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The t-intersection Problem in the Truncated Boolean Lattice

RUDOLF AHLSWEDE†, CHRISTIAN BEY, KONRAD ENGEL AND LEVON H. KHACHATRIAN

Let $I(n,t)$ be the class of all t-intersecting families of subsets of $[n]$ and set $I_k(n,t) = I(n,t) \cap 2^{\binom{[n]}{k}}$, $I_{\leq k}(n,t) = I(n,t) \cap 2^{\binom{[n]}{\leq k}}$.

After the maximal families in $I(n,t)$ [13] and in $I_k(n,t)$ [1, 9] are known we study now maximal families in $I_{\leq k}(n,t)$. We present a conjecture about the maximal cardinalities and prove it in several cases.

More generally cardinalities are replaced by weights and asymptotic estimates are given.

Analogous investigations are made for $I(n,t) \cap C(n,s)$, where $C(n,s)$ is the class of all s-cointersecting families of subsets of $[n]$. In particular we establish an asymptotic form of a conjecture by Bang *et al.* [4].

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1. INTRODUCTION AND NOTATION

Let $\mathbb{N}$ be the set of natural numbers, $[n] := \{1, \ldots, n\}$, and for $i, j \in \mathbb{N}$, $i < j$, let $[i, j] := \{i, i+1, \ldots, j\}$. Let $2^{[n]}$ be the family of all subsets of $[n]$. Also, let

$$\binom{[n]}{k} := \{X \subseteq [n] : |X| = k\}, \qquad \binom{[n]}{\leq k} := \{X \subseteq [n] : |X| \leq k\},$$
$$\binom{[n]}{\geq k} := \{X \subseteq [n] : |X| \geq k\}.$$

A family $\mathcal{F} \subseteq 2^{[n]}$ is called t-*intersecting* (resp. s-*cointersecting*) if, for all $X, Y \in \mathcal{F}$, $|X \cap Y| \geq t$ (resp. $|X \cup Y| \leq n-s$). Let $I(n,t)$ (resp. $C(n,s)$) be the class of all t-intersecting (resp. s-cointersecting) families of subsets of $[n]$. Furthermore, let

$$I_k(n,t) := I(n,t) \cap 2^{\binom{[n]}{k}}, \qquad I_{\leq k}(n,t) := I(n,t) \cap 2^{\binom{[n]}{\leq k}},$$

i.e., the class of t-intersecting families whose members have size equal to k resp. not greater than k, and let $I_{\geq k}(n,t)$, $C_{\leq k}(n,s)$, $C_{\geq k}(n,s)$ be defined analogously.

For a class $\mathcal{K}$ of families, let

$$M(\mathcal{K}) := \max\{|\mathcal{F}| : \mathcal{F} \in \mathcal{K}\}.$$

More generally, if there is given a weight function $\omega : 2^{[n]} \to \mathbb{R}_+$ (the set of all nonnegative reals), let for $\mathcal{F} \subseteq 2^{[n]}$

$$\omega(\mathcal{F}) := \sum_{X \in \mathcal{F}} \omega(X)$$

and

$$M(\mathcal{K}, \omega) := \max\{\omega(\mathcal{F}) : \mathcal{F} \in \mathcal{K}\}.$$

In this paper we study the numbers $M(\mathcal{K})$ for

$$\mathcal{K} \in \{I_{\leq k}(n,t), I_{\geq k}(n,t), C_{\leq k}(n,s), C_{\geq k}(n,s), I(n,t) \cap C_(n,s)\}.$$

†To whom correspondence should be addressed.

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2. RESULTS

First of all, by considering complements

$$M(C_{\geq k}(n,s)) = M(I_{\leq n-k}(n,s)),$$
$$M(C_{\leq k}(n,s)) = M(I_{\geq n-k}(n,s)),$$

so that only three of the five numbers are of interest.

Let, for $r = 0, \dots, \lfloor \frac{n-t}{2} \rfloor$,

$$S(n,t,r) := \{X \in 2^{[n]} : |X \cap [t+2r]| \geq t + r\},$$
$$S_k(n,t,r) := S(n,t,r) \cap \binom{[n]}{k},$$
$$S_{\leq k}(n,t,r) := S(n,t,r) \cap \binom{[n]}{\leq k},$$

and let $S_{\geq k}(n,t,r)$ be defined analogously. By construction, these families are t-intersecting.

The following two results are basic for our investigation.

THEOREM 1 (KATONA [13]). *We have*

$$M(I(n,t)) = \left| S\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right|.$$

THEOREM 2 (AHLSWEDE, KHACHATRIAN [1]). *We have*

$$M(I_k(n,t)) = \max\left\{|S_k(n,t,r)| : r = 0, \dots, \left\lfloor \frac{n-t}{2} \right\rfloor\right\}.$$

Moreover, for $n > 2k - t$, the optimal r is given by

$$\frac{(k-t+1)(t-1)}{n-2k+2t-2} - 1 \leq r \leq \frac{(k-t+1)(t-1)}{n-2k+2t-2}.$$

An easy consequence of Theorem 1 is the following (cf. [6, 8]):

THEOREM 3. *Let $\omega(X) = \omega(Y)$ for all $X, Y \subseteq [n]$ with $|X| = |Y|$ and let $\omega(X) \leq \omega(Y)$ if $|X| + |Y| = n + t - 1$, $|X| \leq |Y|$. Then*

$$M(I(n,t),\omega) = \omega\left(S\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right).$$

Setting

$$\omega(X) := \begin{cases} 1 & \text{if } |X| \geq k \\ 0 & \text{otherwise} \end{cases}$$

we obtain immediately from Theorem 3:

COROLLARY 4. *We have*

$$M(I_{\geq k}(n,t)) = \left| S_{\geq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right|.$$

The determination of $M(I_{\leq k}(n,t))$ is more difficult and, up to now, we can provide only partial results.

PROPOSITION 5. *We have*

$$M(I_{\leq k}(n,1)) = |S_{\geq k}(n,t,0)|.$$

Indeed, this follows easily using complements and the Erdős–Ko–Rado theorem [9]. Hence we suppose throughout $t \geq 2$ when studying $I_{\leq k}(n,t)$.

