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The edge-diametric theorem in Hamming spaces

Christian Bey

Otto-von-Guericke-Universität Magdeburg, Institut für Algebra und Geometrie, Postfach 4120, 39016 Magdeburg, Germany

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Dedicated to the memory of Levon Khachatrian

Abstract

The maximum number of edges spanned by a subset of given diameter in a Hamming space with alphabet size at least three is determined. The binary case was solved earlier by Ahlswede and Khachatrian [A diametric theorem for edges, J. Combin. Theory Ser. A 92(1) (2000) 1–16].

Keywords: Diametric problem; Hamming space; Intersection theorem

1. Introduction

Let $\mathscr{H}_\alpha^n = \{1, \ldots, \alpha\}^n$ be the Hamming space, i.e. $\mathscr{H}_\alpha^n$ is equipped with the Hamming distance $d_{\mathscr{H}}$, where for $a = (a_1, \ldots, a_n)$, $b = (b_1, \ldots, b_n) \in \mathscr{H}_\alpha^n$ we have $d_{\mathscr{H}}(a, b) = |\{i : a_i \neq b_i\}|$. The diameter of a subset $\mathscr{A} \subset \mathscr{H}_\alpha^n$ is defined by

$$\operatorname{diam}(\mathscr{A}) = \max\{d_{\mathscr{H}}(a, b) : a, b \in \mathscr{A}\}.$$

For every integer d with $0 < d < n$ put

$$D(\alpha, n, d) = \{\mathscr{A} \subseteq \mathscr{H}_\alpha^n : \operatorname{diam}(\mathscr{A}) \leqslant d\}.$$

As usual, $\mathscr{H}_\alpha^n$ is also considered as a graph $(\mathscr{V}, \mathscr{E})$ with vertex set $\mathscr{V} = \mathscr{H}_\alpha^n$ and edge set $\mathscr{E} = \{\{a, b\} : d_{\mathscr{H}}(a, b) = 1\}$. For every subset $\mathscr{A} \subseteq \mathscr{V}$ let $\mathscr{E}(\mathscr{A}) = \{\{a, b\} : a, b \in \mathscr{A},\ d_{\mathscr{H}}(a, b) = 1\}$ be the edge set induced by $\mathscr{A}$.

The vertex-resp. edge diametric problem in Hamming space is to determine the function

$$V(\alpha, n, d) := \max_{\mathscr{A} \in D(\alpha,n,d)} |\mathscr{A}| \quad \text{(vertex-diametric function)},$$

resp.

$$E(\alpha, n, d) := \max_{\mathscr{A} \in D(\alpha,n,d)} |\mathscr{E}(\mathscr{A})| \quad \text{(edge-diametric function)}.$$

Clearly, these problems can be formulated in every graph.

E-mail address: christian.bey@mathematik.uni-magdeburg.de.

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Consider for every integer t with $0 < t < n$ the following subsets of $\mathscr{H}_\alpha^n$:

$$\mathscr{F}_i(\alpha, n, t) := \{a \in \mathscr{H}_\alpha^n : |B(a) \cap \{1, \ldots, t+2i\}| \geqslant t+i\}, \quad 0 \leqslant i \leqslant \frac{n-t}{2},$$

where $B(a)$ is defined for every $a = (a_1, \ldots, a_n) \in \mathscr{H}_\alpha^n$ by

$$B(a) = \{i \in \{1, \ldots, n\} : a_i = 1\}.$$

Note that $\mathscr{F}_i(\alpha, n, t)$ is the product of a ball with radius i on the first $t + 2i$ coordinates and a Hamming space on the remaining $n - t - 2i$ coordinates. Thus, $\mathscr{F}_i(\alpha, n, t) \in D(\alpha, n, n-t)$ for all i.

The vertex-diametric problem in binary Hamming spaces was solved by Kleitman:

Theorem 1 (*Kleitman [11]*). $V(2, n, d) = |\mathscr{F}_{\lfloor d/2 \rfloor}(2, n, n-d)|$.

