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On the average size of sets in intersecting Sperner families

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Abstract

We show that the average size of subsets of $[n]$ forming an intersecting Sperner family of cardinality not less than $\binom{n-1}{k-1}$ is at least k provided that $k \leqslant n/2 - \sqrt{n}/2 + 1$. The statement is not true if $n/2 \geqslant k > n/2 - \sqrt{8n+1}/8 + 9/8$. © 2002 Elsevier Science B.V. All rights reserved.

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1. Introduction

Let $[n]$ be the set $\{1,\ldots,n\}$ and $2^{[n]}$ be the power set of $[n]$. A set $\mathscr{F} \subseteq 2^{[n]}$ is called a *Sperner family* (or *antichain*) if there are no inclusion relations between the members of $\mathscr{F}$:

$$A \nsubseteq B \quad \text{for all } A, B \in \mathscr{F}, A \neq B.$$

A family $\mathscr{F} \subseteq 2^{[n]}$ is called *intersecting* if any two members of $\mathscr{F}$ are nondisjoint:

$$A \cap B \neq \emptyset \quad \text{for all } A, B \in \mathscr{F}.$$

In [12], Kleitman and Milner found the following result on the average size of sets in Sperner families:

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Theorem 1. *Let $k \leqslant n/2$ be an integer. If $\mathscr{F} \subseteq 2^{[n]}$ is a Sperner family with $|\mathscr{F}| \geqslant \binom{n}{k}$, then the average size of the sets in $\mathscr{F}$ is at least k.*

Kleitman and Milner gave two proofs for their result, one using replacement operations as in [13], the other using the LYM-inequality and linear duality. See also [3,8,9,10], [1, p. 62] and [2, p. 155].

In this note we address the question whether the corresponding statement of Theorem 1 remains true for intersecting Sperner families. Note that by the Erdős–Ko–Rado Theorem [5] the maximum size of an intersecting Sperner family consisting only of sets of size k, $k \leqslant n/2$, is $\binom{n-1}{k-1}$. Thus, we ask whether the average size of the sets of an intersecting Sperner family $\mathscr{F} \subseteq 2^{[n]}$ is at least k provided that $|\mathscr{F}| \geqslant \binom{n-1}{k-1}$.

This is certainly true if all sets in $\mathscr{F}$ have size not greater than $n/2$, which follows directly from Theorem 1 using a correspondence between intersecting Sperner families in $2^{[n]}$ and Sperner families in $2^{[n-1]}$ ([6], see also [1, Chapter 8]).

Putting no restrictions on the setsizes, we have the following result.

Theorem 2. *Let $k \leqslant (n+2)/2 - \sqrt{n}/2$. If $\mathscr{F} \subseteq 2^{[n]}$ is an intersecting Sperner family with $|\mathscr{F}| \geqslant \binom{n-1}{k-1}$, then the average size of sets in $\mathscr{F}$ is at least k.*

This statement fails if $n/2 \geqslant k > n/2 - \sqrt{8n+1}/8 + 9/8$.

One might ask for the smallest cardinality an intersecting Sperner family must have in order to ensure an average setsize not less than k. An easy general upper bound for this cardinality is given by the next theorem:

Theorem 3. *Let $k \leqslant n/2$. If $\mathscr{F} \subseteq 2^{[n]}$ is an intersecting Sperner family with $|\mathscr{F}| \geqslant \binom{n}{k-1}$, then the average size of sets in $\mathscr{F}$ is at least k.*

The proofs of Theorems 2 and 3 use the known description [4,11] of the convex hull of all profiles of intersecting Sperner families. Recall that the *profile* of a family $\mathscr{F} \subseteq 2^{[n]}$ is the $(n+1)$-dimensional vector having the number of i-sets of $\mathscr{F}$ as its ith entry $(0 \leqslant i \leqslant n)$.

2. Proof of Theorem 3

Our proof follows the one given in [9] for Theorem 1 (see also [1, p. 62]).

Let $\mathscr{F} \subseteq 2^{[n]}$ be an intersecting Sperner family with profile $(f_0, f_1, \ldots, f_n)$. The following LYM-type-inequality was proved in [7]

$$\sum_{i \leqslant n/2} \frac{f_i}{\binom{n}{i-1}} + \sum_{i > n/2} \frac{f_i}{\binom{n}{i}} \leqslant 1. \tag{1}$$

Let $g(i)$ denote the coefficient of f_i in this inequality. It is easy to verify that the sequence $g(i)$, $i = 1, \ldots, n$, is convex. If g is extended to a function on the real interval $[1,n]$ by linear interpolation, then Jensen's inequality, (1) and the hypothesis $|\mathscr{F}|\geqslant\binom{n}{k-1}$ give

$$g\left(\sum_i \frac{f_i}{|\mathscr{F}|}\, i\right) \leqslant \sum_i \frac{f_i}{|\mathscr{F}|}\, g(i) \leqslant \frac{1}{|\mathscr{F}|} \leqslant g(k).$$

Now the theorem follows from the monotonity of the function $g(x)$, $1\leqslant x\leqslant k$.