The following question was the starting point of our investigations:

PROBLEM 6. For which numbers k do we have

$$M(I_{\leq k}(n,t)) = \left|S_{\leq k}\left(n,t,\left\lfloor \frac{n-t}{2} \right\rfloor\right)\right|? \tag{1}$$

Concerning this question we may clearly suppose that $k \geq \lfloor \frac{n+t}{2} \rfloor$ because otherwise $S_{\leq k}(n,t,\lfloor \frac{n-t}{2} \rfloor) = \emptyset$. Problem 6 is answered essentially by the following results:

THEOREM 7. *Let t and c be fixed constants and let $k \leq \frac{n+t}{2} + c\sqrt{n}$. Then* (1) *does not hold if n is large enough.*

THEOREM 8. *Let t be fixed and $k \geq \frac{n+t}{2} + \sqrt{\log n}\sqrt{n}$. Then* (1) *holds if n is large enough.*

THEOREM 9. *Let c be fixed constant and let $k \leq \frac{n+t}{2} + c$. Then there exists $\delta > 0$ such that for $t \leq \delta n$ and n sufficiently large* (1) *does not hold.*

THEOREM 10. *Let $\delta > 0$ be fixed constant and let $t \geq \delta n$. Then there exists $c > 0$ such that for $k \geq \frac{n+t}{2} + c$ and n sufficiently large* (1) *holds.*

Concerning the complete determination of $M(I_{\leq k}(n,t))$ we have the following conjecture:

CONJECTURE 11. *If $k < \frac{n+t}{2}$, then*

$$M(I_{\leq k}(n,t)) = \max\left\{|S_{\leq k}(n,t,r)| : r = 0,\dots,\left\lfloor \frac{n-t}{2} \right\rfloor\right\}. \tag{2}$$

This conjecture is supported by the following results.

THEOREM 12. *Let t and $0 < \epsilon < \frac{1}{2}$ be fixed constants and $k \leq (\frac{1}{2} - \epsilon)n$. Then* (2) *holds for sufficiently large n.*

THEOREM 13. *Let $t = \tau n + o(n)$ and $k = \kappa n + o(n)$ with $0 < \tau < \kappa < \frac{1+\tau}{2}$. Then, as $n \to \infty$*

$$M(I_{\leq k}(n,t)) \sim \max\left\{|S_{\leq k}(n,t,r)| : r = 0,\dots,\left\lfloor \frac{n-t}{2} \right\rfloor\right\}.$$

Studying $M(I(n,t) \cap C(n,s))$ one can clearly suppose throughout that $t + s \leq n$. Given n, t, s and $r \in \{0,\dots,\lfloor \frac{n-t-s}{2} \rfloor\}$, always let

$$q := \left\lfloor \frac{n-t-s}{2} \right\rfloor - r.$$

Note that

$$(t+2r) + (s+2q) = \begin{cases} n & \text{if } 2 \mid n-s-t \\ n-1 & \text{otherwise.} \end{cases}$$

Let, for $r = 0,\dots,\lfloor \frac{n-t-s}{2} \rfloor$,

$$S(n,t,s,r) := \{X \in 2^{[n]} : |X \cap [t+2r]| \geq t+r \text{ and } |X \cap [n-s-2q+1,n]| \leq q\}.$$

Obviously, these families are t-intersecting and s-cointersecting. Verifying a conjecture of Katona, Frankl [10] proved:

THEOREM 14. *We have*

$$M(I(n,1) \cap C(n,s)) = |S(n,1,s,0)|.$$

Moreover, Frankl [11] and Bang *et al*. [4] propose:

CONJECTURE 15. *We have*

$$M(I(n,t) \cap C(n,s)) = \max\left\{|S(n,t,s,r)| : r = 0, \dots, \left\lfloor \frac{n-t-s}{2} \right\rfloor\right\}.$$

In [4] this conjecture is proved for $n - t - s \leq 3$.
From Theorem 1 one easily obtains that for fixed t

$$M(I(n,t)) \sim 2^{n-1} \qquad \text{as } n \to \infty.$$

This gives, applying in a standard way Kleitman's inequality (cf. [7, p. 266]):

PROPOSITION 16. *Let t and s be fixed and let $n \to \infty$. Then*

$$M(I(n,t) \cap C(n,s)) \sim 2^{n-2} \sim \max\left\{|S(n,t,s,r)| : r = 0, \dots, \left\lfloor \frac{n-t-s}{2} \right\rfloor\right\}.$$

In addition, we have the following result:

THEOREM 17. *Let $t = \tau n + o(n)$, $s = \sigma n + o(n)$, $\tau, \sigma > 0$, $\tau + \sigma < 1$ and $n \to \infty$. Then*

$$M(I(n,t) \cap C(n,s)) \sim \max\left\{|S(n,t,s,r)| : r = 0, \dots, \left\lfloor \frac{n-t-s}{2} \right\rfloor\right\}.$$

Thus Conjecture 15 is supported by Proposition 16 and Theorem 17.

3. SHORT PROOFS FOR RESULTS CONCERNING $I_{\leq k}(n,t)$

PROOF OF THEOREM 7. It is easy to see that (1) holds for some k if it holds for some k' with $k' < k$ (see Lemma 19). Hence it is sufficient to prove the assertion for

$$k = \left\lceil \frac{n+t}{2} + c\sqrt{n} \right\rceil.$$

We use the well-known fact that for constants a, b (with $a < b$) and for $n \to \infty$

$$\sum_{\frac{n}{2}+\frac{1}{2}\sqrt{n}a+o(\sqrt{n}) \leq j \leq \frac{n}{2}+\frac{1}{2}\sqrt{n}b+o(\sqrt{n})} \binom{n}{j} \sim (\Phi(b) - \Phi(a))2^n \tag{3}$$

uniformly in $a, b \in \mathbb{R}$, where Φ is the Gaussian distribution. Since

$$\sum_{i=\lfloor\frac{n+t}{2}\rfloor+1}^{k} \binom{n}{i} \leq \left|S_{\leq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right| \leq \sum_{i=\lfloor\frac{n+t}{2}\rfloor}^{k} \binom{n}{i}$$

we have

$$\left|S_{\leq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right| \sim (\Phi(2c) - \Phi(0))2^n = \left(\Phi(2c) - \frac{1}{2}\right)2^n. \tag{4}$$

Now choose $r := \lfloor n^{\frac{1}{4}} \rfloor$. From (3) it follows that

$$\sum_{j=0}^{k-i} \binom{n-t-2r}{j} \sim \Phi(2c) 2^{n-t-2r}$$

uniformly in $i \in [t+r, t+2r]$ and that

$$\sum_{i=t+r}^{t+2r} \binom{t+2r}{i} \sim \Phi(0) 2^{t+2r}.$$

Consequently,

$$\begin{aligned} |S_{\leq k}(n,t,r)| &= \sum_{i=t+r}^{t+2r} \binom{t+2r}{i} \sum_{j=0}^{k-i} \binom{n-t-2r}{j} \\ &\sim \Phi(0) 2^{t+2r} \Phi(2c) 2^{n-t-2r} = \tfrac{1}{2} \Phi(2c) 2^n. \end{aligned} \tag{5}$$