For larger alphabets, the complete solution of the vertex-diametric problem is due to Ahlswede and Khachatrian:

Theorem 2 (*Ahlswede and Khachatrian [2]*). *Let r be the largest nonnegative integer such that $2r \leqslant d$ and $r \leqslant (n - d - 1)/(\alpha - 2)$ are satisfied. Then*

$$V(\alpha, n, d) = |\mathscr{F}_r(\alpha, n, n-d)|.$$

See [2] for a list of previously obtained partial results.

Equivalent versions of the above theorems (in terms of intersection instead of diametry) were obtained by Katona [10] (Theorem 1) and Frankl and Tokushige [9] (Theorem 2).

Ahlswede and Khachatrian also solved the edge-diametric problem in binary Hamming spaces:

Theorem 3 (*Ahlswede and Khachatrian [4]*).

$$E(2, n, d) = \begin{cases} |\mathscr{E}(\mathscr{F}_0(2, n, 1))| & \text{if } d = n-1, \\ |\mathscr{E}(\mathscr{F}_{\lfloor d/2 \rfloor}(2, n, n-d))| & \text{if } d \leqslant n-2. \end{cases}$$

Here we continue these investigations by providing a solution for the edge-diametric problem for all other alphabet sizes:

Theorem 4. *Let $\alpha \geqslant 3$. Then*

$$E(\alpha, n, d) = \begin{cases} |\mathscr{E}(\mathscr{F}_0(\alpha, n, n-d))| & \text{if } d = n-1, n-2, \\ \max\limits_{0 \leqslant i \leqslant d/2} |\mathscr{E}(\mathscr{F}_i(\alpha, n, n-d))| & \text{if } d < n-2. \end{cases}$$

The maximum in Theorem 4 is attained at the largest nonnegative integer i for which $2i \leqslant d$ and

$$(\alpha - 1)\left(1 + \frac{1}{(d-2i)(1-1/\alpha) + i}\right) \leqslant 1 + \frac{n-d-1}{i}$$

are satisfied.

We remark without proof that, up to permutations of the coordinates and of the alphabets in the components, the $\mathscr{F}_i(\alpha, n, n-d)$ are the only optimal configurations.

Our proof of Theorem 4 is based on the powerful methods developed by Ahlswede and Khachatrian in [1,3].

2. Reduction to an intersection problem

Let $\mathbb{N}$ denote the set of positive integers. For $i, j \in \mathbb{N}$ put $[i, j] = \{i, i+1, \ldots, j\}$ and $[i] = [1, i]$. Further, let $2^{[i]} = \{A : A \subseteq [i]\}$ and $\binom{[i]}{k} = \{A \subseteq [i] : |A| = k\}$.

Consider the set $M = [1, \alpha n]$ together with the partition

$$M = M_1 \cup \cdots \cup M_n \quad \text{where } M_i = \{j \in M : j \equiv i \bmod n\}, \quad i = 1, \ldots, n.$$

There is a natural bijection between the Hamming space $\mathscr{H}_\alpha^n$ and the set

$$\mathscr{C}_\alpha^n = \{A \subseteq M : |A \cap M_i| = 1 \text{ for all } i = 1, \ldots, n\},$$

which maps $a = (a_1, \ldots, a_n) \in \mathscr{H}_\alpha^n$ to $A = \bigcup_{i=1}^n \{(a_i - 1)n + i\} \in \mathscr{C}_\alpha^n$. Accordingly, we define for every $\mathscr{A} \subseteq \mathscr{C}_\alpha^n$

$$\mathscr{E}(\mathscr{A}) := \{\{A_1, A_2\} : A_1, A_2 \in \mathscr{A}, |A_1 \triangle A_2| = 2\},$$

where $A_1 \triangle A_2$ denotes the symmetric difference of A_1 and A_2.

Recall that a system of sets $\mathscr{A} \subseteq 2^{\mathbb{N}}$ is called t-intersecting if

$$|A_1 \cap A_2| \geqslant t \quad \text{for all } A_1, A_2 \in \mathscr{A}.$$

For $m \in \mathbb{N}$ let $I(m, t)$ be the set of all t-intersecting systems $\mathscr{A} \subseteq 2^{[m]}$, and put

$$I_\alpha^n(t) = \{\mathscr{A} \subseteq \mathscr{C}_\alpha^n : \mathscr{A} \in I(\alpha n, t)\}.$$

The following equality is now obvious:

$$E(\alpha, n, d) = \max_{\mathscr{A} \in I_\alpha^n(n-d)} |\mathscr{E}(\mathscr{A})|. \tag{1}$$

We continue with the well-known notion of left-shifted set systems [8].

