# OLD AND NEW RESULTS FOR THE WEIGHTED T–INTERSECTION PROBLEM VIA AK–METHODS

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**Dedicated to Professor Rudolf Ahlswede on occasion of his 60th birthday.**

**Abstract:** Let $[n] := \{1, \ldots, n\}$, $2^{[n]}$ be the power set of $[n]$ and $s \in [n]$. A family $\mathcal{F} \subseteq 2^{[n]}$ is called *t-intersecting in* $[s]$ if

$$|X_1 \cap X_2 \cap [s]| \geq t \text{ for all } X_1, X_2 \in \mathcal{F}.$$

Let $\omega : 2^{[n]} \to \mathbb{R}_+$ be a given weight function and

$$M_s(n, t; \omega) := \max\{\omega(\mathcal{F}) : \mathcal{F} \text{ is } t\text{-intersecting in } [s]\}.$$

For several weight functions, the numbers $M_n(n, t; \omega)$ can be determined using three important methods of Ahlswede and Khachatrian: Generating Sets [2], Comparison Lemma [4], and Pushing–Pulling [3]. We survey these methods.

Also, sufficient conditions on $\omega$ for the equality

$$M_s(n, t; \omega) = M_n(n, t; \omega)$$

are presented which simplify the method of Generating Sets. In addition, analogous conditions are given for the case that $|\cap_{X \in \mathcal{F}} X| < t$ is required (nontrivial $t$-intersection).

Applications of these methods include new intersection theorems for chain– and star products.

## INTRODUCTION AND NOTATION

In this paper we give a survey and discuss some new results for and insights into the problem of determining the maximum weight of $t$-intersecting families



*I. Althöfer et al. (eds.), Numbers, Information and Complexity, 45-74.*

of subsets of a finite set. The ingenious, relatively elementary methods were elaborated by Ahlswede and Khachatrian in several papers [2, 1, 4, 3]. Our aim is to provide a unifying approach such that most of the results are covered. Since Erdős, Ko and Rado [11] have initiated the study of such problems in the thirties many results were obtained by several authors. Here we cite only the recent papers which are related to the new AK-methods. More on the history of the results can be found in the corresponding papers. Moreover, in order to avoid too much technical details we describe only one and not all optimal families though in most cases Ahlswede and Khachatrian also proved the uniqueness of the optimal family up to permutation of the elements.

Let $\mathbb{N}$ be the set of natural numbers, $[n] := \{1, \dots, n\}$ and for $i, j \in \mathbb{N}, i < j$, let $[i, j] := \{i, i+1, \dots, j\}$. Let $2^{[n]}$ (resp. $\binom{[n]}{k}$) be the family of all (resp. all $k$-element) subsets of $[n]$. Each subfamily of $\binom{[n]}{k}$ is said to be *$k$-uniform*. A family $\mathcal{F} \subseteq 2^{[n]}$ is called *$t$-intersecting* if $|X_1 \cap X_2| \geq t$ for all $X_1, X_2 \in \mathcal{F}$ (1-intersecting is abbreviated by *intersecting*). We will suppose throughout that $1 \leq t \leq n-1$. Let $I(n, t)$ be the class of all $t$-intersecting families of subsets of $[n]$.

Suppose that we are given a *weight function* $\omega : 2^{[n]} \to \mathbb{R}_+$ (the set of all nonnegative reals). For $\mathcal{F} \subseteq 2^{[n]}$ let

$$\omega(\mathcal{F}) := \sum_{X \in \mathcal{F}} \omega(X).$$

The *weighted $t$-intersection problem* is the problem of determining

$$M(n, t; \omega) := \max\{\omega(\mathcal{F}) : \mathcal{F} \in I(n, t)\}.$$

In several applications the weight function depends only on the size of the subsets, i.e. we have $\omega(X_1) = \omega(X_2)$ if $|X_1| = |X_2|$. In this case we call $\omega$ *size-dependent* and we set $\omega_i := \omega(X)$ for $|X| = i$. Each family $\mathcal{F} \subseteq 2^{[n]}$ may be partitioned into (possibly empty) subfamilies $\mathcal{F}_i := \{X \in \mathcal{F} : |X| = i\}$. We put $f_i := f_i(\mathcal{F}) := |\mathcal{F}_i|$. The vector $(f_0, \dots, f_n)$ is called the *profile* of $\mathcal{F}$. A special case of the weighted $t$-intersection problem is the *size-dependent weighted $t$-intersection problem*: For $\boldsymbol{\omega} = (\omega_0, \dots, \omega_n) \in \mathbb{R}_+^{n+1}$ determine

$$M(n, t; \boldsymbol{\omega}) := \max\left\{\sum_{i=0}^{n} \omega_i f_i : \mathcal{F} \in I(n, t)\right\}.$$

Candidates for the solution are the families

$$\mathcal{S}_r^n := \{X \subseteq [n] : |X \cap [t+2r]| \geq t + r\}, r = 0, \dots, \lfloor \tfrac{n-t}{2} \rfloor$$

which are easily seen to be $t$-intersecting. Some further candidates are

$$\begin{aligned} \mathcal{S}_{r,i}^n := \{X \subseteq [n] : |X \cap [t+2r]| \geq t + r \text{ and } i \leq |X| < n + t - i\} \\ \cup\{X \subseteq [n] : |X| \geq n + t - i\}, r = 0, \dots, \lfloor \tfrac{n-t}{2} \rfloor, i = t + r, \dots, \lfloor \tfrac{n+t}{2} \rfloor. \end{aligned}$$

In the following we will omit the upper index $n$ if the basic set $[n]$ is clear from the context. Note that $\mathcal{S}_r = \mathcal{S}_{r,t+r}$.

For instance, if $n = 8$, $t = 2$, $\omega = (0,0,0,0,1,0,1,0,0)$ we have $\omega(\mathcal{S}_{1,4}) = 45$, whereas $\omega(\mathcal{S}_0) = 30$, $\omega(\mathcal{S}_1) = 39$, $\omega(\mathcal{S}_2) = 43$, $\omega(\mathcal{S}_3) = 28$, so these further candidates should not be forgotten.

In particular, for $t = 1$, Erdős, Frankl and Katona [12] proved:

**Theorem 1.** *The optimum in the size–dependent weighted* 1*-intersection problem is attained at one of the families* $\mathcal{S}_0, \mathcal{S}_{0,2}, \ldots, \mathcal{S}_{0,\lfloor \frac{n+1}{2} \rfloor}$, *and each of these families is optimal for some weight function.* □

Unfortunately, we do not have such a general theorem for $t > 1$. The so far strongest result is given by the celebrated complete intersection theorem of Ahlswede, Khachatrian [2] which solves the case $\omega = e_k$ where $e_k$ is the $n+1$-dimensional unit vector with 1 at coordinate $k$, $k = 0, \ldots, n$.

**Theorem 2 (Complete Intersection Theorem)** *Let* $\omega = e_k$, $k \geq t$. *The optimum in the size–dependent weighted* $t$*-intersection problem is attained at* $\mathcal{S}_r$, $r = 0, \ldots, \lfloor \frac{n-t}{2} \rfloor$, *if*

$$(k - t + 1)\left(2 + \frac{t-1}{r+1}\right) \leq n \leq (k - t + 1)\left(2 + \frac{t-1}{r}\right) \tag{1}$$

*(with the definition* $\infty := \frac{t-1}{0}$ *for all* $t \geq 1$*) and at* $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$ *if*

$$n < (k - t + 1)\left(2 + \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor + 1}\right). \tag{2}$$

□

We note that (2) is equivalent to $n \leq 2k - t$ and that in this case $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor} \supseteq \binom{[n]}{k}$.

Define

$$k_r^n := \begin{cases} t & \text{if } r = -1 \\ \frac{(r+1)n}{t+2r+1} + t - 1 & \text{if } r = 0, \ldots, \lfloor \frac{n-t}{2} \rfloor. \end{cases}$$

We omit again the upper index $n$ if the basic set $[n]$ is clear from the context. Note that $k_{-1} \leq k_0 \leq \ldots \leq k_{\lfloor \frac{n-t}{2} \rfloor}$. It is easy to see that an equivalent formulation of Theorem 2 is the following:

**Theorem 2a.** *The maximum size of a* $k$*-uniform* $t$*-intersecting family in* $2^{[n]}$ *is attained at* $\mathcal{S}_r \cap \binom{[n]}{k}$, *if* $k_{r-1} \leq k \leq k_r$, $r = 0, \ldots, \lfloor \frac{n-t}{2} \rfloor$, *and at* $\binom{[n]}{k}$ *if* $k > k_{\lfloor \frac{n-t}{2} \rfloor}$. □

As a direct consequence we obtain:

**Corollary 3.** *Suppose that for some* $r \in \{0, \ldots, \lfloor \frac{n-t}{2} \rfloor\}$

$$\omega_i = 0 \text{ unless } k_{r-1} \leq i \leq k_r. \tag{3}$$

*Then the optimum in the size–dependent weighted t-intersection problem is attained at $\mathcal{S}_r$.* □

**Remark.** *If we replace condition (3) by*

$$\omega_i = 0 \text{ unless } k_{r-1} \le \ell \le i \le k_r \text{ or } n+t-\ell \le i$$

*then the optimum is attained at $\mathcal{S}_{r,\ell}$ since $\mathcal{S}_{r,\ell}$ contains the maximum possible number of members from $\binom{[n]}{i}$ if $i$ belongs to the first interval and all members from $\binom{[n]}{i}$ if $n+t-\ell \le i$.*

Corollary 3 can be sharpened to the following theorem which will be proved in Section 6 (the essential steps are given in Example 4 and Lemma 19 which also provide an independent proof of Theorem 2).