Since $\Phi(2c) - \frac{1}{2} < \frac{1}{2}\Phi(2c)$ we have by (4) and (5) for sufficiently large n,

$$\left| S_{\leq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right) \right| < |S_{\leq k}(n,t,r)|. \qquad \square$$

PROOF OF THEOREM 9. Analogously to the proof of Theorem 7 we prove the assertion only for

$$k = \left\lceil \frac{n+t}{2} + c \right\rceil.$$

W.l.o.g. we may assume that c is an integer. Moreover, we suppose that $2 \mid n+t$. If $2 \nmid n+t$ the proof can be modified in a straightforward way. We have $k = \frac{n+t}{2} + c$ and put $d := 3(c+2)^2$. Note that for constant integers a and b

$$\frac{\binom{n-a}{\ell}}{\binom{n}{\ell+b}} \sim (1-\ell/n)^a \left(\frac{\ell/n}{1-\ell/n} \right)^b. \tag{6}$$

Let $\tau := \frac{t}{n}$. We take $r := \frac{n-t}{2} - d$ and compare $|S_{\leq k}(n,t,r)|$ with $|S_{\leq k}(n,t,\lfloor \frac{n-t}{2} \rfloor)|$. We have (with $t+r+c+d = k$)

$$|S_{\leq k}(n,t,r)| = \sum_{i=0}^{c+d} \binom{n-2d}{t+r+i} \sum_{j=0}^{c+d-i} \binom{2d}{j}.$$

Using (6) we obtain

$$\frac{|S_{\leq k}(n,t,r)|}{\binom{n}{(n+t)/2}} \sim \left(\frac{1-\tau}{2} \right)^{2d} \sum_{i=0}^{c+d} \left(\frac{1+\tau}{1-\tau} \right)^{d-i} \sum_{j=0}^{c+d-i} \binom{2d}{j}.$$

Analogously,

$$\left| S_{\leq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right) \right| = \sum_{j=0}^{c} \binom{n}{(n+t)/2 + j},$$

$$\frac{\left| S_{\leq k}\left(n, t, \lfloor \frac{n-t}{2} \rfloor\right) \right|}{\binom{n}{(n+t)/2}} \sim \sum_{j=0}^{c} \left(\frac{1+\tau}{1-\tau} \right)^{-j}.$$

For the proof it is enough to show that there are $\epsilon, \delta > 0$ such that for $\tau \le \sigma$, independently of n,

$$\left(\frac{1-\tau}{2}\right)^{2d} \sum_{i=0}^{c+d} \left(\frac{1+\tau}{1-\tau}\right)^{d-i} \sum_{j=0}^{c+d-i} \binom{2d}{j} \ge \sum_{j=0}^{c} \left(\frac{1+\tau}{1-\tau}\right)^{-j} + \epsilon \tag{7}$$

since then for sufficiently large n and $t \le \tau n$

$$|S_{\le k}(n, t, r)| > \left| S_{\le k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right|.$$

Both sides of (7) are continuous functions of τ. Hence it is enough to consider $\tau = 0$ and to prove

$$L := \sum_{i=0}^{c+d} \sum_{j=0}^{c+d-i} \binom{2d}{j} > (c+1)2^{2d} =: R. \tag{8}$$

Let $a \in \{0, \ldots, c-1\}$ and consider on the LHS of (8) the terms with $i = a$ and $i = 2c - a$. We have

$$\begin{aligned}\sum_{j=0}^{c+d-a} \binom{2d}{j} + \sum_{j=0}^{c+d-(2c-a)} \binom{2d}{j} &= \sum_{j=0}^{c+d-a} \binom{2d}{j} + \sum_{j=0}^{d-c+a} \binom{2d}{2d-j} \\ &= \sum_{j=0}^{c+d-a} \binom{2d}{j} + \sum_{j=c+d-a}^{2d} \binom{2d}{j} \\ &> 2^{2d}.\end{aligned}$$

For $i = c$,

$$\sum_{j=0}^{c+d-i} \binom{2d}{j} = \frac{1}{2}2^{2d} + \frac{1}{2}\binom{2d}{d}.$$

Consequently, we have the following estimation for the LHS of (8):

$$L > \left(c + \frac{1}{2}\right)2^{2d} + \frac{1}{2}\binom{2d}{d} + \sum_{i=2c+1}^{c+d} \sum_{j=0}^{c+d-i} \binom{2d}{j}. \tag{9}$$

For $i \ge 2c + 1$,

$$\begin{aligned}\sum_{j=0}^{c+d-i} \binom{2d}{j} &= \sum_{j=0}^{d} \binom{2d}{j} - \sum_{j=c+d-i+1}^{d} \binom{2d}{j} \\ &> \frac{1}{2}2^{2d} + \frac{1}{2}\binom{2d}{d} - (i-c)\binom{2d}{d} \\ &= \frac{1}{2}2^{2d} - \left(i - c - \frac{1}{2}\right)\binom{2d}{d}.\end{aligned}$$

Considering in (8) only the terms with $i = 2c + 1, 2c + 2, 2c + 3$ gives

$$L > (c+1)2^{2d} + 2^{2d} - (3c+4)\binom{2d}{d}.$$

Accordingly, $L > R$ (i.e., (7) holds) if

$$2^{2d} > \binom{2d}{d}(3c+4). \tag{10}$$

It is well-known (cf. [12, p. 283]) that

$$\binom{2d}{d} \le \frac{2^{2d}}{\sqrt{3d+1}}.$$

Hence (10) holds if $\sqrt{3d+1} > 3c+4$. Indeed (using $d = 3(c+2)^2$), $\sqrt{3d+1} > \sqrt{9(c+2)^2} = 3(c+2) > 3c+4$. □

4. Asymptotic Estimates of $M(I_{\le k}(n,t))$ and $M(I(n,t) \cap C(n,s))$

Proof of Theorem 13. For any family $\mathcal{F}$ we use the notation

$$\mathcal{F}_h := \{X \in \mathcal{F} : |X| = h\}.$$

Let $\mathcal{F} \in I_{\le k}(n,t)$. Clearly,

$$|\mathcal{F}| = \sum_{h=0}^{k} |\mathcal{F}_h|. \tag{11}$$

First we estimate each $|\mathcal{F}_h|$. In the following the maximum is always extended over $r \in \{0, \dots, \lfloor \frac{n-t}{2} \rfloor\}$. By Theorem 2,

$$\begin{aligned}
|\mathcal{F}_h| &\le \max\{|S_h(n,t,r)|\} = \max\left\{\sum_{i=0}^{r} \binom{t+2r}{r-i}\binom{n-t-2r}{h-t-r-i}\right\} \\
&\le \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{h-t-r}\sum_{i=0}^{\infty}\left(\frac{r}{t+r+1}\frac{h-t-r}{n-h-r+1}\right)^i\right\} \\
&\le \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}\left(\frac{k-t-r}{n-k-r+1}\right)^{k-h}\frac{1}{1-\frac{r}{t+r+1}\frac{k-t-r}{n-k-r+1}}\right\}. \qquad (12)
\end{aligned}$$