Definition. For any $\mathscr{A} \subseteq 2^{\mathbb{N}}$, any $A \in \mathscr{A}$ and $i, j \in \mathbb{N}$ let

$$S_{i,j}(A) = \begin{cases} \{i\} \cup (A \backslash \{j\}) & \text{if } i \notin A, \quad j \in A, \quad \{i\} \cup (A \backslash \{j\}) \notin \mathscr{A}, \\ A & \text{otherwise} \end{cases}$$

and $S_{i,j}(\mathscr{A}) = \{S_{i,j}(A) : A \in \mathscr{A}\}$.

A family $\mathscr{A} \subseteq 2^{\mathbb{N}}$ is called left-shifted in T (where $T \subseteq \mathbb{N}$) if $S_{i,j}(\mathscr{A}) = \mathscr{A}$ for all $i, j \in T$ with $i < j$.

The shift-operations $S_{i,j}$ have the following easy but important properties.

Lemma 5. *Let* $\mathscr{A} \subseteq 2^{\mathbb{N}}$ *and* $i, j \in \mathbb{N}$. *Then*:

(i) $S_{i,j}(\mathscr{A}) \subseteq \mathscr{C}_\alpha^n$ *whenever* $\mathscr{A} \subseteq \mathscr{C}_\alpha^n$ *and* $i, j \in M_k$ *for some* $k = 1, \ldots, n$,
(ii) $|S_{i,j}(\mathscr{A})| = |\mathscr{A}|$,
(iii) $S_{i,j}(\mathscr{A}) \in I(m, t)$ *whenever* $\mathscr{A} \in I(m, t)$ *and* $i, j \in [m]$,
(iv) $|\mathscr{E}(S_{i,j}(\mathscr{A}))| \geqslant |\mathscr{E}(\mathscr{A})|$.

Let

$$LI_\alpha^n(t) = \{\mathscr{A} \in I_\alpha^n(t) : \mathscr{A} \text{ is left-shifted in every } M_k, \ k = 1, \ldots, n\}.$$

Now (1) and Lemma 5 imply

$$E(\alpha, n, d) = \max_{\mathscr{A} \in LI_\alpha^n(n-d)} |\mathscr{E}(\mathscr{A})|. \tag{2}$$

For any $\mathscr{A} \subseteq 2^{\mathbb{N}}$ let

$$\mathscr{B}(\mathscr{A}) = \{A \cap [n] : A \in \mathscr{A}\}.$$

We note that if $a \in \mathscr{H}_\alpha^n$ and $A \in \mathscr{C}_\alpha^n$ correspond under the bijection between $\mathscr{H}_\alpha^n$ and $\mathscr{C}_\alpha^n$ then $B(a) = A \cap [n]$, where $B(a)$ is defined in Section 1.

The systems in $LI^n_\alpha(n-d)$ have the following easily verified properties:

Lemma 6. *Let $\mathscr{A} \subseteq LI^n_\alpha(t)$. Then*

(i) $\mathscr{B}(\mathscr{A}) \in I(n,t)$, i.e. $|A_1 \cap A_2 \cap [n]| \geqslant t$ *for all* $A_1, A_2 \in \mathscr{A}$,
(ii) $\mathscr{B}(\mathscr{A}) \subseteq 2^{[n]}$ *is an upset, i.e.* $B_1 \in \mathscr{B}(\mathscr{A})$ *and* $B_1 \subseteq B_2 \subseteq [n]$ *imply* $B_2 \in \mathscr{B}(\mathscr{A})$.