We remark that instead of (1) other inequalities [11] might be used as well, which give slight improvements of the bound $\binom{n-1}{k}$. We omit the details.

3. A preliminary lemma

Let us first record the following easily established numerical fact: If (as in the supposition of Theorem 2)

$$k\leqslant \frac{n+2}{2} - \frac{\sqrt{n}}{2},$$

then

$$(n+1-2k)(n+2-2k)\geqslant 2(k-1) \tag{2}$$

and the sequence

$$\left(\frac{n+1}{2} - i\right)\binom{n-1}{i-1}, \quad i=1,\ldots,k \tag{3}$$

is increasing.

Lemma 4. *Let $a<b<c<d$, $e\leqslant n$ natural and δ,ε nonnegative real numbers satisfying $b\leqslant n/2$,*

$$\frac{b-a}{d-b} > \delta, \tag{4}$$

$$(n+1-2b)(d-b)\geqslant b, \tag{5}$$

$$(n+1-2b)(e-b)\geqslant b. \tag{6}$$

Then

$$(c-b+(e-b)\varepsilon)\,\frac{1}{\binom{n}{a}} + (b-a-(d-b)\delta)\frac{1}{\binom{n}{c}}$$

$$\geqslant (c-a+(e-a)\varepsilon-(d-c)\delta+(e-d)\delta\varepsilon)\,\frac{1}{\binom{n}{b}}. \tag{7}$$

Proof. Let $\binom{n}{x}$ for $0 \leqslant x \leqslant n$, x real, be defined by linear interpolation of the sequence $1/\binom{n}{i}$, $i = 0, \ldots, n$, i.e.

$$\frac{1}{\binom{n}{x}} := \frac{1-(x-\lfloor x \rfloor)}{\binom{n}{\lfloor x \rfloor}} + \frac{x-\lfloor x \rfloor}{\binom{n}{\lfloor x \rfloor + 1}}.$$

It is well known that the function $1/\binom{n}{x}$, $0 \leqslant x \leqslant n$, is convex. We will use the function

$$\psi(x) := \frac{b-x}{\binom{n}{b}/\binom{n}{x} - 1}, \quad 0 \leqslant x < b.$$

It is easily established that ψ is increasing on the entire domain $[0, b)$. Indeed, if $x = i + \alpha$, $i < b$ a nonnegative integer, $0 \leqslant \alpha \leqslant 1$ real, we have

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\psi(i+\alpha) = \frac{1 + (b-i-1)\binom{n}{b}/\binom{n}{i} - (b-i)\binom{n}{b}/\binom{n}{i+1}}{\left(\binom{n}{b}\left((1-\alpha)/\binom{n}{i} + \alpha/\binom{n}{i+1}\right) - 1\right)^2},$$

which is nonnegative by the convexity of the sequence $1/\binom{n}{i}$, $i = 0, \ldots, n$. (The monotonity of the sequence $\psi(i)$, $i \neq b$, $i \leqslant n/2$, was already used in [12].) Furthermore, the function ψ is constant on $[b-1, b)$:

$$\psi(x) = \frac{b}{n+1-2b} \quad \text{for } b-1 \leqslant x < b.$$

Now the LHS of (7) is by Jensen's inequality not less than

$$(c-a+(e-b)\varepsilon-(d-b)\delta)\frac{1}{\binom{n}{b^*}},$$

where

$$b^* = b - \frac{(b-a)(e-b)\varepsilon + (c-b)(d-b)\delta}{c-a+(e-b)\varepsilon-(d-b)\delta}.$$

To establish (7), it suffices therefore to show that

$$\frac{\binom{n}{b}}{\binom{n}{b^*}} \geqslant \frac{c-a+(e-a)\varepsilon-(d-c)\delta+(e-d)\delta\varepsilon}{c-a+(e-b)\varepsilon-(d-b)\delta},$$

which is easily seen to be equivalent to

$$\psi(b^*) \leqslant \frac{(b-a)(e-b)\varepsilon+(c-b)(d-b)\delta}{(b-a)\varepsilon+(c-b)\delta+(e-d)\delta\varepsilon}.$$