We will see that almost all numbers $|\mathcal{F}_h|$ can be neglected. Only the values $|\mathcal{F}_h|$ with h near to k give an essential contribution. Clearly, it is enough to extend the maximum only over $r \in \{0, \dots, k-t\}$. Then

$$\frac{r}{t+r+1} \le \frac{k-t}{k+1} = 1 - \frac{\tau}{\kappa} + o(1).$$

Moreover, for large n, $k-t-r < n-k-r+1$, hence

$$\frac{k-t-r}{n-k-r+1} \le \frac{k-t}{n-k+1} = \frac{\kappa-\tau}{1-\kappa} + o(1) < 1.$$

Choose α such that $\frac{\kappa-\tau}{1-\kappa} < \alpha < 1$. Then, for any $\epsilon > 0$ and any h with $h \le k - \epsilon n$,

$$|\mathcal{F}_h| \le \frac{1}{(1-\tau/\kappa)\alpha}\alpha^{\epsilon n}\max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}\right\}$$

and

$$\sum_{h \le k-\epsilon n} |\mathcal{F}_h| \le \frac{1}{(1-\tau/\kappa)\alpha} n\alpha^{\epsilon n}\max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}\right\}. \tag{13}$$

We put $\epsilon := n^{-\frac{1}{2}}$. Now let h be near to k, i.e., $k - h \le \epsilon n$. By Theorem 2, $\max\{|S_h(n,t,r)|\}$ is attained at some $r = r(k)$ with

$$\frac{(\kappa - \epsilon - \tau)\tau n}{1 - 2\kappa + 2\epsilon + 2\tau} - o(n) \le r \le \frac{(\kappa - \tau)\tau n}{1 - 2\kappa + 2\tau} + o(n).$$

Then, uniformly for $k - \epsilon n \le h \le k$,

$$\frac{r}{t + r + 1} = \frac{\kappa - \tau}{1 - (\kappa - \tau)} + o(1),$$
$$\frac{k - t - r}{n - k - r + 1} = \frac{\kappa - \tau}{1 - (\kappa - \tau)} + o(1).$$

Let $\omega := \frac{\kappa - \tau}{1 - (\kappa - \tau)}$. From (12) we obtain

$$|\mathcal{F}_h| \le \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}(\omega + o(1))^{k-h}\frac{1}{1 - \omega^2 - o(1)}\right\}$$

and, consequently,

$$\sum_{k - \epsilon n < h \le k} |\mathcal{F}_h| \le \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1 + o(1)) \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}\right\}. \tag{14}$$

Since $n\alpha^{\epsilon n} = o(1)$, we finally get from (11), (13) and (14)

$$|\mathcal{F}| \le \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1 + o(1)) \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}\right\}. \tag{15}$$

On the other hand, using more or less the same estimations, one can derive $\max\{|S_{\le k}(n,t,r)|\} \ge \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1 + o(1)) \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{k-t-r}\right\}$ which proves together with (15) the assertion. □

PROOF OF THEOREM 17. Let $\mathcal{F} \in I(n,t) \cap C(n,s)$. First let $2 \mid n + t + s$ and let $k := \frac{n+t-s}{2}$. We divide $\mathcal{F}$ into two subfamilies

$$\mathcal{F}' := \bigcup_{h=0}^{k} \mathcal{F}_h, \qquad \mathcal{F}'' := \bigcup_{h=k+1}^{n} \mathcal{F}_h$$

and put

$$\mathcal{F}''' := \{[n] \setminus X : X \in \mathcal{F}''\}.$$

Obviously, $\mathcal{F}' \in I_{\le k}(n,t)$, $\mathcal{F}''' \in I_{\le n-k-1}(n,s)$. Using the notations from Theorem 13 we have (for $\mathcal{F}'$ and $\mathcal{F}'''$)

$$\omega = \frac{1 - \tau - \sigma}{1 + \tau + \sigma}$$

and get the estimations

$$|\mathcal{F}'| \le \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1 + o(1)) \max\left\{\binom{t+2r}{r}\binom{n-t-2r}{(n-t-s)/2-r}\right\},$$

$$|\mathcal{F}'''| \le \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1 + o(1)) \max\left\{\binom{s+2q}{q}\binom{n-s-2q}{(n-t-s)/2-1-q}\right\},$$

$$= \frac{\omega}{1-\omega}\frac{1}{1-\omega^2}(1 + o(1)) \max\left\{\binom{s+2q}{q}\binom{n-s-2q}{(n-t-s)/2-q}\right\},$$

and, with $r := \frac{n-t-s}{2} - q$,

$$|\mathcal{F}'''| \leq \frac{\omega}{1-\omega}\frac{1}{1-\omega^2}(1+o(1))\max\left\{\binom{t+2r}{r}\binom{n-t-2r}{(n-t-s)/2-r}\right\}.$$

Consequently,

$$|\mathcal{F}| = |\mathcal{F}'| + |\mathcal{F}'''| \leq \frac{1}{(1-\omega)^2}(1+o(1))\max\left\{\binom{t+2r}{r}\binom{n-t-2r}{(n-t-s)/2-r}\right\}.$$

Again, in a similar way, one can derive that

$$\max\left\{|S(n,t,s,r)| : r = 0, \ldots, \frac{n-t-s}{2}\right\}$$
$$\geq \frac{1}{(1-\omega)^2}(1+o(1))\max\left\{\binom{t+2r}{r}\binom{n-t-2r}{(n-t-s)/2-r}\right\}$$

which proves the assertion.