**Theorem 3a.** *Suppose that for some $r \in \{0, \dots, \lfloor \frac{n-t}{2} \rfloor - 1\}$*

$$\omega_i = 0 \text{ unless } k_{r-1} \le i \le k_{r+1}.$$

*Then $M(n,t,\boldsymbol{\omega})$ is attained at $\mathcal{S}_r$ or $\mathcal{S}_{r+1}$.* □

Let $s \in [n]$. It is easy to see that

$$\mathcal{S}_r^n = \{X \cup Y : X \in \mathcal{S}_r^s \text{ and } Y \subseteq [s+1,n]\}, r = 0, \dots, \lfloor \tfrac{s-t}{2} \rfloor,$$

and that for each $i$-element subset $X$ of $[s]$ there exist $\binom{n-s}{k-i}$ $k$-element subsets $Z$ of $[n]$ such that $Z \cap [s] = X$. Since the $t$-intersection in $\mathcal{S}_r^n$ is already realized in $[s]$ if $r \le \lfloor \frac{s-t}{2} \rfloor$ we obtain a further consequence of Theorem 2a.

**Corollary 3b.** *Let $s \in [n]$ and let $\boldsymbol{\omega}$ be defined by $\omega_i = \binom{n-s}{k-i}$. Then $M(s,t,\boldsymbol{\omega})$ is attained at $\mathcal{S}_r^s$ if $k_{r-1}^n \le k \le k_r^n$, $r = 0, \dots, \lfloor \frac{s-t}{2} \rfloor$.* □

We mention that we cannot conclude automatically that $M(s,t,\boldsymbol{\omega})$ is attained at $\mathcal{S}^s_{\lfloor \frac{s-t}{2} \rfloor}$ if $k > k^n_{\lfloor \frac{s-t}{2} \rfloor}$.

For a succeeding application we change the notation a little bit. We put $m := n$, $s := n$, and $\boldsymbol{\nu} := \boldsymbol{\omega}$. Then a special case of Corollary 3b reads:

**Corollary 3c.** *The optimum $M(n,t,\boldsymbol{\nu})$ with $\nu_i = \binom{m-n}{k-i}$ is attained at $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$ if $k^m_{\lfloor \frac{n-t}{2} \rfloor - 1} \le k \le k^m_{\lfloor \frac{n-t}{2} \rfloor}$, i.e. if*

$$(k-t+1)\left(2 + \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor + 1}\right) \le m \le (k-t+1)\left(2 + \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor}\right). \quad (4)$$

□

## OPTIMALITY OF THE LAST CANDIDATE FAMILY

Corollary 3c gives a first example of a non–trivial weight function for which $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$ is optimal. Here we look for other weight functions with the same property.

The following theorem is due to Katona [15].

**Theorem 4.** *Let* $t \le i \le \lfloor \frac{n+t-1}{2} \rfloor$. *If* $\omega_i = \omega_{n+t-i-1} = 1$ *and* $\omega_j = 0$ *for* $j \notin \{i, n+t-i-1\}$ *then* $M(n,t;\boldsymbol{\omega})$ *is attained at* $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$, *i.e.*

$$M(n,t;\boldsymbol{\omega}) = \begin{cases} \binom{n}{n+t-i-1} & \text{if } i < \frac{n+t-1}{2} \\ \binom{n-1}{\frac{n+t-1}{2}} & \text{if } i = \frac{n+t-1}{2}. \end{cases} \tag{5}$$

□

As an easy consequence of Theorem 4 Engel, Frankl [10] obtained the following:

**Theorem 5.** *If* $\omega_i \le \omega_{n+t-i-1}$, $i = t, \ldots, \lfloor \frac{n+t-1}{2} \rfloor$, *then* $M(n,t;\boldsymbol{\omega})$ *is attained at* $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$. □

From this theorem we may easily derive that Corollary 3c remains true for all integers $m$ with

$$n \le m \le 2k - t + 1 \tag{6}$$

since for these $m$ the inequalities

$$\binom{m-n}{k-i} \le \binom{m-n}{k-n-t+i+1}, i = 0, \ldots, \lfloor \tfrac{n+t-1}{2} \rfloor,$$

are true.

Theorem 5 contains a fundamental result of Katona [15] as a special case ($\omega_i = 1$ for all $i$):

**Theorem 6.** *Among all t-intersecting families in* $2^{[n]}$ *the family* $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$ *has maximum size.* □

In order to apply the strong Corollary 3c, Ahlswede and Khachatrian developed a method which is given in the next theorem.

**Theorem 7 (Comparison Lemma)** *Let* $\mathcal{P}$ *be a set of points in* $\mathbb{R}_+^{n+1-t}$ *whose coordinates are indexed by* $t, t+1, \ldots, n$. *Let* $\nu \in \mathbb{R}_+^{n+1-t}$ *be a given positive weight vector. Suppose that there is some* $\boldsymbol{f}^* \in \mathcal{P}$ *such that*

$$\nu \cdot \boldsymbol{f}^* = \max\{\nu \cdot \boldsymbol{f} : \boldsymbol{f} \in \mathcal{P}\},$$

*and for some* $p \in [t, n]$

$$f_i^* = 0 \quad \text{if } t \le i < p \tag{7}$$

$$f_i \le f_i^* \quad \text{if } p \le i \le n \text{ and } \boldsymbol{f} \in \mathcal{P}. \tag{8}$$

*Let* $\omega \in \mathbb{R}_+^{n+1-t}$ *be another positive weight vector with the property*

$$\frac{\nu_i}{\nu_{i+1}} \ge \frac{\omega_i}{\omega_{i+1}}, i = t, \ldots, n-1. \tag{9}$$

*Then also*

$$\boldsymbol{\omega} \cdot \boldsymbol{f}^* = \max\{\boldsymbol{\omega} \cdot \boldsymbol{f} : \boldsymbol{f} \in \mathcal{P}\}.$$

Probably this theorem is not as well–known as it should be. Thus we reprove it here:

**Proof.** First we consider the special case

$$\frac{\nu_i}{\nu_{i+1}} = \frac{\omega_i}{\omega_{i+1}} \text{ for all } i = p, \ldots, n-1.$$

Then

$$\omega_i \begin{cases} = \frac{\omega_p}{\nu_p}\nu_i & \text{if } i = p, \ldots, n \\ \leq \frac{\omega_p}{\nu_p}\nu_i & \text{if } i = t, \ldots, p-1. \end{cases}$$

Consequently, for all $\boldsymbol{f} \in \mathcal{P}$ (using (7))

$$\begin{aligned} \boldsymbol{\omega} \cdot (\boldsymbol{f}^* - \boldsymbol{f}) &= \sum_{i=t}^{p-1} \omega_i(f_i^* - f_i) + \sum_{i=p}^{n} \omega_i(f_i^* - f_i) \\ &\geq \sum_{i=t}^{p-1} \frac{\omega_p}{\nu_p}\nu_i(f_i^* - f_i) + \sum_{i=p}^{n} \frac{\omega_p}{\nu_p}\nu_i(f_i^* - f_i) \\ &= \frac{\omega_p}{\nu_p}\boldsymbol{\nu} \cdot (\boldsymbol{f}^* - \boldsymbol{f}) \geq 0. \end{aligned}$$

Now we prove the general case by induction on the smallest number $s(\boldsymbol{\omega})$ such that

$$\frac{\nu_i}{\nu_{i+1}} = \frac{\omega_i}{\omega_{i+1}} \text{ for all } i = s(\boldsymbol{\omega}), \ldots, n-1.$$

Just before we treated the case $s(\boldsymbol{\omega}) = p$. Let us look at the induction step. Suppose that

$$\frac{\nu_q}{\nu_{q+1}} > \frac{\omega_q}{\omega_{q+1}} \text{ but } \frac{\nu_i}{\nu_{i+1}} = \frac{\omega_i}{\omega_{i+1}} \text{ for all } i = q+1, \ldots, n-1,$$

i.e. $s(\boldsymbol{\omega}) = q + 1$. Let

$$\alpha := \frac{\nu_q}{\nu_{q+1}}\frac{\omega_{q+1}}{\omega_q}$$

and let $\boldsymbol{\omega}'$ be defined by

$$\omega_i' := \begin{cases} \omega_i & \text{if } i = q+1, \ldots, n \\ \alpha\omega_i & \text{if } i = t, \ldots, q. \end{cases}$$