If in addition $\mathscr{A} = \{A \in \mathscr{C}^n_\alpha : A \cap [n] \in \mathscr{B}(\mathscr{A})\}$ *then*

(iii)

$$|\mathscr{E}(\mathscr{A})| = \binom{\alpha}{2} \sum_{B \in \mathscr{B}(\mathscr{A})} (n-|B|)(\alpha-1)^{n-|B|-1}.$$

We note that (iii) applies for every $\mathscr{A} \subseteq LI^n_\alpha(n-d)$ with $E(\alpha,n,d) = |\mathscr{E}(\mathscr{A})|$.
By (2) and Lemma 6 we obtain a further reduction:

$$E(\alpha,n,d) = \binom{\alpha}{2} \max_{\mathscr{B} \in I(n,n-d)} \sum_{B \in \mathscr{B}} (n-|B|)(\alpha-1)^{n-|B|-1}. \tag{3}$$

Given a family $\mathscr{B} \subseteq 2^{[n]}$ and nonnegative real numbers (weights) ω_i, $i = 0, \ldots, n$, put

$$\omega(\mathscr{B}) = \sum_{B \in \mathscr{B}} \omega_{|B|} = \sum_{i=0}^{n} \left| \mathscr{B} \cap \binom{[n]}{i} \right| \omega_i$$

and

$$M(n,t,\omega) = \max_{\mathscr{B} \in I(n,t)} \omega(\mathscr{B}).$$

We consider the weights

$$\omega_i = (n-i)(\alpha-1)^{n-i-1}. \tag{4}$$

Then, according to (3) we have

$$E(\alpha,n,d) = \binom{\alpha}{2} M(n,n-d,\omega). \tag{5}$$

Let $\mathscr{F}_i(n,t) = \{B \subseteq [n] : |B \cap [t+2i]| \geqslant t+i\} \subseteq 2^{[n]}$, and recall the families $\mathscr{F}_i(\alpha,n,t) \subseteq \mathscr{H}^n_\alpha$ defined in the previous section. With Lemma 6(iii) we obtain

$$|\mathscr{E}(\mathscr{F}_i(\alpha,n,t))| = \binom{\alpha}{2} \omega(\mathscr{F}_i(n,t)). \tag{6}$$

Finally, (5) and (6) show that Theorem 4 is equivalent to the following:

Theorem 4′. *Let* $\alpha \geqslant 3$ *and* $\omega_i = (n-i)(\alpha-1)^{n-i-1}$, $i = 0, \ldots, n$. *Then*

$$M(n,t,\omega) = \begin{cases} \omega(\mathscr{F}_0(n,t)) & \text{if } t = 1, 2, \\ \max\limits_{0 \leqslant i \leqslant (n-t)/2} \omega(\mathscr{F}_i(n,t)) & \text{if } t > 2. \end{cases}$$

We remark that the cases $t = 1$ or n sufficiently large follow from results of Frankl [7, Theorem 5.2, Remark 5.3 and Theorem 5.4].

3. Auxiliary results

Lemma 7. *Let $\mathscr{S} \subseteq 2^{[m]}$ be a nonempty system of sets such that*

(i) *$\mathscr{S}$ is complement-closed, i.e. $A \in \mathscr{S}$ implies $\overline{A} = [m]\backslash A \in \mathscr{S}$ and*
(ii) *$\mathscr{S}$ is convex, i.e. $A, C \in \mathscr{S}$ and $A \subseteq B \subseteq C$ imply $B \in \mathscr{S}$.*

Furthermore, let $v_0 > v_1 > \cdots > v_m > 0$ be real numbers. Put

$$\frac{1}{Q} = \min\left\{\frac{v_{i+1}}{v_i} : 0 \leqslant i < m\right\}.$$

Then there exists an intersecting subsystem $\mathscr{I} \subseteq \mathscr{S}$ such that

$$v(\mathscr{I}) \geqslant \frac{1}{1+Q}\, v(\mathscr{S}). \tag{7}$$

Proof. We follow the construction of an intersecting system $\mathscr{I}$ given in [4].
If $\mathscr{S} = 2^{[m]}$ we may take $\mathscr{I} = \{A \subseteq [m] : m \in A\}$ since then

$$(1+Q)v(\mathscr{I}) = \sum_{A\subseteq[m-1]} v_{|A|+1} + Q\, v_{|A|+1} \geqslant \sum_{A\subseteq[m-1]} v_{|A|+1} + v_{|A|} = v(\mathscr{S})\,.$$