Now let $2 \nmid n+t+s$. Here we put $k := \frac{n+t-s-1}{2}$. With the same approach we get

$$|\mathcal{F}'| \leq \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1+o(1))\max\left\{\binom{t+2r}{r}\binom{n-t-2r}{(n-t-s-1)/2-r}\right\},$$

$$|\mathcal{F}'''| \leq \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1+o(1))\max\left\{\binom{s+2q}{q}\binom{n-s-2q}{(n-t-s-1)/2-q}\right\},$$

$$= \frac{1}{1-\omega}\frac{1}{1-\omega^2}(1+o(1))\max\left\{\binom{t+2r+1}{r}\binom{n-1-t-2r}{(n-t-s-1)/2-r}\right\}.$$

It is not difficult to verify that the maximum on both RHS is attained at some r with

$$r \sim \frac{\tau}{2}\frac{1-\tau-\sigma}{\tau+\sigma}n.$$

This easily implies

$$|\mathcal{F}| = |\mathcal{F}'| + |\mathcal{F}'''| \leq \frac{2}{(1-\omega)^2}(1+o(1))\max\left\{\binom{t+2r}{r}\binom{n-1-t-2r}{(n-t-s-1)/2-r}\right\}.$$

But the RHS is obviously also a lower bound for

$$\max\left\{|S(n,t,s,r)| : r = 0, \ldots, \frac{n-t-s-1}{2}\right\}. \qquad \square$$

5. Comparison Methods and Proofs of Theorems 8 and 10

In this section we work with *size-dependent* weight functions, i.e., functions $\omega : 2^{[n]} \to \mathbb{R}_+$ for which there are numbers $\omega_0, \ldots, \omega_n$ such that $\omega(X) = \omega_i$ for all $X \subseteq [n]$ with $|X| = i$, $i = 0, \ldots, n$. We call $\boldsymbol{\omega} := (\omega_0, \ldots, \omega_n)$ the *weight vector*.

A corollary of the comparison lemma [2] is the following result proved in [6]:

THEOREM 18. *Let ω be size-dependent. Then*

$$M(I(n,t),\omega) = \omega\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right)$$

if

$$\max\left\{\frac{\omega_i}{\omega_{i+1}} : i = t, \ldots, n-1\right\} < 1 + \frac{t-1}{\lfloor\frac{n-t}{2}\rfloor}.$$

REMARK. Using a continuity argument it is easy to see that the relation '$<$' in the above condition can be replaced by '$\leq$'.

In the next lemmas we present conditions for how the weight function can be changed without changing the optimal solution.

LEMMA 19. *Let ω be size-dependent and suppose that $M(I(n,t),\omega)$ is attained at $S(n,t,\lfloor\frac{n-t}{2}\rfloor)$. Let ω' be a new size-dependent weight defined by either one of the following assignments:*

$$\omega_i' := \begin{cases} \omega_i - \lambda & \text{if } i = u \\ \omega_i + \lambda\frac{\binom{n}{u}}{\binom{n}{\ell}} & \text{if } i = \ell \\ \omega_i & \text{otherwise,} \end{cases} \tag{16}$$

where $0 < \lambda \leq \omega_u$ and, $\frac{n+t}{2} \leq \ell < u \leq n$ or $0 \leq \ell < u < \lfloor\frac{n+t}{2}\rfloor$,

$$\omega_i' := \begin{cases} \omega_i + \delta & \text{if } i = \ell \\ \omega_i & \text{otherwise,} \end{cases} \tag{17}$$

where $\delta > 0$ and $\frac{n+t}{2} \leq \ell \leq n$.

Then $M(I(n,t),\omega')$ is also attained at $S(n,t,\lfloor\frac{n-t}{2}\rfloor)$.

PROOF. Let ω' be given by (16). Note that

$$\omega'\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right) = \omega\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right).$$

Let $\mathcal{F}$ be an optimal family for ω'. W.l.o.g. we may assume that $\mathcal{F}$ is a *filter* (or *upset*), i.e., $X \in \mathcal{F}$, $X \subseteq Y$ imply $Y \in \mathcal{F}$. By the normalized matching property of the Boolean lattice (cf. [7, p. 149]) we have

$$\frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} \leq \frac{|\mathcal{F}_u|}{\binom{n}{u}}.$$

It follows that

$$\begin{aligned} \omega'(\mathcal{F}) &= \omega(\mathcal{F}) + \lambda\frac{\binom{n}{u}}{\binom{n}{\ell}}|\mathcal{F}_\ell| - \lambda|\mathcal{F}_u| = \omega(\mathcal{F}) + \lambda\binom{n}{u}\left(\frac{|\mathcal{F}_\ell|}{\binom{n}{\ell}} - \frac{|\mathcal{F}_u|}{\binom{n}{u}}\right) \\ &\leq \omega(\mathcal{F}) \leq \omega\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right) = \omega'\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right). \end{aligned}$$

Now let ω' be given by (17) and let $\mathcal{F}$ be an optimal family for ω'. Then

$$\begin{aligned} \omega'(\mathcal{F}) &= \omega(\mathcal{F}) + \delta|\mathcal{F}_\ell| \leq \omega(\mathcal{F}) + \delta\binom{n}{\ell} \\ &\leq \omega\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right) + \delta\binom{n}{\ell} = \omega'\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right). \end{aligned}$$

□

LEMMA 20. *Let ω be size-dependent and suppose that $M(I(n,t),\omega)$ is attained at $S(n,t,\lfloor\frac{n-t}{2}\rfloor)$. Let $\lambda > 0$, $0 \leq \ell < \lfloor\frac{n+t}{2}\rfloor$ and let ω' be a new size-dependent weight defined by*

$$\omega_i' := \begin{cases} \omega_i + \lambda & \text{if } i = \ell \\ \omega_i + \lambda\frac{\ell - t + 1}{\ell} & \text{if } i = n + t - \ell - 1 \\ \omega_i & \text{otherwise.} \end{cases}$$

Then $M(I(n,t),\omega')$ is also attained at $S(n,t,\lfloor\frac{n-t}{2}\rfloor)$.

PROOF. Obviously,

$$\omega'\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right)=\omega\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right)+\lambda\frac{\ell-t+1}{\ell}\binom{n}{n+t-\ell-1}.$$

Let $\mathcal{F}$ be an optimal family for ω'. From Katona's theorem concerning shadows of t-intersecting families (cf. [7, p. 301]) it follows that

$$|\mathcal{F}_{n+t-\ell-1}|\le\binom{n}{n+t-\ell-1}-\frac{\ell}{\ell-t+1}|\mathcal{F}_\ell|.$$

Accordingly,

$$\begin{aligned}\omega'(\mathcal{F})&=\omega(\mathcal{F})+\lambda|\mathcal{F}_\ell|+\lambda\frac{\ell-t+1}{\ell}|\mathcal{F}_{n+t-\ell-1}|\\&\le\omega(\mathcal{F})+\lambda\frac{\ell-t+1}{\ell}\left(\frac{\ell}{\ell-t+1}|\mathcal{F}_\ell|+\binom{n}{n+t-\ell-1}-\frac{\ell}{\ell-t+1}|\mathcal{F}_\ell|\right)\\&\le\omega\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right)+\lambda\frac{\ell-t+1}{\ell}\binom{n}{n+t-\ell-1}=\omega'\left(S\left(n,t,\left\lfloor\frac{n-t}{2}\right\rfloor\right)\right).\end{aligned}$$