Then $s(\boldsymbol{\omega}') = q$ and $\boldsymbol{\omega}'$ satisfies (9) with $\boldsymbol{\omega}$ replaced by $\boldsymbol{\omega}'$. By the induction hypothesis and (8), for all $\boldsymbol{f} \in \mathcal{P}$

$$\boldsymbol{\omega} \cdot (\boldsymbol{f}^* - \boldsymbol{f}) = \frac{1}{\alpha}\boldsymbol{\omega}' \cdot (\boldsymbol{f}^* - \boldsymbol{f}) + \left(1 - \frac{1}{\alpha}\right) \sum_{i=q+1}^{n} \omega_i(f_i^* - f_i) \geq 0.$$

□

In order to apply Theorem 7 via Corollary 3c to some other size–dependent weighted $t$-intersection problems we delete from the profiles of the $t$-intersecting families the coordinates $0, \ldots, t-1$ (they are obviously zero), we take $\boldsymbol{f}^*$ as the (reduced) profile of $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$, and put $p := \lfloor \frac{n+t}{2} \rfloor$. Then (7) and (8) are satisfied (note (5) in the case $2 \nmid n+t$). Note that $\nu_i = \binom{m-n}{k-i} = 0$ if $k < i$ or $m - n < k - i$ and that

$$\frac{\nu_i}{\nu_{i+1}} = \frac{m-n+1}{k-i} - 1 \text{ if } k - m + n \leq i \leq k-1.$$

Thus we have:

**Corollary 8.** *The optimum in the size–dependent $t$-intersection problem with weight vector $\omega$ being positive at coordinates $t, \ldots, n$ is attained at $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$ if there are integers $m$ and $k$ such that*

$$n \leq \min\{m, k, m+t-k\}, \qquad (10)$$

*(4) (resp. (6)) is satisfied, and*

$$\frac{\omega_i}{\omega_{i+1}} \leq \frac{m-n+1}{k-i} - 1 \qquad (11)$$

*for all $i = t, \ldots, n-1$.* □

In general, it is not easy to find such integers $m$ and $k$ satisfying (4), (10), and (11). One idea is to look for "large" numbers $m$ and $k$. In order to avoid long and tedious computations we use the following easy lemma.

**Lemma 9.** *Let $a_j, b_j, c_j, d_j \in \mathbb{R}_+, j = 1, \ldots, p$, and $n$ be a fixed number. If*

$$\max\{a_1, \ldots, a_p\} < \min\{c_1, \ldots, c_p\}$$

*then there are positive integers $m$ and $k$ not less than $n$ such that*

$$a_j k + b_j \leq m \leq c_j k + d_j, j = 1, \ldots, p.$$

□

**Corollary 10.** *The optimum in the size–dependent $t$-intersection problem with weight vector $\omega$ being positive at coordinates $t, \ldots, n$ is attained at $\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$ if*

$$\max\left\{\frac{\omega_i}{\omega_{i+1}}, i = t, \ldots, n-1\right\} < 1 + \frac{t-1}{\lfloor \frac{n-t}{2} \rfloor}.$$

**Proof.** The inequalities (4), (10), and (11) can be written in the form

$$a_j k + b_j \leq m \leq c_j k + d_j, j = 1, 2, 3, \ldots, n-t+2$$

where

$$
\begin{aligned}
a_1 &= 2+\frac{t-1}{\lfloor\frac{n-t}{2}\rfloor+1}, \quad c_1 = 2+\frac{t-1}{\lfloor\frac{n-t}{2}\rfloor},\\
a_2 &= 1, \quad c_2=\infty,\\
a_j &= \frac{\omega_i}{\omega_{i+1}}+1, \quad c_j=\infty, \quad j=i-t+3=3,\dots,n-t+2.
\end{aligned}
$$

Corollary 8 and Lemma 9 yield the result. □

We mention that the case $\omega_i = \alpha^i$, i.e. $\frac{\omega_i}{\omega_{i+1}} = \frac{1}{\alpha}$ = constant was considered by Ahlswede, Khachatrian in [4], see also Example 5.

The idea of looking for large numbers $m$ and $k$ does not always work. Sometimes one has luck since there exist "small" numbers $m$ and $k$.

**Corollary 11.** *Let $\omega_i = \binom{\ell-1}{\ell-i}, i = 0,\dots,n$, and let $\ell \geq n$. Then $M(n,t;\omega)$ is attained at $S_{\lfloor\frac{n-t}{2}\rfloor}$.*

**Proof.** We apply Corollary 8 with $k := n$, $m := 2n$. Then (10) is satisfied, (4) is equivalent to

$$
\begin{aligned}
2n-\frac{2(t-1)}{n-t+2} \leq 2n \leq\ & 2n+\frac{2(t-1)}{n-t} \quad \text{if } 2 \mid n+t,\\
2n \leq 2n \leq\ & 2n+\frac{2(t-1)}{n-t-1} \quad \text{if } 2 \nmid n+t,
\end{aligned}
$$

hence (4) is satisfied, and finally (11) is equivalent to

$$\frac{\ell}{\ell-i} \leq \frac{n+1}{n-i} \text{ for all } i=t,\dots,n-1$$

which is obviously satisfied since $\ell \geq n$. □

## A NEW APPLICATION – PRODUCTS OF INFINITE CHAINS

Let $N_\ell(n,\infty) := \{\boldsymbol{a} = (a_1,\dots,a_n) : a_i \in \mathbb{N}, i=1,\dots,n, \sum_{i=1}^n a_i = \ell\}$. A family $\mathcal{F} \subseteq N_\ell(n,\infty)$ is called (statically) *t-intersecting* if for all $\boldsymbol{a},\boldsymbol{b} \in \mathcal{F}$ there exist $t$ coordinates $i_1,\dots,i_t$ such that $a_{i_j}, b_{i_j} \geq 1$ holds for $j=1,\dots,t$. Let

$$M_\ell(n,\infty,t) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq N_\ell(n,\infty),\ \mathcal{F} \text{ is } t\text{-intersecting}\}.$$

The set $N_\ell(n,\infty)$ can be viewed as the $\ell$-th level in the direct product of $n$ chains $0 \lessdot 1 \lessdot \cdots$ or as the family of all $\ell$-element multisets over the basic set $[n]$. The property of being $t$-intersecting means in the case of multisets that any two members of the family have at least $t$ different elements from the basic set in common.

Define for $\boldsymbol{a} \in N_\ell(n,\infty)$ (resp. $\mathcal{F} \subseteq N_\ell(n,\infty)$) the *support* of $\boldsymbol{a}$ (resp. of $\mathcal{F}$) by $\operatorname{supp}(\boldsymbol{a}) := \{i : a_i > 0\}$ (resp. $\operatorname{supp}(\mathcal{F}) := \{\operatorname{supp}(\boldsymbol{a}) : \boldsymbol{a} \in \mathcal{F}\}$). Obviously $\mathcal{F} \subseteq N_\ell(n,\infty)$ is $t$-intersecting iff $\operatorname{supp}(\mathcal{F}) \subseteq 2^{[n]}$ is $t$-intersecting.

A fundamental combinatorial formula (combinations with repetitions) says that for each fixed $i$-element subset $X$ of $[n]$

$$|\{\boldsymbol{a} \in N_\ell(n,\infty) : \mathrm{supp}\,(\boldsymbol{a}) \subseteq X\}| = \binom{i+\ell-1}{\ell}.$$

Deleting from each coordinate of supp $(\boldsymbol{a})$ a one leads to a bijection between the sets $\{\boldsymbol{a} \in N_\ell(n,\infty) : \mathrm{supp}\,(\boldsymbol{a}) = X\}$ and $\{\boldsymbol{a} \in N_{\ell-i}(n,\infty) : \mathrm{supp}\,(\boldsymbol{a}) \subseteq X\}$. Hence, for each fixed $i$-element subset $X$ of $[n]$

$$|\{\boldsymbol{a} \in N_\ell(n,\infty) : \mathrm{supp}\,(\boldsymbol{a}) = X\}| = \binom{i+\ell-i-1}{\ell-i} = \binom{\ell-1}{\ell-i}. \qquad (12)$$

We define the weight vector $\boldsymbol{\omega}$ by $\omega_i = \binom{\ell-1}{\ell-i}$ and obtain easily

$$M_\ell(n,\infty,t) = M(n,t;\boldsymbol{\omega}). \qquad (13)$$

Let $\mathcal{F}_r := \{\boldsymbol{a} \in N_\ell(n,\infty) : \mathrm{supp}\,(\boldsymbol{a}) \in \mathcal{S}_r\}$. Clearly, $\mathcal{F}_r$ is $t$-intersecting. Define

$$\ell_r := \begin{cases} t & \text{if } r=-1 \\ \frac{r+1}{t+r}(n+t-2)+t-1 & \text{if } r = 0,\ldots,\lfloor\frac{n-t}{2}\rfloor. \end{cases}$$

It is not difficult to verify that $\ell_{-1} \le \ell_0 \le \cdots \le \ell_{\lfloor\frac{n-t}{2}\rfloor}$.