Assume now that $\mathscr{S} \neq 2^{[m]}$ and choose a set $B \in \mathscr{S}$ such that $B\backslash\{i\} \notin \mathscr{S}$ for some element $i \in [m]$. Consider the partition $\mathscr{S} = \mathscr{S}_1 \cup \mathscr{S}_2 \cup \mathscr{S}_3 \cup \mathscr{S}_4$, where

$$\begin{aligned}
\mathscr{S}_1 &= \{A \in \mathscr{S} : i \in A \text{ and } A\backslash\{i\} \in \mathscr{S}\},\\
\mathscr{S}_2 &= \{A \in \mathscr{S} : i \notin A \text{ and } A \cup \{i\} \in \mathscr{S}\},\\
\mathscr{S}_3 &= \{A \in \mathscr{S} : i \in A \text{ and } A\backslash\{i\} \notin \mathscr{S}\},\\
\mathscr{S}_4 &= \{A \in \mathscr{S} : i \notin A \text{ and } A \cup \{i\} \notin \mathscr{S}\}.
\end{aligned}$$

Clearly, $\overline{\mathscr{S}_1} = \{\overline{A} : A \in \mathscr{S}_1\} = \mathscr{S}_2$ and $\overline{\mathscr{S}_3} = \{\overline{A} : A \in \mathscr{S}_3\} = \mathscr{S}_4$, and hence $|\mathscr{S}_1| = |\mathscr{S}_2|$ and $|\mathscr{S}_3| = |\mathscr{S}_4|$. Further, $\mathscr{S}_1 = \{A \cup \{i\} : A \in \mathscr{S}_2\}$ and therefore

$$v(\mathscr{S}_1) = \sum_{A\in\mathscr{S}_1} v_{|A|} = \sum_{A\in\mathscr{S}_2} v_{|A|+1} \geqslant \frac{1}{Q} \sum_{A\in\mathscr{S}_2} v_{|A|} = \frac{1}{Q} v(\mathscr{S}_2). \tag{8}$$

Now $\mathscr{S}_1 \cup \mathscr{S}_3$ is clearly an intersecting system. It is easily verified that also the system $\mathscr{S}_1 \cup \mathscr{S}_4$ is intersecting. We may assume that

$$v(\mathscr{S}_1 \cup \mathscr{S}_3) \leqslant \frac{1}{1+Q}\, v(\mathscr{S}),$$

since otherwise we are done. Then necessarily

$$v(\mathscr{S}_2 \cup \mathscr{S}_4) \geqslant \frac{Q}{1+Q}\, v(\mathscr{S}). \tag{9}$$

Since $B \in \mathscr{S}_3$ we have $\mathscr{S}_4 = \overline{\mathscr{S}_3} \neq \emptyset$. Hence $v(\mathscr{S}_4) > (1/Q)v(\mathscr{S}_4)$, and with (8) and (9) we obtain

$$v(\mathscr{S}_1 \cup \mathscr{S}_4) > \frac{1}{Q}\, v(\mathscr{S}_2 \cup \mathscr{S}_4) \geqslant \frac{1}{1+Q}\, v(\mathscr{S}). \qquad \square$$

Remark. The proof shows that there is always an intersecting subsystem $\mathscr{I} \subseteq \mathscr{S}$ for which strict inequality in (7) holds unless $\mathscr{S} = 2^{[m]}$ and v_{i+1}/v_i is constant for $i = 0, \ldots, m-1$.

Corollary 8. *Let $\mathscr{S} \subseteq 2^{[m]}$ be a nonempty system of sets as in Lemma 7. Further let $\alpha > 1$ and $c > 0$ be reals. Then there exists an intersecting subsystem $\mathscr{I} \subseteq \mathscr{S}$ with*

$$\sum_{A\in\mathscr{I}} \frac{m-|A|+c}{(\alpha-1)^{|A|}} \geqslant \left(\frac{1}{1+(\alpha-1)(1+1/c)}\right) \sum_{A\in\mathscr{S}} \frac{m-|A|+c}{(\alpha-1)^{|A|}}. \tag{10}$$

For $m > 1$ there is an intersecting subsystem $\mathscr{I} \subseteq \mathscr{S}$ such that strict inequality in (10) *holds.*

We continue with further auxiliary results. Recall the families $\mathscr{F}_i(n,t)$ which we abbreviate here and in the next section by $\mathscr{F}_i$. Recall also the weights ω_i which were defined in (4).