Since by (4) we have $b^* < b$, it suffices to show by the above-mentioned properties of ψ that the RHS of the last inequality is not less than $b/(n+1-2b)$. If $\varepsilon = 0$ resp. $\delta = 0$ this is just (5) resp. (6). If $\delta, \varepsilon > 0$, we want to show that

$$\frac{b-a}{\delta}((n+1-2b)(e-b)-b) + \frac{c-b}{\varepsilon}((n+1-2b)(d-b)-b)$$
$$\geqslant b(e-d).$$

However, using (4), the last inequality follows from

$$(d-b)(n+1-2b)\left(e-b+\frac{c-b}{\varepsilon}\right)\geqslant b(e-d)+b(d-b)+b\left(\frac{c-b}{\varepsilon}\right)$$

$$=b\left(e-b+\frac{c-b}{\varepsilon}\right),$$

which is (5). □

4. Proof of Theorem 2

We start with the first statement of Theorem 2. Our proof method follows the proofs of Theorem 1 given in [2,10].

Let $\mathscr{P}\subseteq\mathbb{R}^{n+1}$ be the convex hull of all profiles of intersecting Sperner families in $2^{[n]}$. The extreme points of the polytope $\mathscr{P}$ were determined in [5]. They are

$$z=(0,0,\ldots,0),$$

$$v_j=\left(0,0,\ldots,\binom{n}{j},\ldots,0\right),\quad j>n/2,$$

$$w_i=\left(0,0,\ldots,\binom{n-1}{i-1},\ldots,0\right),\quad i\leqslant n/2,$$

$$w_{ij}=\left(0,0,\ldots,\binom{n-1}{i-1},\ldots,\binom{n-1}{j},\ldots,0\right),\quad i\leqslant n/2,\ i+j>n,$$

where the nonzero entries of v_j, w_i resp. w_{ij} occur at the coordinates j, i resp. i and j.

If $\mathscr{F}\subseteq 2^{[n]}$ is an intersecting Sperner family with $|\mathscr{F}|\geqslant\binom{n-1}{k-1}$, then the profile $(f_0,f_1,\ldots,f_n)$ of $\mathscr{F}$ lies in the intersection of $\mathscr{P}$ and the halfspace given by

$$\sum_{i=0}^{n}x_i\geqslant\binom{n-1}{k-1}. \qquad (8)$$

We denote this new polytope by $\mathscr{P}'$. The average size of sets in $\mathscr{F}$ will be at least k iff the profile of $\mathscr{F}$ satisfies the linear inequality

$$\sum_{i=0}^{n}(i-k)f_i\geqslant 0. \qquad (9)$$

Hence, it is enough to verify (9) (under the hypothesis of Theorem 2) only for the extreme points of $\mathscr{P}'$. Obviously, each extreme point of $\mathscr{P}'$ is a convex combination of two extreme points of $\mathscr{P}$. Consequently, it is sufficient to prove the following implication: If p_1 and p_2 are extreme points of $\mathscr{P}$ such that $\alpha p_1+(1-\alpha)p_2$ satisfies (8) for some $0\leqslant\alpha\leqslant 1$, then $\alpha p_1+(1-\alpha)p_2$ satisfies also (9).

Let us write

$$p_1 = \left(0, 0, \ldots, \sigma_{i_1}\binom{n-1}{i_1-1}, \ldots, \sigma_{j_1}\binom{n-1}{j_1}, \ldots, \sigma_{\ell_1}\binom{n}{\ell_1}, \ldots, 0\right)$$

(the nonzero entries located at coordinates i_1, j_1 and ℓ_1) with $i_1 \leqslant n/2$, $i_1 + j_1 > n$, $\ell_1 > n/2$ and

$$(\sigma_{i_1}, \sigma_{j_1}, \sigma_{\ell_1}) \in \{(0,0,0), (1,0,0), (0,0,1), (1,1,0)\}.$$

An analogous notation with $(\sigma_{i_2}, \sigma_{j_2}, \sigma_{\ell_2})$ instead of $(\sigma_{i_1}, \sigma_{j_1}, \sigma_{\ell_1})$ is used for p_2 (where for simplicity of notation the variables $\sigma_{i_2}, \sigma_{j_2}, \sigma_{\ell_2}$ are considered to be different from $\sigma_{i_1}, \sigma_{j_1}, \sigma_{\ell_1}$).