**Theorem 12.** [1] *We have*

$$M_\ell(n,\infty,t) = \begin{cases} |\mathcal{F}_r| & \textit{if } \ell_{r-1} \le \ell \le \ell_r \\ & \textit{for some } r \in \{0,\ldots,\lfloor\frac{n-t}{2}\rfloor\} \\ |\mathcal{F}_{\lfloor\frac{n-t}{2}\rfloor}| & \textit{if } \ell > \ell_{\lfloor\frac{n-t}{2}\rfloor}. \end{cases}$$

**Proof.** In view of (12) and (13) we only have to show that $M(n,t;\boldsymbol{\omega})$ is attained at $\mathcal{S}_r$ if $\ell_{r-1} \le \ell \le \ell_r$, $r \in \{0,\ldots,\lfloor\frac{n-t}{2}\rfloor\}$ and at $\mathcal{S}_{\lfloor\frac{n-t}{2}\rfloor}$ if $\ell > \ell_{\lfloor\frac{n-t}{2}\rfloor}$.

First let for some $r \in \{0,\ldots,\lfloor\frac{n-t}{2}\rfloor\}$

$$\ell_{r-1} \le \ell \le \ell_r. \qquad (14)$$

We put in Corollary 3b $k := \ell$, $s := n$, $n := n+\ell-1$. Then we obtain our weight function $\omega_i = \binom{n-s}{k-i} = \binom{\ell-1}{\ell-i}$, and by Corollary 3b, $M(n,t;\boldsymbol{\omega})$ is attained at $\mathcal{S}_r$ if

$$k_{r-1}^{n+\ell-1} \le \ell \le k_r^{n+\ell-1}. \qquad (15)$$

Using the definition of $k_r^n$ it is easy to prove the equivalence of (14) and (15).

Now let $\ell > \ell_{\lfloor\frac{n-t}{2}\rfloor}$. A simple computation shows that this inequality is equivalent to

$$\ell > \begin{cases} n-1+4\frac{t-1}{n+t} & \text{if } 2 \mid n+t \\ n-1+2\frac{t-1}{n+t-1} & \text{if } 2 \nmid n+t, \end{cases}$$

[1]We thank U. Leck for stimulating the study of $M_\ell(n,\infty,t)$.

hence to $\ell \geq n$. The assertion follows directly from Corollary 11. □

Instead of $N_\ell(n,\infty)$ one can consider $N_\ell(n,k) := \{\boldsymbol{a} = (a_1, \dots a_n) : a_i \in \{0,1,\dots,k\}, i = 1,\dots,n, \sum_{i=1}^n a_i = \ell\}$ – the $\ell$-th level in the direct product of $n$ chains $0 < 1 < \cdots < k$. We take the same $t$-intersection property as before and define

$$M_\ell(n,k,t) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq N_\ell(n,k),\ \mathcal{F} \text{ is } t\text{-intersecting}\}.$$

It seems very difficult to determine this number in general. The following asymptotic result of the authors generalizes previous work from [10].

**Theorem 13.** *For every $t > 1$ (resp. for $t = 1$) there exist real numbers $0 = \lambda_{t,-1} < \lambda_{t,0} < \lambda_{t,1} < \cdots$ (resp. $0 = \lambda_{1,-1} < \lambda_{1,0}$) with $\lim_{r\to\infty} \lambda_{t,r} = \lambda_{1,0}$ such that the following holds. If $k$, $t$ and $\lambda$ are fixed and $n$ tends to infinity then*

*a)* $M_{\lfloor \lambda n \rfloor}(n,k,t) \sim |\mathcal{F}_r|$ *if* $\lambda_{t,r-1} < \lambda \leq \lambda_{t,r}$,

*b)* $M_{\lfloor \lambda n \rfloor}(n,k,t) \sim \frac{1}{2}|N_{\lfloor \lambda n \rfloor}(n,k)|$ *if* $\lambda = \lambda_{1,0}$,

*c)* $M_{\lfloor \lambda n \rfloor}(n,k,t) \sim |N_{\lfloor \lambda n \rfloor}(n,k)|$ *if* $\lambda > \lambda_{1,0}$.

*Here, of course,* $\mathcal{F}_r := \{\boldsymbol{a} \in N_{\lfloor \lambda n \rfloor}(n,k) : \operatorname{supp}(\boldsymbol{a}) \in \mathcal{S}_r\}$. □

The proof uses Corollary 3 and (in the case $\lambda = \lambda_{t,r}$) Theorem 3a. See [8] for details.

## THE METHOD OF RESTRICTED INTERSECTION

In this section we present a method which can be considered as one key for the proof of many intersection theorems, in particular also of Theorem 2. It is based of but simplifies the original method of generating sets by Ahlswede and Khachatrian [2].

Let $s \in [n]$ and $\mathcal{F} \in 2^{[n]}$. We call $\mathcal{F}$ *$t$-intersecting in* $[s]$ (briefly *$s$-$t$-intersecting*) if

$$|X_1 \cap X_2 \cap [s]| \geq t \text{ for all } X_1, X_2 \in \mathcal{F}.$$

Let $I_s(n,t)$ be the class of all $s$-$t$-intersecting families in $2^{[n]}$. Given a weight function $\omega : 2^{[n]} \to \mathbb{R}_+$, the *weighted $s$-$t$-intersection problem* is the problem of determining

$$M_s(n,t;\omega) := \max\{\omega(\mathcal{F}) : \mathcal{F} \in I_s(n,t)\}.$$

We define a new weight function $\omega_{n\to s} : 2^{[s]} \to \mathbb{R}_+$ by

$$\omega_{n\to s}(X) := \omega(\{Z \subseteq [n] : Z \cap [s] = X\}).$$

Note that $\omega_{n\to s}$ is size–dependent if so is $\omega$. We then put for $|X| = i$, $i = 0,\dots,s$,

$$\omega_{n\to s}(i) := \omega_{n\to s}(X).$$

Obviously,

$$M(s,t;\omega_{n\to s}) = M_s(n,t;\omega) \leq M(n,t;\omega). \tag{16}$$

Moreover, for $s_1 < s_2 < s_3$ and $X \subseteq [s_1]$

$$\omega_{s_3\to s_1}(X) = (\omega_{s_3\to s_2})_{s_2\to s_1}(X). \tag{17}$$

Using (16) and (17) one can derive that

$$\begin{aligned} M(s,t;\omega_{n\to s}) &= M_s(s+1,t;\omega_{n\to s+1}) \\ &\leq M(s+1,t;\omega_{n\to s+1}) = M_{s+1}(s+2,t;\omega_{n\to s+2}) \leq \dots \\ &\leq M(n-1,t;\omega_{n\to n-1}) = M_{n-1}(n,t;\omega) \leq M(n,t;\omega). \end{aligned} \tag{18}$$

In the following we will study the question when the inequality in (16) does hold as an equality. Because of (18) it is enough to look for sufficient conditions for the equality

$$M_{n-1}(n,t;\omega) = M(n,t;\omega). \tag{19}$$

First recall the shifting–operation $s_{i,j} : 2^{2^{[n]}} \to 2^{2^{[n]}}$ defined for $i,j \in [n]$ by

$$s_{i,j}(\mathcal{F}) := \{s_{i,j}(X) : X \in \mathcal{F}\} \cup \{X : X \in \mathcal{F}, s_{i,j}(X) \in \mathcal{F}\},$$

where (with the same notation) $s_{i,j} : 2^{[n]} \to 2^{[n]}$ is given by

$$s_{i,j}(X) := \begin{cases} X \setminus \{j\} \cup \{i\}, & \text{if } j \in X \text{ and } i \notin X \\ X & \text{otherwise.} \end{cases}$$

It is well–known (cf. [13]) and can be easily checked that

$$s_{i,j}(\mathcal{F}) \text{ is } t\text{-intersecting iff } \mathcal{F} \text{ is } t\text{-intersecting.} \tag{20}$$

When studying (19) we will apply only $s_{i,n}, i \in [n-1]$. Obviously, $s_{i,n}(\mathcal{F}) = \mathcal{F}$ or $s_{i,n}(\mathcal{F})$ contains less members having $n$ as an element than $\mathcal{F}$. Iterated application of $s_{i,n}$ (with all possible $i$'s) yields a family $\mathcal{F}'$ with the property

$$s_{i,n}(\mathcal{F}') = \mathcal{F}' \text{ for all } i \in [n-1].$$

We call families with this property *n-shifted.* Let $I^*(n,t)$ be the class of all $n$-shifted $t$-intersecting families, and

$$M^*(n,t;\omega) := \max\{\omega(\mathcal{F}) : \mathcal{F} \in I^*(n,t)\}.$$

**Supposition 1.** *For all $i \in [n-1]$ and $A \subseteq [n]$*

$$\omega(A) \leq \omega(s_{i,n}(A)).$$

It is easy to see that under this Supposition $M^*(n,t;\omega) = M(n,t;\omega)$. In the following we require the weight function $\omega$ to satisfy Supposition 1. Note that this is always true if $\omega$ is size–dependent.

Now assume that

$$M(n,t;\omega) > M_{n-1}(n,t;\omega). \tag{21}$$

We will look for further suppositions such that a contradiction can be obtained.