Lemma 9. *The sequence $\omega(\mathscr{F}_i)$, $i = 0, \ldots, \lfloor (n-t)/2 \rfloor$ is unimodal. More precisely, for $0 < r \leqslant (n-t)/2$ we have*

$$\omega(\mathscr{F}_{r-1}) \leqslant \omega(\mathscr{F}_r)$$

if and only if

$$(\alpha-1)\left(1+\frac{1}{(n-t-2r)(1-1/\alpha)+r}\right) \leqslant 1+\frac{t-1}{r}.$$

Proof. The lemma follows by comparing the two numbers

$$\begin{aligned}
\omega(\mathscr{F}_{r-1}\backslash\mathscr{F}_r) &= \binom{t+2r-2}{t+r-1} \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} \omega_{j+t+r-1} \\
&= \binom{t+2r-2}{t+r-1} \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} (n-t-r+1-j)(\alpha-1)^{n-t-r-j} \\
&= \binom{t+2r-2}{t+r-1} (\alpha-1)^r \alpha^{n-t-2r} \left((n-t-2r)\left(1-\frac{1}{\alpha}\right)+r+1\right)
\end{aligned}$$

and

$$\begin{aligned}
\omega(\mathscr{F}_r\backslash\mathscr{F}_{r-1}) &= \binom{t+2r-2}{t+r-2} \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} \omega_{j+t+r} \\
&= \binom{t+2r-2}{t+r-2} (\alpha-1)^{r-1} \alpha^{n-t-2r} \left((n-t-2r)\left(1-\frac{1}{\alpha}\right)+r\right). \qquad \square
\end{aligned}$$

We need also the following numerical fact.

Lemma 10. *Let $\alpha > 1$ and $1 \leqslant r \leqslant (n-t)/2$. Then $\omega(\mathscr{F}_{r-1}) \leqslant \omega(\mathscr{F}_r)$ implies*

$$(\alpha-1)\left(1+\frac{1}{(n-t-2r)(1-1/\alpha)+s}\right) \leqslant 1+\frac{t-1}{s} \tag{11}$$

for every positive integer $s \leqslant r$. Moreover, for $t > 2$ we have strict inequality in (11) *unless $\omega(\mathscr{F}_{r-1}) = \omega(\mathscr{F}_r)$ and $s = r$.*

Proof. Put $c = (n-t-2r)(1-1/\alpha)$. It suffices to show that

$$\left(1+\frac{t-1}{s}\right) \Big/ \left(1+\frac{t-1}{s-1}\right) \leqslant \left(1+\frac{1}{c+s}\right) \Big/ \left(1+\frac{1}{c+s-1}\right)$$

for every integer $s \leqslant r$. Note that the LHS is decreasing in t, the RHS is increasing in c, and equality holds for $(t,c) = (2,0)$. $\square$

4. Proof of Theorem 4

We prove Theorem 4′. It follows essentially from the following two lemmas whose proofs utilize the methods from [1] ("generating sets") and [3] ("pushing–pulling").

Let $LI(n,t)$ denote the set of all left-shifted t-intersecting systems $\mathscr{B} \subseteq 2^{[n]}$. Put $\mathscr{F}_{\lfloor (n-t)/2 \rfloor + 1} = \mathscr{F}_{-1} = \emptyset$.

Lemma 11. *Let r be the smallest nonnegative integer such that $\omega(\mathscr{F}_r) > \omega(\mathscr{F}_{r+1})$. Then every $\mathscr{B} \in LI(n,t)$ with $\omega(\mathscr{B}) = M(n,t,\omega)$ is t-intersecting in $[t+2r]$, i.e. $|B_1 \cap B_2 \cap [t+2r]| \geqslant t$ for all $B_1, B_2 \in \mathscr{B}$.*

This is a special case of [5, Lemma 29] (let their (α, k) be our $(\alpha, n-1)$), and is thus not reproved here. Note that in [5] only the existence of a system $\mathscr{B}$ having the properties of Lemma 11 is stated (even under the weaker requirement $\omega(\mathscr{F}_r) \geqslant \omega(\mathscr{F}_{r+1})$; this already would suffice for the following arguments), and that the proof there gives indeed the above stronger statement (which, however, is only needed for uniqueness considerations).

