From graphs to metrics. Uniqueness and rigidity

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The main results of the present talk were proved in:
O. Dovgoshey and R. Shanin, *Uniqueness of best proximity pairs and rigidity of semimetric spaces*, arXiv:2201.04380v2[mathGN], 2 April 2022.

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Let (X, d) be a metric space. Recall that a set $A \subseteq X$ is said to be *proximinal* in (X, d) if, for every $x \in X$, there exists $a_0 \in A$ such that

$$d(x, a_0) = \inf\{d(x, a) \colon a \in A\}.$$

The point a_0 is called a *best approximation* to x in A.

A bipartite graph G with given parts A and B is *proximinal* if there exists a metric space (X, d) such that A and B are disjoint proximinal subsets of X, and the equivalence

$$\big(\{a, b\} \in E(G)\big) \Leftrightarrow \big(d(a, b) = \operatorname{dist}(A, B)\big)$$

is valid for all $a \in A$ and every $b \in B$. In this case we write $G = G_{X,d}(A, B)$ and say that G is proximinal for the space (X, d).

A metric space (X, d) is *strongly rigid* if

$$d(x, y) = d(u, v) \neq 0$$

implies $\{x, y\} = \{u, v\}$ for all x, y, u, $v \in X$.

A metric space is *weakly rigid* if every three-point subspace of this space is strongly rigid.

In the previous seminar we discussed

Theorem 1

Let G be a bipartite graph with fixed parts A and B. Then following statements are equivalent.

(i) G is proximinal for a strongly rigid metric space.

(ii) The following conditions are simultaneously fulfilled:

(ii_1) The inequalities $|E(G)| \leqslant 1$ and $|V(G)| \leqslant \mathfrak{c}$ hold, where $\mathfrak{c}$ is the cardinality of the continuum.

(ii_2) If G is a null graph, then A and B are infinite.

Definition 2

A metric space (X, d) belongs to the class **UBPP** (Unique Best Proximity Pair) if the inequality $|E(G)| \leq 1$ holds whenever G is a proximinal graph for (X, d).

Theorem 1 implies that every strongly rigid metric space is a **UBPP**-space.

The main purpose of the present talk is to describe the structure of UBP-spaces.

We will do it using the concepts of digraph and weak similarity of metric spaces.

A *digraph* D is a nonempty set $V(D)$ of *vertices* together with a (possible empty) set $E(D)$ of ordered pairs of distinct vertices of D.

A digraph D_1 is isomorphic to a digraph D_2 if there exists a bijection $f\colon V(D_1) \to V(D_2)$ such that $(u, v) \in E(D_1)$ if and only if $(f(u), f(v)) \in E(D_2)$.

Definition 3

Let (X, d) be a finite metric space with $|X| \geqslant 2$. Then we write $Di = Di_X$ for the digraph with the vertex set $V(Di)$, consisting of all two-point subsets of X and such that, for $u = \{p, q\} \in V(Di)$ and $v = \{l, m\} \in V(Di)$, the relationship

$$(u, v) \in E(Di)$$

holds if and only if $d(p, q) > d(l, m)$ and, for every $\{x, y\} \in V(Di)$, the double inequality

$$d(p, q) \geqslant d(x, y) \geqslant d(l, m)$$

implies either $\{x, y\} = \{p, g\}$ or $\{x, y\} = \{l, m\}$.

Example 4

Let (X,d), $X = \{p,q,l,m\}$ be the rectangle depicted below, then (X,d) is weakly rigid but $(X,d) \notin$ **UBPP**.

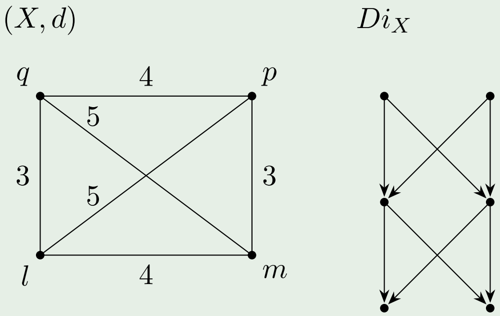

Figure 1. The rectangle (X,d) and its digraph Di_X.

Example 5

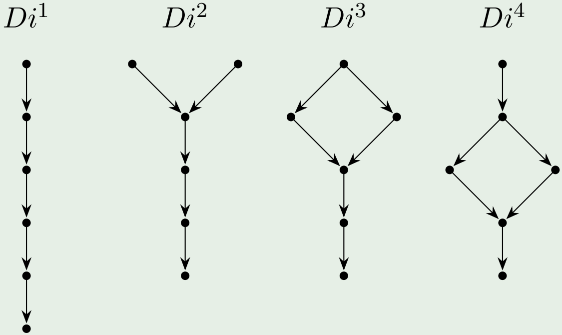

Figure 2. The digraphs Di of all four-point **UBPP**-spaces.

Example 6

The digraphs Di_{X^*} and Di_{Z^*}, Di_{Y^*} of the four-point metric spaces (X^*, ρ^*), (Y^*, Δ^*) and (Z^*, δ^*) are isomorphic to Di^4.