Choose among all optimal $t$-intersecting families, i.e. $\omega(\mathcal{F}) = M(n,t;\omega)$, one for which the set

$$\mathcal{R} := \mathcal{R}(\mathcal{F}) := \{X \in \mathcal{F} : n \in X, X \setminus \{n\} \notin \mathcal{F}\} \tag{22}$$

has minimum cardinality (note that $\mathcal{R} \neq \emptyset$ since otherwise $\mathcal{F}$ would be already $t$-intersecting in $[n-1]$ in contradiction to (21)). We may assume that $\mathcal{F}$ has the following property:

$$n \notin X \in \mathcal{F}, X \subseteq Y \text{ implies } Y \in \mathcal{F}. \tag{23}$$

Then, by Supposition 1, the choice of $\mathcal{F}$, and (23), $\mathcal{F}$ is $n$-shifted. Let

$$\mathcal{R}_i := \{X \in \mathcal{R} : |X| = i\} \text{ and } \mathcal{R}'_i := \{X \setminus \{n\} : X \in \mathcal{R}_i\}. \tag{24}$$

It is not too difficult to verify that $|X \cap Y| \geq t$ for all $X \in \mathcal{R}'_i$ and $Y \in \mathcal{F} \setminus \mathcal{R}_{n+t-i}$ (use that $\mathcal{F}$ is $n$-shifted $t$-intersecting). Hence, for any $i \in [t, \frac{n+t}{2})$ the two families

$$\begin{aligned} \mathcal{F}_{1,i} &:= (\mathcal{F} \setminus \mathcal{R}_{n+t-i}) \cup \mathcal{R}'_i, \\ \mathcal{F}_{2,i} &:= (\mathcal{F} \setminus \mathcal{R}_i) \cup \mathcal{R}'_{n+t-i} \end{aligned} \tag{25}$$

are $t$-intersecting and we have

$$\omega(\mathcal{F}_{1,i}) \geq \omega(\mathcal{F}) \quad \text{iff } \omega(\mathcal{R}'_i) \geq \omega(\mathcal{R}_{n+t-i}), \tag{26}$$

$$|\mathcal{R}(\mathcal{F}_{1,i})| < |\mathcal{R}(\mathcal{F})| \quad \text{iff } \mathcal{R}_i \neq \emptyset \text{ or } \mathcal{R}_{n+t-i} \neq \emptyset, \tag{27}$$

$$\omega(\mathcal{F}_{2,i}) \geq \omega(\mathcal{F}) \quad \text{iff } \omega(\mathcal{R}'_{n+t-i}) \geq \omega(\mathcal{R}_i), \tag{28}$$

$$|\mathcal{R}(\mathcal{F}_{2,i})| < |\mathcal{R}(\mathcal{F})| \quad \text{iff } \mathcal{R}_i \neq \emptyset \text{ or } \mathcal{R}_{n+t-i} \neq \emptyset. \tag{29}$$

Hence we obtain a contradiction if (26) and (27) or if (28) and (29) hold since otherwise $\mathcal{F}_{1,i}$ or $\mathcal{F}_{2,i}$ would be "better" than $\mathcal{F}$.

This leads us to the following second supposition (with the definition $\mathcal{A} \cup \{a\} := \{A \cup \{a\} : A \in \mathcal{A}\}$, $\mathcal{A} \subseteq 2^{[n]}$).

**Supposition 2.** *For all $j \in [t, \frac{n+t}{2})$, $\mathcal{A} \subseteq \binom{[n-1]}{j-1}$, $\mathcal{B} \subseteq \binom{[n-1]}{n+t-j-1}$ not both inequalities are valid:*

$$\begin{aligned} \omega(\mathcal{A}) &< \omega(\mathcal{B} \cup \{n\}), \\ \omega(\mathcal{B}) &< \omega(\mathcal{A} \cup \{n\}). \end{aligned}$$

Under Supposition 2 we obtain $\mathcal{R}_i = \emptyset$ for all $i \in [t,n] \setminus \{\frac{n+t}{2}\}$ which yields the contradiction $\mathcal{R} = \emptyset$ in the case $2 \nmid n+t$. Thus we need a further supposition such that $\mathcal{R}_{\frac{n+t}{2}} \neq \emptyset$ leads to a contradiction.

**Case $t = 1$:**

Let $A \in \mathcal{R}_{\frac{n+t}{2}}$ and $A' := A \setminus \{n\}$. Let $B' := [n-1] \setminus A'$, $B := B' \cup \{n\}$. If $B \notin \mathcal{R}_{\frac{n+t}{2}}$ (which implies $B \notin \mathcal{F}$) then $\mathcal{F}' := \mathcal{F} \cup \{A'\}$ is also $t$-intersecting, but $\omega(\mathcal{F}') \geq \omega(\mathcal{F})$ and $|\mathcal{R}(\mathcal{F}')| < |\mathcal{R}(\mathcal{F})|$, a contradiction. Thus $B \in \mathcal{R}_{\frac{n+t}{2}}$. Let

$$\begin{aligned} \mathcal{F}_1 &:= (\mathcal{F} \setminus \{B\}) \cup \{A'\}, \\ \mathcal{F}_2 &:= (\mathcal{F} \setminus \{A\}) \cup \{B'\}. \end{aligned} \tag{30}$$

Obviously $\mathcal{F}_1$ and $\mathcal{F}_2$ are $t$-intersecting and

$$\begin{aligned} \omega(\mathcal{F}_1) &\geq \omega(\mathcal{F}) \quad \text{iff } \omega(A') \geq \omega(B), \\ \omega(\mathcal{F}_2) &\geq \omega(\mathcal{F}) \quad \text{iff } \omega(B') \geq \omega(A), \\ |\mathcal{R}(\mathcal{F}_i)| &< |\mathcal{R}(\mathcal{F})|, \quad i = 1, 2. \end{aligned}$$

This leads us to the following supposition which yields in the case $t = 1$ the desired contradiction.

**Supposition 3.1.** *For all $A \in \binom{[n-1]}{\frac{n-1}{2}}$ not both inequalities are valid:*

$$\begin{aligned} \omega(A) &< \omega([n] \setminus A), \\ \omega([n-1] \setminus A) &< \omega(A \cup \{n\}). \end{aligned}$$

Supposition 3.1 is true if $\omega(A) \geq \omega(A \cup \{n\})$ for all $A \subseteq \binom{[n-1]}{\frac{n-1}{2}}$.

**Case** $t \geq 1$ and the weight function $\omega$ is size dependent, i.e. there is some $\omega$ such that $\omega(X) = \omega_j$ for all $X$ with $|X| = j$.
We have by double counting

$$\sum_{j=1}^{n-1} \sum_{X \in \mathcal{R}_{\frac{n+t}{2}} : j \notin X} \omega(X) = \sum_{X \in \mathcal{R}_{\frac{n+t}{2}}} \sum_{j \in [n-1] : j \notin X} \omega(X) = \frac{n-t}{2} \omega(\mathcal{R}_{\frac{n+t}{2}}).$$

Hence there is some $j \in [n-1]$ such that

$$\sum_{X \in \mathcal{R}_{\frac{n+t}{2}} : j \notin X} \omega(X) \geq \frac{n-t}{2(n-1)} \omega(\mathcal{R}_{\frac{n+t}{2}}). \tag{31}$$

Let $\mathcal{T} := \{X \in \mathcal{R}_{\frac{n+t}{2}} : j \notin X\}$ and $\mathcal{T}' := \{X \setminus \{n\} : X \in \mathcal{T}\}$. By the size–dependence of $\omega$, (31) is equivalent to

$$|\mathcal{T}| \omega_{\frac{n+t}{2}} \geq \frac{n-t}{2(n-1)} \left| \mathcal{R}_{\frac{n+t}{2}} \right| \omega_{\frac{n+t}{2}}. \tag{32}$$

It is easy to see that

$$\mathcal{F}_1 := \left( \mathcal{F} \setminus \mathcal{R}_{\frac{n+t}{2}} \right) \cup \mathcal{T} \cup \mathcal{T}' \tag{33}$$

is $t$-intersecting,

$$|\mathcal{R}(\mathcal{F}_1)| < |\mathcal{R}(\mathcal{F})| \text{ if } \mathcal{R}_{\frac{n+t}{2}} \neq \emptyset,$$

and that $\omega(\mathcal{F}_1) \geq \omega(\mathcal{F})$ is equivalent to the following inequalities

$$\begin{aligned} \omega(\mathcal{T}) + \omega(\mathcal{T}') &\geq \omega(\mathcal{R}_{\frac{n+t}{2}}), \\ |\mathcal{T}| \left(\omega_{\frac{n+t}{2}} + \omega_{\frac{n+t}{2}-1}\right) &\geq |\mathcal{R}_{\frac{n+t}{2}}| \omega_{\frac{n+t}{2}}. \end{aligned} \tag{34}$$

Thus we obtain the desired contradiction in the case $\mathcal{R}_{\frac{n+t}{2}} \neq \emptyset$ if (34) holds. Finally we claim that the following supposition for our candidate families is sufficient for (34):

**Supposition 3.2.** *We have*

$$\omega(\mathcal{S}_{\frac{n-t}{2}-1}) \geq \omega(\mathcal{S}_{\frac{n-t}{2}}). \tag{35}$$

Indeed, we have

$$\begin{aligned} \mathcal{S}_{\frac{n-t}{2}-1} \setminus \mathcal{S}_{\frac{n-t}{2}} &= \left\{X \subseteq [n-2] : |X| = \frac{n+t}{2} - 1\right\}, \\ \mathcal{S}_{\frac{n-t}{2}} \setminus \mathcal{S}_{\frac{n-t}{2}-1} &= \left\{X \cup \{n-1, n\} : X \subseteq [n-2], |X| = \frac{n+t}{2} - 2\right\}. \end{aligned}$$

Hence (35) is equivalent to the following inequalities:

$$\begin{aligned} \binom{n-2}{\frac{n+t}{2}-1} \omega_{\frac{n+t}{2}-1} &\geq \binom{n-2}{\frac{n+t}{2}-2} \omega_{\frac{n+t}{2}}, \\ \omega_{\frac{n+t}{2}-1} &\geq \frac{n+t-2}{n-t} \omega_{\frac{n+t}{2}}. \end{aligned} \tag{36}$$

From (32) and (36) we obtain (34).