We remark that the existence of a system $\mathscr{B}$ with the properties in Lemma 11 follows more generally for all weights which satisfy $\omega_i \geqslant \omega_{i+1}$, $i = t, \ldots, n-1$, and $\omega(\mathscr{F}_r) \geqslant \cdots \geqslant \omega(\mathscr{F}_{\lfloor (n-t)/2 \rfloor})$, see [6, Theorem 15].

Lemma 12. *Let $t \geqslant 2$. Let r be the largest integer such that $t+2r \leqslant n$ and $\omega(\mathscr{F}_{r-1}) < \omega(\mathscr{F}_r)$. Then every $\mathscr{B} \in LI(n,t)$ with $\omega(\mathscr{B}) = M(n,t,\omega)$ is invariant under exchanging coordinates in $[t+2r]$, i.e. $S_{i,j}(\mathscr{B}) = \mathscr{B}$ for all $i, j \in [t+2r]$.*

Proof. If $t = 2$ then $r = 0$ and $\omega(\mathscr{F}_0) > \omega(\mathscr{F}_1)$ by Lemma 9 (note that $\alpha \geqslant 3$). Lemma 11 shows that $\mathscr{B}$ is invariant in $[t]$.

Let $t > 2$. We consider two cases, first let $\omega(\mathscr{F}_r) > \omega(\mathscr{F}_{r+1})$.

By Lemma 11 every $\mathscr{B} \in LI(n,t)$ with $\omega(\mathscr{B}) = M(n,t,\omega)$ is t-intersecting in $[t+2r]$. Let $\mathscr{A} = \{B \cap [t+2r] : B \in \mathscr{B}\}$. Then, since $\omega(\mathscr{B}) = M(n,t,\omega)$ and $\omega_i > 0$ for $i = t, \ldots, n-1$, necessarily

$$\mathscr{B} = \{B \subseteq [n] : B \cap [t+2r] \in \mathscr{A}\},$$

and thus

$$\omega(\mathscr{B}) = \omega'(\mathscr{A})$$

where the new weights $\omega'_t, \ldots, \omega'_{t+2r}$ are given by

$$\omega'_i = \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} \omega_{i+j} = \sum_{j=0}^{n-t-2r} \binom{n-t-2r}{j} (n-i-j)(\alpha-1)^{n-i-j-1}$$
$$= (\alpha-1)^{t+2r-1} \alpha^{n-t-2r} \left(\frac{(n-t-2r)(1-1/\alpha) + t + 2r - i}{(\alpha-1)^i} \right).$$

Further, we clearly have $\mathscr{A} \in LI(t+2r, t)$. Let $\ell \leqslant t+2r$ be the largest integer such that $\mathscr{A}$ is invariant under exchanging coordinates in $[\ell]$. Assume that

$$\ell < t + 2r. \tag{12}$$

Consider the sets

$$\mathscr{A}' = \{A \in \mathscr{A} : S_{\ell+1,i}(A) \notin \mathscr{A} \text{ for some } 1 \leqslant i \leqslant \ell\},$$
$$\mathscr{A}^* = \{A \cap [\ell+2, t+2r] : A \in \mathscr{A}'\}.$$

Clearly, $\mathscr{A}' \neq \emptyset$ and hence $\mathscr{A}^* \neq \emptyset$. Now we state the following facts which (among others) follow from the pushing–pulling method [3,4] (see also [6]):

(i) $\ell \geqslant t$ and $\ell + t$ is even.

(ii) $\mathscr{A}^*$ is complement-closed (i.e. $A \in \mathscr{A}^*$ implies $[\ell+2, n] \backslash A \in \mathscr{A}^*$), and $\mathscr{A}^*$ is convex (i.e. $A_1 \subseteq A_2 \subseteq A_3$ and $A_1, A_3 \in \mathscr{A}^*$ imply $A_2 \in \mathscr{A}^*$).

