,\ p_s \rangle p_s \right\|_{C[-1,1]} \ \le \ C\,\|f\|_{C[-1,1]}\,. \tag{24}$$

PROOF. In view of the reproduction property with respect to the orthogonal system of the Chebyshev polynomials note that (23) holds true for

$$\begin{aligned} q \in \Pi_{\mu} \quad & \text{if} \quad \mu < 2^{\eta+1}-1\,, \\ q \in \Pi_{\mu-1} \quad & \text{if} \quad \mu = 2^{\eta+1}-1\,, \\ q \in \Pi_{\mu-2} \quad & \text{if} \quad \mu = 2^{\eta+1}\,. \end{aligned}$$

Furthermore, with $\mu = 2^j + (\lambda-1)2^{j-\eta+1} + s$ as in (4) it holds that $p_\mu \in W_{2^j,\lambda}^{2^{j-\eta}}$, and by the definition of V_N^M and Lemma 3.4, we have

$$\Pi_{2^j+\lambda 2^{j-\eta+1}-3\cdot 2^{j-\eta}} \subset V_{2^j+(\lambda-1)2^{j-\eta+1}}^{2^{j-\eta}} \subset \ \text{span}\ \{p_s : s = 0,\ldots,\mu\}\,.$$

Hence, we have (23) for all q with $\deg q \le \mu \cdot d$ where

$$\begin{aligned} d &= \min\left(\frac{2^{\eta+1}-2}{2^{\eta+1}}, \frac{2^j+\lambda 2^{j-\eta+1}-3\cdot 2^{j-\eta}}{2^j+(\lambda-1)\cdot 2^{j-\eta+1}+s}\right) \\ &= \min\left(1-\frac{2}{2^{\eta+1}}, 1-\frac{2^{j-\eta}+s}{2^j+(\lambda-1)\cdot 2^{j-\eta+1}+s}\right) \\ &> \min\left(1-\frac{1}{2^{\eta}}, 1-\frac{3}{2^{\eta}-2}\right) \\ &> 1-\varepsilon\,. \end{aligned}$$

Now we prove (24). Let us first mention the case $\mu \le 2^{\eta+1}$ where we use the classical estimate for the Lebesgue constant of the Fourier-Chebyshev sum

$$\left\| \sum_{s=0}^{\mu} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} \ <\ C \log \mu\, \|f\|_{C[-1,1]} \ <\ C \log 2^{\eta}\, \|f\|_{C[-1,1]} \ <\ C\, \|f\|_{C[-1,1]}\ .$$

For $\mu > 2^{\eta+1}$, we split the partial sum on the left-hand side. For $\lambda > 0$ we write

$$\begin{aligned}
\left\| \sum_{s=0}^{\mu} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} \ \le\ & \left\| \sum_{s=0}^{2^j} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} + \\
& + \sum_{\ell=1}^{\lambda-1} \left\| \sum_{s=2^j+(\ell-1)2^{j-\eta+1}+1}^{2^j+\ell 2^{j-\eta+1}} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} + \\
& + \left\| \sum_{s=2^j+(\lambda-1)2^{j-\eta+1}+1}^{\mu} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} .
\end{aligned}$$

If $\lambda = 0$, then $\mu \le 2^j$. Here we estimate

$$\left\| \sum_{s=0}^{\mu} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} \ \le\ \left\| \sum_{s=0}^{2^j} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} + \left\| \sum_{s=\mu+1}^{2^j} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} .$$

From (21) and (22) with $N = 2^j$ and $M = 2^{j-\eta}$ one concludes that this splitting coincides with the splitting of the orthogonal projection to the orthogonal projections of f into V_N^M and $W_{N,\ell}^M$, $\ell = 1, \ldots, \lambda - 1$ and the partial sum of the orthogonal projection into the last subspace $W_{N,\lambda}^M$, respectively. Hence we can write for $\lambda > 0$

$$\begin{aligned}
\left\| \sum_{s=0}^{\mu} \langle f\ ,\ p_s \rangle\, p_s \right\|_{C[-1,1]} \ \le\ & \left\| \frac{2}{N} \sum_{s=0}^{N} \frac{1}{1+\delta_{s,0}+\delta_{s,N}} \langle f\ ,\ \phi_{N,s}^M \rangle\, \phi_{N,s}^M \right\|_{C[-1,1]} + \\
& + \sum_{\ell=1}^{\lambda-1} \left\| \frac{1}{M} \sum_{s=2^j+(\ell-1)2^{j-\eta+1}}^{2^j+\ell 2^{j-\eta+1}} \langle f\ ,\ \psi_{N,\ell,s}^M \rangle\, \psi_{N,\ell,s}^M \right\|_{C[-1,1]} + \\
& + \left\| \frac{1}{M} \sum_{s=2^j+(\lambda-1)2^{j-\eta+1}+1}^{\mu} \langle f\ ,\ \psi_{N,\lambda,s}^M \rangle\, \psi_{N,\lambda,s}^M \right\|_{C[-1,1]} \\
\le\ & \|f\|_{C[-1,1]} \left(\max_{0\le s\le N} \langle 1\ ,\ |\phi_{N,s}^M| \rangle \left\| \frac{2}{N} \sum_{s=0}^{N} \frac{|\phi_{N,s}^M|}{1+\delta_{s,0}+\delta_{s,N}} \right\|_{C[-1,1]} + \right. \\
& \left. + \sum_{\ell=1}^{\lambda} \max_{1\le s\le 2M} \langle 1\ ,\ |\psi_{N,\ell,s}^M| \rangle \left\| \frac{1}{M} \sum_{s=1}^{2M} |\psi_{N,\ell,s}^M| \right\|_{C[-1,1]} \right),
\end{aligned}$$

and analogously for $\lambda = 0$

$$\left\| \sum_{s=0}^{\mu} \langle f \ , \ p_s \rangle \ p_s \right\|_{C[-1,1]} \leq \|f\|_{C[-1,1]} \left(\max_{0 \leq s \leq N} \langle 1 \ , \ |\phi_{N,s}^M| \rangle \ \left\| \frac{2}{N} \sum_{s=0}^{N} \frac{|\phi_{N,s}^M|}{1 + \delta_{s,0} + \delta_{s,N}} \right\|_{C[-1,1]} + \right.$$
$$\left. + \max_{0 \leq s < 2M} \langle 1 \ , \ |\psi_{N,0,s}^M| \rangle \ \left\| \frac{1}{M} \sum_{s=1}^{2M} |\psi_{N,0,s}^M| \right\|_{C[-1,1]} \right) .$$

Now, applying Lemma 3.6 we obtain (24), and finally (7). □

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