Herewith we proved the following theorem:

**Theorem 14.** *We have $M_{n-1}(n,t;\omega) = M(n,t;\omega)$ if Suppositions 1, 2, and 3.1 (if t=1) or Suppositions 2 and 3.2 (if $\omega$ is size dependent) are true. In the case $n+t$ odd already Suppositions 1 and 2 (resp. only Supposition 2) are sufficient.* □

Note for all $\ell \in [t+2r+1, n]$ the following two facts:

1) If $\omega(Z) \leq \omega(s_{i,\ell}(Z))$ for all $i \in [\ell-1]$, $Z \subseteq [n]$ then also $\omega_{n\to\ell}(X) \leq \omega_{n\to\ell}(s_{i,\ell}(X))$ for all $i \in [\ell-1]$, $X \subseteq [\ell]$ .

2) $\omega(\mathcal{S}_\varrho^n) = \omega_{n\to\ell}(\mathcal{S}_\varrho^\ell)$ for all $\varrho \in [0, \lfloor\frac{\ell-t}{2}\rfloor]$.

Hence, iterated application of Theorem 14 together with (18) yields:

**Theorem 15.** *Let* $r \in \{0, \dots, \lfloor \frac{n-t-1}{2} \rfloor\}$. *We have*

$$M_{t+2r}(n,t;\omega) = M(n,t;\omega)$$

*if* $t = 1$ *and conditions (i), (ii), and (iii.1) are satisfied, or if* $t \geq 1$, $\omega$ *is size–dependent and conditions (ii) and (iii.2) are satisfied, where*

*(i) For all* $\ell \in [t+2r+1, n]$, $i \in [\ell - 1]$, $A \subseteq [n]$

$$\omega(A) \leq \omega(s_{i,\ell}(A)).$$

*(ii) For all* $\ell \in [t+2r+1, n]$, $i \in [t, \frac{\ell+t}{2})$, $\mathcal{A} \subseteq \binom{[\ell-1]}{i-1}$, $\mathcal{B} \subseteq \binom{[\ell-1]}{\ell+t-i-1}$ *not both inequalities are valid:*

$$\begin{array}{rcl} w_{n\to\ell}(\mathcal{A}) & < & \omega_{n\to\ell}(\mathcal{B} \cup \{\ell\}), \\ w_{n\to\ell}(\mathcal{B}) & < & \omega_{n\to\ell}(\mathcal{A} \cup \{\ell\}). \end{array}$$

*(iii.1) For all* $\varrho \in \left[r+1, \lfloor \frac{n-1}{2} \rfloor\right]$, $A \in \binom{[2\varrho]}{\varrho}$ *not both inequalities are valid:*

$$\begin{array}{rcl} \omega_{n\to 2\varrho+1}(A) & < & \omega_{n\to 2\varrho+1}([2\varrho+1] \setminus A), \\ \omega_{n\to 2\varrho+1}([2\varrho] \setminus A) & < & \omega_{n\to 2\varrho+1}(A \cup \{2\varrho+1\}). \end{array}$$

*(iii.2)*

$$\omega(\mathcal{S}_r) \geq \omega(\mathcal{S}_{r+1}) \geq \dots \geq \omega(\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}).$$

□

**Remark.**

*a) The following condition (iv) is sufficient for (ii) and (iii.1).*

*(iv) For all* $\ell \in [t+2r+1, n]$, $A \in 2^{[n]}$

$$\omega(A) \geq \omega(A \cup \{\ell\}).$$

*b) In the case of size–dependence the following conditions (ii') and (iii.2') are sufficient for (ii) and (iii.2), respectively (recall (35), (36)).*

*(ii') For all* $\ell \in [t+2r+1, n]$, $i \in [t, \frac{\ell+t}{2})$

$$\omega_{n\to\ell}(i-1)\, \omega_{n\to\ell}(\ell+t-i-1) \geq \omega_{n\to\ell}(i)\, \omega_{n\to\ell}(\ell+t-i).$$

*(iii.2') For all* $\varrho \in \left[r+1, \lfloor \frac{n-t}{2} \rfloor\right]$

$$\omega_{n\to t+2\varrho}(t+\varrho-1) \geq \left(1 + \frac{t-1}{\varrho}\right) \omega_{n\to t+2\varrho}(t+\varrho).$$

**Example 1.** Let $\boldsymbol{\omega} = \boldsymbol{e}_k$, $k \geq t$. Then $\omega_{n\to\ell}(i) = \binom{n-\ell}{k-i}$. A simple computation shows that *(ii')* is satisfied if $n > 2k - t$ and *(iii.2')* is satisfied if $k \leq k_r^n$.

**Example 2.** [2] Let $\boldsymbol{\omega} = \boldsymbol{e}_k + \boldsymbol{e}_{k+1}$, $k \geq t$. Then $\omega_{n\to\ell}(i) = \binom{n-\ell}{k-i} + \binom{n-\ell}{k+1-i} = \binom{n-\ell+1}{k+1-i}$. As in the previous example, *(ii')* is satisfied if $n+1 > 2(k+1) - t$ (i.e. $k+1 \leq k^{n+1}_{\lfloor\frac{n+1-t}{2}\rfloor}$) and *(iii.2')* is satisfied if $k+1 \leq k_r^{n+1}$. Together with Corollary 3c we obtain that $\mathcal{S}_r$ is optimal if $k_{r-1}^{n+1} \leq k+1 \leq k_r^{n+1}$, $r = 0, \ldots, \lfloor\frac{n-t}{2}\rfloor$. If $k+1 > k^{n+1}_{\lfloor\frac{n-t}{2}\rfloor}$ then $k \geq k^n_{\lfloor\frac{n-t}{2}\rfloor-1}$, hence $\mathcal{S}_{\lfloor\frac{n-t}{2}\rfloor}$ is optimal.

**Example 3.** Let $\omega_i = \binom{\ell-1}{\ell-i}$, $\ell \geq t$ (compare with (12)). Then $\omega_{n\to s}(i) = \binom{n-s+\ell-1}{\ell-i}$. A simple computation shows that *(ii')* is satisfied if $n > \ell - t + 1$ and *(iii.2')* is satisfied if $\ell \leq \ell_r$.

**Example 4.** Let $\omega_k = 0$ unless $k \leq k_r$. Note that then $\omega_k \neq 0$ implies $n > 2k - t$. We have $\omega_{n\to\ell}(i) = \sum_{k=0}^{k_r} \omega_k \binom{n-\ell}{k-i}$. Then *(ii')* reads:

$$\sum_{j,k=0}^{k_r} \omega_j\omega_k \binom{n-\ell}{j-i+1}\binom{n-\ell}{k-\ell-t+i+1} \geq \sum_{j,k=0}^{k_r} \omega_j\omega_k \binom{n-\ell}{j-i}\binom{n-\ell}{k-\ell-t+i}.$$

Using $j + k - t < n$ it is not difficult to verify that

$$\binom{n-\ell}{j-i+1}\binom{n-\ell}{k-\ell-t+i+1} \geq \binom{n-\ell}{j-i}\binom{n-\ell}{k-\ell-t+i}$$

for all $0 \leq j, k \leq k_r$. Consequently, *(ii')* is satisfied. Using Example 1 it is easy to show that also *(iii.2')* is satisfied.

**Example 5.** Let $\alpha$ be a positive real number and $\omega_i = \alpha^{-i}$. Then $\omega_{n\to\ell}(i) = \alpha^{-i}(1+\alpha^{-1})^{n-\ell}$ and *(ii')* is satisfied if $\alpha \geq 1$. Further, *(iii.2')* is satisfied if $\alpha \geq 1 + \frac{t-1}{r+1}$. Together with Corollary 10 we obtain that $\mathcal{S}_r$ is optimal if $1 + \frac{t-1}{r} > \alpha \geq 1 + \frac{t-1}{r+1}$.