Let $k \leqslant (n+2)/2 - \sqrt{n}/2$. We want to show that for $0 \leqslant \alpha \leqslant 1$ the following is true: If

$$\alpha\sigma_{i_1}\binom{n-1}{i_1-1} + (1-\alpha)\sigma_{i_2}\binom{n-1}{i_2-1} + \alpha\sigma_{j_1}\binom{n-1}{j_1}$$
$$+ (1-\alpha)\sigma_{j_2}\binom{n-1}{j_2} + \alpha\sigma_{\ell_1}\binom{n}{\ell_1} + (1-\alpha)\sigma_{\ell_2}\binom{n}{\ell_2} \geqslant \binom{n-1}{k-1}, \qquad (10)$$

then

$$(i_1-k)\alpha\sigma_{i_1}\binom{n-1}{i_1-1} + (i_2-k)(1-\alpha)\sigma_{i_2}\binom{n-1}{i_2-1} + (j_1-k)\alpha\sigma_{j_1}\binom{n-1}{j_1}$$
$$+ (j_2-k)(1-\alpha)\sigma_{j_2}\binom{n-1}{j_2} + (\ell_1-k)\alpha\sigma_{\ell_1}\binom{n}{\ell_1}$$
$$+ (\ell_2-k)(1-\alpha)\sigma_{\ell_2}\binom{n}{\ell_2} \geqslant 0. \qquad (11)$$

Since (11) trivially holds if $i_1, i_2 \geqslant k$, we may assume that $i_1 < k$ and $\sigma_{i_1} = 1$. Then necessarily $\sigma_{\ell_1} = 0$.

Case 1: $i_2 \leqslant k$ or $\sigma_{i_2} = 0$.

By lower estimating $(j_1 - k)$, $(j_2 - k)$, $(\ell_1 - k)$ and $(\ell_2 - k)$ to $(n+1)/2 - k$ and using (10) we have that the LHS of (11) is not less than

$$\left(\frac{n+1}{2} - k\right)\binom{n-1}{k-1}$$
$$- \left(\frac{n+1}{2} - i_1\right)\alpha\binom{n-1}{i_1-1} - \left(\frac{n+1}{2} - i_2\right)(1-\alpha)\sigma_{i_2}\binom{n-1}{i_2-1},$$

which in both cases $i_2 \leqslant k$ and $\sigma_{i_2} = 0$ is nonnegative by the monotonity (3).

Case 2: $i_2 > k$ and $\sigma_{i_2} = 1$.

We have then $\sigma_{\ell_1} = \sigma_{\ell_2} = 0$ and $\binom{n-1}{i_2-1} \geqslant \binom{n-1}{k-1}$.

Case 2.1: $\binom{n-1}{i_1-1} + \sigma_{j_1}\binom{n-1}{j_1} \geqslant \binom{n-1}{i_2-1}$.

Then necessarily $\sigma_{j_1} = 1$. Using the last two inequalities we have that the LHS of (11) is not less than

$$(j_1 - k)\alpha\binom{n-1}{j_1} + (i_1 - k)\alpha\binom{n-1}{i_1-1}$$

$$\geqslant (j_1 - k)\alpha\binom{n-1}{k-1} - (j_1 - i_1)\alpha\binom{n-1}{i_1-1},$$

which is again nonnegative by the monotonity (3).

Case 2.2: $\binom{n-1}{i_1-1} + \sigma_{j_1}\binom{n-1}{j_1} < \binom{n-1}{i_2-1}$.

By eliminating α in (10) and (11), it suffices to show that

$$\left(k - i_1 - (j_1 - k)\sigma_{j_1}\frac{\binom{n-1}{j_1}}{\binom{n-1}{i_1-1}}\right)\frac{1}{\binom{n-1}{i_2-1}}$$

$$+\left(i_2 - k + (j_2 - k)\sigma_{j_2}\frac{\binom{n-1}{j_2}}{\binom{n-1}{i_2-1}}\right)\frac{1}{\binom{n-1}{i_1-1}}$$

$$\geqslant \left(i_2 - i_1 + (j_2 - i_1)\sigma_{j_2}\frac{\binom{n-1}{j_2}}{\binom{n-1}{i_2-1}} - (j_1 - i_2)\sigma_{j_1}\frac{\binom{n-1}{j_1}}{\binom{n-1}{i_1-1}}\right.$$

$$\left. + (j_2 - j_1)\sigma_{j_1}\sigma_{j_2}\frac{\binom{n-1}{j_1}}{\binom{n-1}{i_1-1}}\frac{\binom{n-1}{j_2}}{\binom{n-1}{i_2-1}}\right)\frac{1}{\binom{n-1}{k-1}}.$$

We apply Lemma 4 with $a := i_1 - 1,\ b := k - 1,\ c := i_2 - 1,\ d := j_1 - 1,\ e := j_2 - 1,$ $\delta := \sigma_{j_1}\binom{n-1}{j_1}/\binom{n-1}{i_1-1}$, $\varepsilon := \sigma_{j_2}\binom{n-1}{j_2}/\binom{n-1}{i_2-1}$ and $n := n - 1$.