(iii) For all intersecting subsystems $\mathscr{I}^*$ of $\mathscr{A}^*$,

$$\sum_{A\in\mathscr{I}^*} \omega'_{|A|+(\ell+t)/2} \leqslant \frac{\ell-t+2}{2(\ell+1)} \sum_{A\in\mathscr{A}^*} \omega'_{|A|+(\ell+t)/2}. \tag{13}$$

In view of (i) and (12) we may write $\ell = t+2s-2$ for an integer s with $1\leqslant s\leqslant r$. By (ii) we may apply Corollary 8 to the set system $\mathscr{A}^* \subseteq 2^{[t+2s,t+2r]}$; this gives an intersecting system $\mathscr{A}_1^* \subseteq \mathscr{A}^*$ such that

$$\sum_{A\in\mathscr{A}_1^*} \frac{m-|A|+c}{(\alpha-1)^{|A|}} \geqslant \frac{1}{1+(\alpha-1)(1+1/c)} \sum_{A\in\mathscr{A}^*} \frac{m-|A|+c}{(\alpha-1)^{|A|}} \tag{14}$$

for $m = t+2r-\ell-1 = 2(r-s)+1$ and any constant c. We put

$$c = (n-t-2r)\left(1-\frac{1}{\alpha}\right)+s$$

in order to get from (14)

$$\sum_{A\in\mathscr{A}_1^*} \omega'_{|A|+(\ell+t)/2} \geqslant \frac{1}{1+(\alpha-1)(1+1/c)} \sum_{A\in\mathscr{A}^*} \omega'_{|A|+(\ell+t)/2}. \tag{15}$$

Now, recalling our choice of r, we obtain from Lemma 10

$$\frac{1}{1+(\alpha-1)(1+1/c)} > \frac{1}{2+(t-1)/s} = \frac{\ell-t+2}{2(\ell+1)}, \tag{16}$$

which in view of (13) and (15) shows that the intersecting system $\mathscr{A}_1^* \subseteq \mathscr{A}^*$ contradicts fact (iii).

Thus, the assumption (12) is false, i.e. $\mathscr{A}$ and hence also $\mathscr{B}$ is invariant in $[t+2r]$.

Replacing r by $r+1$ in the above arguments (including Lemma 10) except in (12), in the condition $1\leqslant s\leqslant r$ and in the conclusion that $\mathscr{A}$ is invariant in $[t+2r]$, yields a proof in the case $\omega(\mathscr{F}_r) = \omega(\mathscr{F}_{r+1})$. □

Proof of Theorem 4′. Let $\mathscr{B} \in I(n,t)$ with $M(n,t,\omega) = \omega(\mathscr{B})$. According to Lemma 5 we can assume that $\mathscr{B} \in LI(n,t)$.

Let r be the largest integer such that $t+2r\leqslant n$ and $\omega(\mathscr{F}_{r-1}) \leqslant \omega(\mathscr{F}_r)$ hold.

For $t=1$ and $t=2$ we obtain from Lemma 9 (note $\alpha\geqslant 3$) that $r=0$. Then Lemma 11 gives $\mathscr{B} \subseteq \mathscr{F}_0$, i.e. $M(n,t,\omega) = \omega(\mathscr{F}_0)$.

Let $t>2$. By Lemma 11, $\mathscr{B}$ is t-intersecting in $[t+2r]$. If $\omega(\mathscr{F}_{r-1}) < \omega(\mathscr{F}_r)$ then Lemma 12 shows that $\mathscr{B}$ is also invariant in $[t+2r]$, which clearly implies $\mathscr{B} \subseteq \mathscr{F}_r$, i.e. $M(n,t,\omega) = \omega(\mathscr{F}_r)$. If $\omega(\mathscr{F}_{r-1}) = \omega(\mathscr{F}_r)$ then Lemma 12 shows that $\mathscr{B}$ is invariant in $[t+2r-2]$. It follows $\mathscr{B} \subseteq \mathscr{F}_{r-1}$ or $\mathscr{B} \subseteq \mathscr{F}_r$, i.e. $M(n,t,\omega) = \max\{\omega(\mathscr{F}_{r-1}), \omega(\mathscr{F}_r)\}$. □

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