This example has the following application: For $\alpha, n \in \mathbb{N}$ consider the set $H_\alpha^n := \{\boldsymbol{a} = (a_1, \ldots, a_n) : a_i \in \{1, \ldots, \alpha\}\}$. On $H_\alpha^n$ one has the Hamming metric $d_H$ which for two tuples $\boldsymbol{a}$, $\boldsymbol{b}$ counts the number of different coordinates: $d_H(\boldsymbol{a}, \boldsymbol{b}) = |\{i : a_i \neq b_i\}|$. As usual, for a subset $\mathcal{F}$ of $H_\alpha^n$ the diameter $d(\mathcal{F})$ is the maximum possible distance between two elements of $\mathcal{F}$. Let $d \in \mathbb{N}$. We are interested in the following diametric problem: Determine the maximum cardinality of a set $\mathcal{F} \subseteq H_\alpha^n$ with diameter $d$ or less. The complete solution was given by Ahlswede and Khachatrian [4]. Independently, Frankl and Tokushige [14] proved the following $t$-intersection version.

[2]This result was communicated to us by L. Khachatrian.

Call a set $\mathcal{F} \subseteq H_\alpha^n$ *t-intersecting* if any two tuples of $\mathcal{F}$ agree in at least $t$ coordinates. Obviously, subsets of $H_\alpha^n$ with diameter at most $d$ are $(n-d)$-intersecting and vice versa. Thus the diametric problem is equivalent to:

$$\text{Determine } M(n,\alpha,t) := \max\{|\mathcal{F}| : \mathcal{F} \subseteq H_\alpha^n,\ \mathcal{F} \text{ is } t\text{-intersecting}\}. \tag{37}$$

Define for $i,j \in [\alpha]$, $c \in [n]$ the operation $s_{i,j,c} : 2^{H_\alpha^n} \to 2^{H_\alpha^n}$ by

$$s_{i,j,c}(\mathcal{F}) := \{s_{i,j,c}(\boldsymbol{a}) : \boldsymbol{a} \in \mathcal{F}\} \cup \{\boldsymbol{a} \in \mathcal{F} : s_{i,j,c}(\boldsymbol{a}) \in \mathcal{F}\}, \tag{38}$$

where (with the same notation) $s_{i,j,c} : H_\alpha^n \to H_\alpha^n$ is given by

$$s_{i,j,c}(\boldsymbol{a}) := \begin{cases} (a_1,\dots,a_{c-1},i,a_{c+1},\dots,a_n), & \text{if } a_c = j \\ \boldsymbol{a} & \text{otherwise.} \end{cases} \tag{39}$$

It is easy to verify that this operation respects the $t$-intersection property. Furthermore, if $s_{i,j,c}(\mathcal{F}) = \mathcal{F}$ for all $i,j,c$, $i<j$, then any two tuples of $\mathcal{F}$ have entry 1 in at least $t$ common coordinates. It follows that the determination of $M(n,t;\omega)$ with $\omega_i := (\alpha-1)^{n-i}$ suffices for (37). Thus, Example 5 shows that one of the candidates $\mathcal{S}_r$ is optimal. We refer to [4] for more details and background.

Let us generalize the previous application. For $\boldsymbol{\alpha} = (\alpha_1,\dots,\alpha_n) \in \mathbb{N}^n$ consider the set $F_{\boldsymbol{\alpha}} := \{\boldsymbol{a} = (a_1,\dots,a_n) : a_i \in \{0,\dots,\alpha_i\}\}$. We define an order relation on $F_{\boldsymbol{\alpha}}$ by $\boldsymbol{a} \le \boldsymbol{b}$ iff $a_i = 0$ or $a_i = b_i$ for all $i = 1,\dots,n$. Then $F_{\boldsymbol{\alpha}}$ is a ranked partially ordered set, isomorphic to the direct product of $n$ stars $0 < \alpha_1, \alpha_2, \dots, \alpha_i$, $i = 1,\dots,n$. Let $N_k(\boldsymbol{\alpha})$ be the $k$-th level of $F_{\boldsymbol{\alpha}}$, i.e. $N_k(\boldsymbol{\alpha}) = \{\boldsymbol{a} \in F_{\boldsymbol{\alpha}} : |\{i : a_i > 0\}| = k\}$ and define $W_k(\boldsymbol{\alpha}) := |N_k(\boldsymbol{\alpha})|$ (note that if $\alpha := \alpha_1 = \alpha_2 = \cdots = \alpha_n$ then $N_n(\boldsymbol{\alpha}) = H_\alpha^n$). A family $\mathcal{F} \subseteq F_{\boldsymbol{\alpha}}$ is called $t$-intersecting if for all $\boldsymbol{a},\boldsymbol{b} \in \mathcal{F}$ there exist $t$ coordinats $i_1,\dots,i_t$ such that $a_{i_j} = b_{i_j} > 0$ holds for $j = 1,\dots,t$ (i.e. the infimum of $\boldsymbol{a}$ and $\boldsymbol{b}$ in $F_{\boldsymbol{\alpha}}$ has rank at least $t$). Define

$$M_k(n,\boldsymbol{\alpha},t) = \max\{|\mathcal{F}| : \mathcal{F} \subseteq N_k(\boldsymbol{\alpha}),\ \mathcal{F} \text{ is } t\text{-intersecting}\}.$$

For $K \subseteq [n]$ let $\pi_K : \mathbb{N}^n \to \mathbb{N}^{n-|K|}$ be the projection map onto the coordinates that are not contained in $K$. Define $\omega : 2^{[n]} \to \mathbb{R}_+$ by

$$\omega(\{i_1,\dots,i_m\}) = W_{k-m}\left(\pi_{\{i_1,\dots,i_m\}}(\alpha_1-1,\dots,\alpha_n-1)\right). \tag{40}$$

Using the operation $s_{i,j,c} : 2^{F_{\boldsymbol{\alpha}}} \to 2^{F_{\boldsymbol{\alpha}}}$ defined by (38) and (39) one can derive that

$$M_k(n,\boldsymbol{\alpha},t) = M(n,t;\omega). \tag{41}$$

**Example 6.** Let $\alpha := \alpha_1 = \ldots = \alpha_n \ge 2$, $n > k$, and $\omega$ as in (40). Then $\omega$ is size-dependent. Let us use the abbreviation

$$N_k\left((\alpha-1)^a, \alpha^b\right) := N_k(\underbrace{\alpha-1,\dots,\alpha-1}_{a},\underbrace{\alpha,\dots,\alpha}_{b}),$$

and similar for $W_k\left((\alpha-1)^a, \alpha^b\right)$. Then

$$\omega_{n\to\ell}(i) = W_{k-i}\left((\alpha-1)^{\ell-i}, \alpha^{n-\ell}\right).$$

It is easy to see that $\omega_{n\to\ell}(i-1) \geq \omega_{n\to\ell}(i)$ holds for all $i$, hence $(ii')$ is satisfied. Furthermore, purely numerical considerations show that $\omega(\mathcal{S}_0) \geq \omega(\mathcal{S}_1)$ holds iff $(iii.2)$ holds for $r = 0$ iff

$$n \geq \left\lfloor \frac{(k-t+\alpha)(t+1)}{\alpha} \right\rfloor. \tag{42}$$

It follows that in this case the family $\mathcal{S}_0$ is optimal for (41). See [6] for details.

In the general case we do not know whether always one of the families $\mathcal{S}_r$ is optimal. However, one can prove the following [5]: If $\alpha, t$ are constant and $n$ is sufficiently large then it holds (for all $k$)

$$M_k(n, \boldsymbol{\alpha}, t) = \max_r \omega(\mathcal{S}_r).$$

**Example 7.** Let $t = 1$, $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_n$, and $\omega$ as in (40). Then $(i)$ is clearly satisfied. It is easy to see that $(iv)$ is satisfied if $\alpha_{t+2r+1} \geq 2$. It follows that $\mathcal{S}_0$ is optimal for (41) (with $t = 1$) if $\alpha_2 \geq 2$. Using Theorem 1 one can also deal the general case $1 = \alpha_1 = \ldots = \alpha_m < \alpha_{m+1} \leq \cdots \leq \alpha_n$, see [7] for details.

## NONTRIVIAL T-INTERSECTION

A family $\mathcal{F} \subseteq 2^{[n]}$ is called *nontrivial t-intersecting* ( resp. *nontrivial t-intersecting in [s]*, briefly *nontrivial s-t-intersecting*, $s \in [n]$) if it is $t$-intersecting (resp. $s$-$t$-intersecting), and if

$$\left|\bigcap_{X\in\mathcal{F}} X\right| < t. \tag{43}$$

Let $\tilde{I}(n,t)$ (resp. $\tilde{I}_s(n,t)$) denote the class of all such families.

Suppose we are given a $t$-intersecting family $\mathcal{F}$ such that $|X| > t$ for all $X \in \mathcal{F}$ (e.g. a $k$-uniform $t$-intersecting family with $k > t$). Then the family $\mathcal{F} \cup \{[n] \setminus \{i\} : i \in n\}$ is nontrivial $t$-intersecting. Thus, dealing with optimal nontrivial $t$-intersecting families with respect to some weight function $\omega$, the intersection in (43) should include only sets $X \in \mathcal{F}$ with $\omega(X) > 0$.