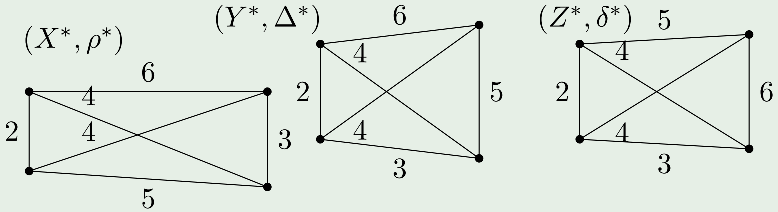

Figure 3.
$(X^*, \rho^*) \notin$ **UBPP**, $(Y^*, \Delta^*) \in$ **UBPP** and $(Z^*, \delta^*) \in$ **UBPP**.

For every metric space (X, d), we denote by $D(X)$ the set of all distances between points of X,

$$D(X) = \left\{ d(x, y) \colon x, y \in X \right\}.$$

Definition 7

Let (X, d) and (Y, ρ) be metric spaces. A mapping $\Phi \colon X \to Y$ is a *weak similarity* if Φ is bijective and there is a bijective strictly increasing function $\psi \colon D(Y) \to D(X)$ such that the equality

$$d(x, y) = \psi\left(\rho\big(\Phi(x), \Phi(y)\big)\right)$$

holds for all x, $y \in X$.
We say that two metric spaces are *weakly similar* is there is a weak similarity of these spaces.

Example 8

Let (X, d) and (Y, ρ) be metric spaces. A bijective mapping $\Phi\colon X \to Y$ is a *similarity*, if there is $r > 0$, the *ratio* of Φ, such that

$$\rho\big(\Phi(x), \Phi(y)\big) = rd(x, y)$$

for all x, $y \in X$. Every similarity is a weak similarity.

Theorem 9

Let (X, d) be a metric space. Then the following statements are equivalent.

(i) $(X, d) \in \mathbf{UBPP}$.

(ii) *(X, d) is a weakly rigid, and, for every four-point $Y \subseteq X$, the digraph Di_Y is isomorphic to the one of the digraphs Di^1, Di^2, Di^3, Di^4, and (X, d) does not contain any four-point subspace, which is weakly similar to the metric space (X^*, ρ^*) depicted in Figure 4.*

Theorem 9 admits the equivalent reformulation.

Theorem 10

Let (X, d) be a weakly rigid metric space. Then $(X, d) \in \mathbf{UBPP}$ if and only if every four-point subspace Y of (X, d) satisfies at least one of the conditions:

(i) *Di_Y is isomorphic to the one of the digraphs Di^1, Di^2, Di^3.*

(ii) *Y is weakly similar to (Y^*, Δ^*).*

(iii) *Y is weakly similar to (Z^*, δ^*).*

Corollary 11

Let (X, d) be a metric space with $|X| \neq 3$. Then $(X, d) \in$ **UBPP** *if and only if we have $(Y, d|_{Y\times Y}) \in$* **UBPP** *for every $Y \subseteq X$ with $|Y| \leqslant 4$.*

Open problems

Problem 12

Describe the structure of metric spaces (X, d) *for which every* $x \in X$ *has at most* k *best approximations in every proximinal* $A \subseteq X$ *with a given integer* $k \geqslant 2$.

Problem 13

Describe the structure of metric spaces (X, d) *for which every proximinal graph* $G_{X,d}(A, B)$ *has at most* k *edges with given integer* $k \geqslant 2$.

We now turn our attention to functions that preserve metrics.

Definition 14

A function $f : [0, \infty) \to [0, \infty)$ is called metric preserving if for every metric space (X, ρ) the composition $f \circ \rho$ is still a metric on X.

For general metric spaces, the structure of metric preserving functions has been firstly studied by W. Wilson in 1935, "On certain type of continuous transformations of metric spaces", *Amer. J. Math.*, **57**, 62–68, and systematically discussed by J. Doboš, *Metric Preserving Functions*, Štroffec, Košice, 1998.

Example 15

The function $f : [0, \infty) \to [0, \infty)$,

$$f(t) = \frac{t}{1+t},$$

is a canonical example of a metric preserving function. It is easy to prove that f also preserves the proximinal graphs: If a bipartite graph G with fixed parts A, B is proximinal for a metric space (X, d),

$$G = G_{X,d}(A, B),$$

then G is still proximinal the for metric space (X, ρ),

$$G = G_{X,\rho}(A, B)$$

with $\rho(x, y) = \frac{d(x,y)}{1+d(x,y)}$, $x, y \in X$.

The following simple theorem is new.

Theorem 16

The following conditions are equivalent for every function $f : [0, \infty) \to [0, \infty)$.

(i) *The function f preserves metrics and proximinal graphs.*

(ii) *The equality $f(0) = 0$ holds, f is strictly increasing, and we have*

$$f(x + y) \leq f(x) + f(y)$$

for all $x, y \in [0, \infty)$.

Thank you for your attention!