□

PROOF OF THEOREM 8. Obviously, it is enough to prove the assertion for

$$k:=\left\lceil\frac{n+t}{2}+\sqrt{\log n}\sqrt{n}\right\rceil$$

(e.g., apply Lemma 19 with (17)). Let

$$q:=1+\frac{t-1}{\lfloor\frac{n-t}{2}\rfloor}.$$

We consider the size-dependent weight ω defined by

$$\omega_i:=\begin{cases}1 & \text{if } i<\frac{n+t}{2}\\ \frac{1}{q} & \text{if } i\ge\frac{n+t}{2}.\end{cases}\tag{18}$$

By Theorem 18 (and the succeeding remark), we know that $M(I(n,t),\omega)$ is attained at $S(n,t,\lfloor\frac{n-t}{2}\rfloor)$.

Now we apply Lemma 19 with (16) for $\ell=\lceil\frac{n+t}{2}\rceil$ and $u=k+1,k+2,\ldots,n$. This gives the new weight vector ω':

$$\omega'_i:=\begin{cases}1 & \text{if } i<\frac{n+t}{2}\\ \frac{1}{q}\left(1+\frac{1}{\binom{n}{\lceil(n+t)/2\rceil}}\sum_{u=k+1}^{n}\binom{n}{u}\right) & \text{if } i=\left\lceil\frac{n+t}{2}\right\rceil\\ \frac{1}{q} & \text{if } \left\lceil\frac{n+t}{2}\right\rceil<i\le k\\ 0 & \text{if } i>k.\end{cases}$$

It is known (cf. [12, p. 284]) that, as $n\to\infty$,

$$\binom{n}{\lceil(n+t)/2\rceil}\sim\frac{2^{n+1}}{\sqrt{2\pi n}},\tag{19}$$

and, with $x = o(n^{\frac{1}{6}})$, $x \to \infty$,

$$\sum_{u > \frac{n}{2} + x\frac{\sqrt{n}}{2}} \binom{n}{u} \sim \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} 2^n.$$

The last formula with $x = 2\sqrt{\log n}$ implies

$$\sum_{u=k+1}^{n} \binom{n}{u} \lesssim \frac{1}{2\sqrt{\pi\sqrt{\log n}}} \frac{2^n}{n^2}. \tag{20}$$

By (19) and (20) we have for sufficiently large n

$$\frac{1}{\binom{n}{\lceil (n+t)/2 \rceil}} \sum_{u=k+1}^{n} \binom{n}{u} < \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor} = q - 1$$

which implies that $\omega_i' \le 1$ for $i = \lceil \frac{n+t}{2} \rceil, \dots, k$. Hence, by again applying Lemma 8 with (17) we obtain that for large n

$$M(I_{\le k}(n,t)) = \left| S_{\le k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right) \right|. \qquad \square$$

PROOF OF THEOREM 10. We use the same method as in the proof of Theorem 8, but here we put

$$k := \left\lceil \frac{n+t}{2} \right\rceil + c,$$

where c is an integer. Recalling (18) we have to show that there exists c such that for large n

$$\frac{1}{q} \left(1 + \frac{1}{\binom{n}{\lceil (n+t)/2 \rceil}} \sum_{u=k+1}^{n} \binom{n}{u} \right) \le 1,$$

or, equivalently,

$$\sum_{u=k+1}^{n} \binom{n}{u} \le (q-1) \binom{n}{\lceil (n+t)/2 \rceil}. \tag{21}$$

We have

$$\frac{1}{q} > \frac{\binom{n}{\lceil (n+t)/2 \rceil + 1}}{\binom{n}{\lceil (n+t)/2 \rceil}} > \cdots > \frac{\binom{n}{n}}{\binom{n}{n-1}}$$

and consequently

$$\sum_{u=k+1}^{n} \binom{n}{u} < \binom{n}{\lceil (n+t)/2 \rceil} \sum_{u=k+1}^{n} q^{-(u - \lceil \frac{n+t}{2} \rceil)} < \binom{n}{\lceil (n+t)/2 \rceil} q^{-(c+1)} \frac{1}{1 - q^{-1}}.$$

Therefore,

$$\frac{(1/q)^{c+1}}{1 - 1/q} \le q - 1,$$

or, equivalently,

$$q^c \ge \frac{1}{(q-1)^2} \tag{22}$$

is sufficient for (21). Using

$$q^c \geq c(q-1)$$

we see that

$$c \geq \frac{1}{(q-1)^3}$$

is sufficient for (22). However, for $t \geq \delta n$, the last condition certainly holds (for large n) if

$$c > \left(\frac{1-\delta}{2\delta}\right)^3. \qquad \square$$

6. Proof of Theorem 12

LEMMA 21. *Let*

$$a_{k,n} = \frac{1}{\binom{n}{k}} \sum_{j=0}^{k} \binom{n}{j}.$$

Then $a_{k,n}$ is increasing in k (for $k = 0, \ldots, n$).

PROOF. For fixed n we have $a_{k,n} \leq a_{k+1,n}$ iff

$$\binom{n}{k} \sum_{j=0}^{k+1} \binom{n}{j} - \binom{n}{k+1} \sum_{j=0}^{k} \binom{n}{j} \geq 0.$$

However, this inequality is true since the LHS is not less than

$$\sum_{j=0}^{k} \left(\binom{n}{k}\binom{n}{j+1} - \binom{n}{k+1}\binom{n}{j} \right)$$

and each term of the last sum is nonnegative by the log-concavity of the binomial coefficients. $\square$

LEMMA 22. *Let $k < \frac{n+t}{2}$. Then the sequence*

$$|S_{\leq k}(n,t,0)|, |S_{\leq k}(n,t,1)|, \ldots, \left|S_{\leq k}\left(n, t, \left\lfloor \frac{n-t}{2} \right\rfloor\right)\right|$$

is unimodal.