Since (11) holds if $\sigma_{j_1} = 1$ and

$$(j_1 - k)\binom{n-1}{j_1} \geqslant (k - i_1)\binom{n-1}{i_1-1},$$

we may assume the opposite; thus condition (4) from Lemma 4 is satisfied. Finally, conditions (5) and (6) follow from $j_1, j_2 \geqslant (n+1)/2$ and (2).

This completes the proof of the first statement of Theorem 2. In order to show the second statement, consider an intersecting Sperner family $\mathscr{F} \subseteq 2^{[n]}$ with $f_{k-1} = \binom{n-1}{k-2}$, $f_{n+2-k} = \binom{n-1}{k-1} - \binom{n-1}{k-2}$ and $f_i = 0$ for $i \neq k-1, n+2-k$. Since $\binom{n-1}{k-1} - \binom{n-1}{k-2} \leqslant \binom{n-1}{n+2-k} = \binom{n-1}{k-3}$ for $n/2 \geqslant k > n/2 - \sqrt{8n+1}/8 + 9/8$, such a family can be taken as a subfamily of one realizing the profile $w_{k-1,n+2-k}$. It is now easily checked that the inequality (9) fails exactly for our choice of k. □

Remark. We conjecture that the first statement of Theorem 2 remains valid for all $k<n/2-\sqrt{8n}/8$. However, our proof method will not give this result: There is a constant $c>\sqrt{8}/4$ such that for $k=\lfloor n/2-c\sqrt{n}/2\rfloor$ and large n, the polytope $\mathscr{P}'$ contains a point which does not satisfy the inequality (9). Indeed, take a suitable convex combination $\alpha w_{i_1}+(1-\alpha)w_{i_2,j_2}$, where e.g. $i_1=\lfloor \frac{n}{2}-0.8\frac{\sqrt{n}}{2}\rfloor$, $k=\lfloor \frac{n}{2}-0.76\frac{\sqrt{n}}{2}\rfloor$, $i_2=\lfloor \frac{n}{2}-0.3\frac{\sqrt{n}}{2}\rfloor$, $j_2=\lfloor \frac{n}{2}+0.31\frac{\sqrt{n}}{2}\rfloor$.

References

[1] I. Anderson, Combinatorics of Finite Sets, Clarendon Press, Oxford, 1987.

[2] K. Engel, Sperner Theory, Cambridge University Press, Cambridge, 1997.

[3] P.L. Erdős, U. Faigle, W. Kern, On the average rank of LYM-sets, Discrete Math. 144 (1995) 11–22.

[4] P.L. Erdős, P. Frankl, G.O.H. Katona, Intersecting Sperner families and their convex hulls, Combinatorica 4 (1984) 21–34.

[5] P. Erdős, C. Ko, R. Rado, Intersection theorems for systems of finite sets, Quart. J. Math. Oxford Ser. 12 (1961) 313–320.

[6] C. Greene, A.J.W. Hilton, Some results on Sperner families, J. Combin. Theory Ser. A 26 (1979) 202–209.

[7] C. Greene, G.O.H. Katona, D.J. Kleitman, Extensions of the Erdős–Ko–Rado theorem, Stud. Appl. Math. 55 (1976) 1–8.

[8] C. Greene, D.J. Kleitman, Proof techniques in the theory of finite sets, in: G.-C. Rota (Ed.), Studies in Combinatorics, MAA Studies in Math., Vol. 17, Mathematical Association of America, Washington, D.C., 1978, pp. 22–79.

[9] G.R. Grimmett, A generalization of a theorem of Kleitman and Milner, Bull. London Math. Soc. 5 (1973) 157–158.

[10] G.O.H. Katona, Extremal problems for finite sets and convex hulls—A survey, Discrete Math. 164 (1997) 175–185.

[11] G.O.H. Katona, G. Schild, Linear inequalities describing the class of intersecting Sperner families of subsets, I, in: R. Bodendieck, R. Henn (Eds.), Topics in Combinatorics and Graph Theory, Essays in Honour of Gerhard Ringel, Physica-Verlag, Heidelberg, 1990, pp. 413–420.

[12] D.J. Kleitman, E.C. Milner, On the average size of sets in a Sperner family, Discrete Math. 6 (1973) 141–147.

[13] E. Sperner, Ein Satz über Untermengen einer endlichen Menge, Math. Z. 27 (1928) 544–548.