PROOF. By considering $|S_{\leq k}(n,t,r) \setminus S_{\leq k}(n,t,r+1)|$ and $|S_{\leq k}(n,t,r+1) \setminus S_{\leq k}(n,t,r)|$ we see that

$$|S_{\leq k}(n,t,r)| \leq |S_{\leq k}(n,t,r+1)|$$

is equivalent to

$$(t+r)\binom{n-t-2r-2}{k-t-r} \leq (t-1) \sum_{i=0}^{k-t-r} \binom{n-t-2r-2}{i}. \qquad (23)$$

We will show that $|S_{\leq k}(n,t,r)| \leq |S_{\leq k}(n,t,r+1)|$ implies $|S_{\leq k}(n,t,r-1)| \leq |S_{\leq k}(n,t,r)|$. It suffices to prove that for all r with $0 < r < \lfloor \frac{n-t}{2} \rfloor$

$$\binom{n-t-2r}{k-t-r+1} \sum_{i=0}^{k-t-r} \binom{n-t-2r-2}{i} \leq \binom{n-t-2r-2}{k-t-r} \sum_{i=0}^{k-t-r+1} \binom{n-t-2r}{i},$$

or, (substituting $a = n - t - 2r - 2$, $b = k - t - r$) that for all a, b with $2b < a + 2$

$$\binom{a+2}{b+1} \sum_{i=0}^{b} \binom{a}{i} \leq \binom{a}{b} \sum_{i=0}^{b+1} \binom{a+2}{i}.$$

Subtracting

$$2\binom{a}{b} \sum_{i=0}^{b} \binom{a}{i}$$

from the last inequality gives

$$\left(\binom{a}{b-1} + \binom{a}{b+1}\right) + \sum_{i=0}^{b} \binom{a}{i} \leq \binom{a}{b} \left(\sum_{i=0}^{b+1} \binom{a}{i} + \sum_{i=0}^{b-1} \binom{a}{i} \right). \tag{24}$$

Using $2b \leq a + 1$ one verifies easily that for $i = 0, 1, \ldots, b$

$$\frac{\binom{a}{b-1} + \binom{a}{b+1}}{\binom{a}{b}} \leq \frac{\binom{a}{i-1} + \binom{a}{i+1}}{\binom{a}{i}}$$

from which (24) follows. □

Proof of Theorem 12

Step 1. Let the weight vector ω be defined by

$$\omega_i := \begin{cases} 1 & \text{if } i \leq k \\ 0 & \text{if } i > k. \end{cases}$$

Let $r^* = r^*(n, k)$ be the least r such that

$$|\omega(S(n, t, r))| \geq |\omega(S(n, t, r + 1))| \geq \cdots. \tag{25}$$

By Lemma 22 we know that $|S_{\leq k}(n, t, r^*)| = \max\{|S_{\leq k}(n, t, r)| : r = 0, \ldots, \lfloor \frac{n-t}{2} \rfloor\}$. In addition, we have

$$\omega_i = 0 \quad \text{if } i \geq \frac{n+t}{2}. \tag{26}$$

Given an arbitrary weight vector satisfying (25) and (26) it follows by the method of generating sets [1] that

$$M(I(n, t), \omega) = M(I(t + 2r^*, t), \omega'),$$

where the weight vector ω' is given by

$$\omega'_i := \sum_{j=0}^{n-t-2r^*} \omega_{i+j} \binom{n - t - 2r^*}{j}$$

for $i = 0, \ldots, t + 2r^*$ (cf. [6, Theorem 15 and Example 4]). Hence, in our case, we have

$$M(I_{\leq k}(n, t)) = M(I(t + 2r^*, t), \omega'),$$

where

$$\omega'_i = \sum_{j=0}^{k-i} \binom{n - t - 2r^*}{j}$$

for $i = 0, \ldots, t + 2r^*$.

Step 2. From Step 1 we know that there is an optimal family $\mathcal{F}$ (i.e., $\mathcal{F} \in I_{\leq k}(n,t)$, $|\mathcal{F}| = M(I_{\leq k}(n,t))$) which has the following property:

$$X \in \mathcal{F} \text{ implies } Y \in \mathcal{F} \text{ for all } Y \in \binom{[n]}{\leq k} \text{ with } Y \cap [t+2r^*] = X \cap [t+2r^*]. \tag{27}$$

W.l.o.g. we assume that $\mathcal{F}$ is *left-compressed*, i.e., $(X \setminus \{i\}) \cup \{j\} \in \mathcal{F}$ for all $i, j \in [n]$ with $i > j$, $i \in X$, $j \notin X$. We will prove by pushing–pulling [3] that $\mathcal{F}$ is invariant in $[t+2r^*]$, i.e., $(X \setminus \{i\}) \cup \{j\} \in \mathcal{F}$ for all $i, j \in [t+2r^*]$, $i \in X$, $j \notin X$. Assume the contrary. Let

$$\begin{aligned} \ell &= \max\{i : \mathcal{F} \text{ is invariant in } [i]\} \\ \mathcal{L} &= \{X \in \mathcal{F} : \ell+1 \notin X, (X \setminus \{i\}) \cup \{\ell+1\} \notin \mathcal{F} \text{ for some } i \in X \cap [\ell]\} \\ \mathcal{L}^* &= \{X \cap [\ell+2, n] : X \in \mathcal{L}\}. \end{aligned}$$

Furthermore, let $\mathcal{L}_i = \{X \in \mathcal{L} : |X \cap [\ell] = i\}$, $\mathcal{L}_i^* = \{X \cap [\ell+2, n] : X \in \mathcal{L}_i\}$. By our assumption we have $\ell < t+2r^*$. The following facts follow from the pushing–pulling method (cf. [6]):

(i) $\mathcal{L}$ is nonempty and invariant in $[\ell]$.
(ii) $\ell \geq t$, $2 \mid \ell + t$, $\mathcal{L}_i = \emptyset$ for $i \in [\ell] \setminus \{\frac{\ell+t}{2}\}$.
(iii) For all intersecting subfamilies $\mathcal{T}^*$ of $\mathcal{L}^*_{\frac{\ell+t}{2}}$,

$$\frac{\sum_{X \in \mathcal{T}^*} \omega_{|X|+\frac{\ell+t}{2}}}{\sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}}} \omega_{|X|+\frac{\ell+t}{2}}} \leq \frac{\ell - t + 2}{2(\ell+1)}.$$

It is easy to see that $\ell = t + 2r^* - 2$ is impossible (e.g., since $\mathcal{L} \neq \emptyset$ we have $t + 2r^* \notin X$ for some $X \in \mathcal{L}^*_{\frac{\ell+t}{2}}$ which implies $\mathcal{F} = S_{\leq k}(n, t, r^* - 1)$ in contradiction to the choice of $\mathcal{F}$ and r^*) Hence $\ell \leq t+2r^*-4$. We show that the family $\mathcal{T}^* = \{X \in \mathcal{L}^*_{\frac{\ell+t}{2}} : n \in X\}$ contradicts fact (iii). Indeed, recalling (27), this will follow from the next inequality (we classify the members X of $\mathcal{L}^*_{\frac{\ell+t}{2}}$ and $\mathcal{T}^*$ with respect to $i = |X \cap [\ell+2, t+2r^*]|$).

CLAIM. *If $k \leq (\frac{1}{2} - \epsilon)n$ and n is sufficiently large then we have for all ℓ, i with $\ell \leq t + 2r^* - 4$, $2 \mid \ell + t$, $0 \leq i \leq t + 2r^* - \ell - 1$*

$$\sum_{j=0}^{k-\frac{\ell+t}{2}-i-1} \binom{n-t-2r^*-1}{j} > \frac{\ell-t+2}{2(\ell+1)} \sum_{j=0}^{k-\frac{\ell+t}{2}-i} \binom{n-t-2r^*}{j}.$$

This inequality is easily seen to be equivalent to

$$\frac{n-t-2r^*}{n-t-2r^*-k+\frac{\ell+t}{2}+i} \frac{\sum_{j=0}^{k-\frac{\ell+t}{2}-i} \binom{n-t-2r^*}{j}}{\binom{n-t-2r^*}{k-\frac{\ell+t}{2}-i}} > \frac{\ell+1}{t-1}. \tag{28}$$

Since $\ell \leq t + 2r^* - 4$ it suffices to show that the LHS of (28) is greater than

$$\frac{t+2r^*-3}{t-1}.$$

For every r let

$$\kappa_r = \frac{r}{t+2r-1} \qquad \text{and} \qquad m_r = \frac{\kappa_{r-1}+\kappa_r}{2}.$$

Note that $r = (t-1)\frac{\kappa_r}{1-2\kappa_r}$ and that κ_r is strictly increasing and $\lim_{r\to\infty}\kappa_r = \frac{1}{2}$. We consider the finite set

$$R := \{r \in \mathbb{N} : \kappa_r \leq \tfrac{1}{2} - \epsilon\}.$$

Since for $\kappa < \frac{1}{2}$, $c \in \mathbb{N}$ constant

$$\lim_{n\to\infty} \frac{1}{\binom{n}{\lfloor \kappa n\rfloor + c}} \sum_{j=0}^{\lfloor \kappa n\rfloor + c} \binom{n}{j} = \frac{1-\kappa}{1-2\kappa}$$

(cf. [5]), we have for sufficiently large n and all r, ℓ, i with $r \in R$, $\ell \leq t + 2r - 4$, $2 \mid \ell + t$, $0 \leq i \leq t + 2r - \ell - 1$

$$\frac{n-t-2r}{n-t-2r-\lfloor m_r n\rfloor + \frac{\ell+t}{2} + i} \frac{\sum_{j=0}^{\lfloor m_r n\rfloor - \frac{\ell+t}{2} - i} \binom{n-t-2r}{j}}{\binom{n-t-2r}{\lfloor m_r n\rfloor - \frac{\ell+t}{2} - i}} > \frac{1}{1-2\kappa_{r-1}} = \frac{t+2r-3}{t-1}. \quad (29)$$

Analogously, we have for sufficiently large n and all $r \in R$

$$\frac{\sum_{j=0}^{\lfloor m_{r+1} n\rfloor - t - r} \binom{n-t-2r-2}{j}}{\binom{n-t-2r-2}{\lfloor m_{r+1} n\rfloor - t - r}} < \frac{1-\kappa_{r+1}}{1-2\kappa_{r+1}} = \frac{t+r}{t-1}. \quad (30)$$

Now let n be such that (29) and (30) are satisfied and let r be determined by

$$\lfloor m_r n\rfloor \leq k < \lfloor m_{r+1} n\rfloor.$$

By (23), Lemma 21 and (30) we have

$$|S_{\leq k}(n,t,r)| > |S_{\leq k}(n,t,r+1)|,$$

hence, by Lemma 22, $r^* \leq r$. Lemma 21 and (29) now imply that (28) is satisfied. □

References

1. R. Ahlswede and L. H. Khachatrian, The complete intersection theorem for systems of finite sets, *Europ. J. Combinatorics*, **18** (1997), 125–136.
2. R. Ahlswede and L. H. Khachatrian, The diametric theorem in Hamming spaces—optimal anti-codes, *Adv. Appl. Math.*, **20** (1998), 429–449.
3. R. Ahlswede and L. H. Khachatrian, A pushing–pulling method: new proofs for intersection theorems, *Combinatorica*, **19** (1999), 1–15.
4. C. Bang, H. Sharp and P. Winkler, On families of finite sets with bounds on unions and intersections, *Discrete Math.*, **45** (1983), 123–126.
5. E. A. Bender, Asymptotic methods in enumeration, *SIAM Rev.*, **16** (1974), 485–515.
6. C. Bey and K. Engel, Old and new results for the weighted t-intersection problem via AK-methods, *Numbers, Information and Complexity*. Special volume in honour of R. Ahlswede on the occasion of his 60th birthday, I. Althöfer, N. Cai, G. Dueck, L. H. Khachatrian, M. Pinsker, A. Sárközy, I. Wegener and Z. Zhang (eds), Kluwer Academic, Boston, 2000, pp. 45–74.
7. K. Engel, *Sperner Theory*, Cambridge University Press, Cambridge, 1997.
8. K. Engel and P. Frankl, An Erdős–Ko–Rado theorem for integer sequences of given rank, *Europ. J. Combinatorics*, **7** (1986), 215–220.
9. P. Erdős, C. Ko and R. Rado, Intersection theorems for systems of finite sets, *Q. J. Math. Oxford Ser.*, **12** (1961), 313–320.
10. P. Frankl, The proof of a conjecture of G. O. H. Katona, *J. Comb. Theory Ser. A*, **19** (1975), 208–213.

11. P. Frankl, Extremal set systems, in: *Handbook of Combinatorics*, Vol. 2, R. L. Graham, M. Grötschel and L. Lovász (eds), Elsevier (North-Holland), Amsterdam, 1995, pp. 1293–1329.
12. G. P. Gavrilov and A. A. Sapozhenko, *Problems and Exercises in Discrete Mathematics*, 2nd edn, Moskva Nauka, Moscow, 1992.
13. G. O. H. Katona, Intersection theorems for systems of finite sets, *Acta Math. Acad. Sci. Hung.*, **15** (1964), 329–337.

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RUDOLF AHLSWEDE AND LEVON H. KHACHATRIAN
Fakultät für Mathematik,
Universität Bielefeld,
Postfach 100131,
33501 Bielefeld,
Germany
E-mail: hollmann@mathematik.uni-bielefeld.de

AND

CHRISTIAN BEY AND KONRAD ENGEL
Fachbereich Mathematik,
Universität Rostock,
18051 Rostock,
Germany