We require the weight function $\omega : 2^{[n]} \to \mathbb{R}_+$ to satisfy the following supposition:

**Supposition 4.** *$\omega(X) > 0$ implies $\omega(Y) > 0$ for all $Y \in 2^{[n]}$ with $|Y| = |X|$.*

Let $\Omega := \Omega(\omega) := \{i : \omega([i]) > 0\}$ and for $\mathcal{F} \in 2^{[n]}$ let

$$\mathcal{F}_\Omega := \bigcup_{i\in\Omega} \mathcal{F}_i = \{X \in \mathcal{F} : \omega(X) > 0\}.$$

Note that for $s \in [n-1]$ the new weight function $\omega_{n \to s}$ satisfies Supposition 4 if so does $\omega$. Moreover, we have

$$i \in \Omega(\omega_{n \to s}) \text{ iff } [i, i+n-s] \cap \Omega(\omega) \neq \emptyset. \tag{44}$$

Finally, we define

$$\begin{aligned} \tilde{M}(n,t;\omega) &:= \max\{\omega(\mathcal{F}) : \mathcal{F}_\Omega \in \tilde{I}(n,t)\}, \\ \tilde{M}_s(n,t;\omega) &:= \max\{\omega(\mathcal{F}) : \mathcal{F}_\Omega \in \tilde{I}_s(n,t)\}. \end{aligned}$$

Note that

$$\begin{aligned} \tilde{M}(s,t;\omega_{n\to s}) &= \tilde{M}_s(s+1,t;\omega_{n\to s+1}) \\ &\leq \tilde{M}(s+1,t;\omega_{n\to s+1}) = \tilde{M}_{s+1}(s+2,t;\omega_{n\to s+2}) \leq \dots \\ &\leq \tilde{M}(n-1,t;\omega_{n\to n-1}) = \tilde{M}_{n-1}(n,t;\omega) \leq \tilde{M}(n,t;\omega). \end{aligned} \tag{45}$$

In this section we shall study the problem of the determination of these numbers. We will see that the method of restricted intersection works as well.

We may always suppose that $n \geq t+2$ since obviously $\tilde{I}(n,n-1) = \tilde{I}(n,n) = \emptyset$.

Let us look at candidates for optimal families. Clearly

$$(\mathcal{S}_1)_\Omega, \dots, \left(\mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}\right)_\Omega$$

are nontrivial $t$-intersecting if $\Omega \neq \{n\}$. Furthermore, for $T \subseteq [n]$ let

$$\mathcal{G}'_T := \{X \subseteq [n] : T \subseteq X\} \cup \{[n] \setminus \{j\} : j \in T\}$$

and

$$\mathcal{G}_T := \begin{cases} \mathcal{G}'_T & \text{if } |T| > t, \\ \mathcal{G}'_T \setminus \{T\} & \text{if } |T| = t. \end{cases}$$

It is easy to see that

$$\mathcal{G}_T \supseteq \mathcal{G}_{T'} \text{ if } T \subsetneq T',\ |T| \geq t, \tag{46}$$

where equality holds iff $|T| = t$ and $n = t+2$.

Note that $(\mathcal{G}_T)_\Omega \in \tilde{I}(n,t)$ if $n-1 \in \Omega$.

**Theorem 16.** *We have for $n \geq t+2$*

$$\tilde{M}(n,t;\omega) = \begin{cases} \tilde{M}_{n-1}(n,t;\omega) & \text{if } n-1 \notin \Omega \\ \max\left\{\tilde{M}_{n-1}(n,t;\omega),\ \omega(\mathcal{G}_T) : T \in \binom{[n-1]}{t}\right\} & \text{if } n-1 \in \Omega \end{cases}$$

*if Supposition 4 and the suppositions from Theorem 14 are satisfied.*

**Proof.** We proceed as in the previous section. Assume that

$$\tilde{M}(n,t;\omega) > \tilde{M}_{n-1}(n,t;\omega).$$

Choose among all $t$-intersecting families $\mathcal{F}$ with $\mathcal{F}_\Omega \in \tilde{I}(n,t)$ and $\omega(\mathcal{F}) = \tilde{M}(n,t;\omega)$ one for which

$$\mathcal{R} := \mathcal{R}(\mathcal{F}) := \{X \in \mathcal{F} : n \in X, X \setminus \{n\} \notin \mathcal{F}\}$$

has minimum cardinality. Note that $\mathcal{R} \neq \emptyset$. We may asume that $\mathcal{F}$ has property (23).

**Claim:** $\mathcal{F}$ is $n$-shifted.

Assume the contrary. Then the set

$$I_\Omega := \{i \in [n] : s_{i,n}(\mathcal{F}_\Omega) \neq \mathcal{F}_\Omega\}$$

is not empty since otherwise every family $s_{i,n}(\mathcal{F})$ with $s_{i,n}(\mathcal{F}) \neq \mathcal{F}$ would be "better" than $\mathcal{F}$. Also, we have for all $i \in I_\Omega$

$$\left| \bigcap_{X \in s_{i,n}(\mathcal{F}_\Omega)} X \right| = t. \tag{47}$$

Let $S := \bigcap_{X \in \mathcal{F}_\Omega} X$. Then (47) implies (for all $i \in I_\Omega$) $i, n \notin S$, $|S| = t-1$ and

$$\bigcap_{X \in s_{i,n}(\mathcal{F}_\Omega)} X = S \cup \{i\}. \tag{48}$$

It follows that for all $i \in I_\Omega$ there exist sets $X_i, Y_i \in \mathcal{F}_\Omega$ such that $X_i \cap \{i,n\} = \{n\}$, $Y_i \cap \{i,n\} = \{i\}$. Further, for all $i \in I_\Omega$ and $Z \in \mathcal{F}_\Omega$ we have $Z \cap \{i,n\} \neq \emptyset$. Since $\mathcal{F}$ has maximum weight we must have (for all $i \in I_\Omega$)

$$Z \in \mathcal{F}_\Omega \text{ if } S \cup \{i,n\} \subseteq Z, \, |Z| \in \Omega. \tag{49}$$

Note that for all $i \in I_\Omega$ we have $|X_i| \leq n-2$ or $|Y_i| \leq n-2$ since otherwise $[n-1], [n] \setminus \{i\} \in \mathcal{F}_\Omega$ in contradiction to (48).

Clearly $I_\Omega \subseteq [n-1] \setminus S$. Note that $|[n-1] \setminus S| \geq 2$ since $n \geq t+2$.

**Case** $|I_\omega| = 1$ :

Let $I_\Omega = \{i\}$. We have $j \in X_i$ for all $j \in [n-1] \setminus (S \cup \{i\})$ since otherwise $i, n \notin s_{j,n}(X_i) \in \mathcal{F}_\Omega$ which contradicts (48). Thus $X_i = [n] \setminus \{i\}$, in particular $n-1 \in \Omega$. By property (23) (applied to $Y_i \in \mathcal{F}_\Omega$) we have also $[n-1] \in \mathcal{F}_\Omega$, a contradiction to (48).

**Case** $I_\Omega = [n-1] \setminus S$ :

Let $i \in I_\Omega$. We have $Y_i = [n-1]$ since otherwise $j, n \notin Y_i \in \mathcal{F}_\Omega$ for some $j \in I_\Omega$, a contradiction to (48). In particular, $|Y_i| = n-1 \in \Omega$. By (49) we have also $[n] \setminus \{i\} \in \mathcal{F}_\Omega$, again a contradiction to (48).

**Case** $|I_\Omega| \geq 2$ and $I_\Omega \subsetneq [n-1] \setminus S$ :

Let $i, j \in I_\Omega$, $i \neq j$, and $k \in [n-1] \setminus (S \cup I_\Omega)$. Then (49) implies that there exists a set $Z \in \mathcal{F}_\Omega$ such that $S \cup \{i,n\} \subseteq Z$ and $j, k \notin Z$. But then $j, n \notin s_{k,n}(Z) \in \mathcal{F}_\Omega$, a contradiction to (48).

Hence in all three cases a contradiction is obtained, this proves the claim.

Let now

$$T := \bigcap_{X \in (\mathcal{F}\setminus\mathcal{R})_\Omega} X, \; \tau := |T|.$$

**Case $\tau < t$ :**
In (25), (30), and (33) we constructed families by deleting some members of $\mathcal{R}$ from $\mathcal{F}$ and adding some new members such that the new families are still $t$-intersecting. Since we did not change $\mathcal{F}\setminus\mathcal{R}$ the new families are even nontrivial $t$-intersecting and we may argue exactly as in the proof of Theorem 14.
**Case $\tau \geq t$ :**
For all $X \in \mathcal{R}_\Omega$ we have

$$\text{either } T \subseteq X \text{ or } X = [n] \setminus \{j\} \text{ for some } j \in T. \tag{50}$$

Indeed, otherwise there would exist two elements $i \in [n]$, $j \in T$ such that $i, j \notin X$. Since $\mathcal{F}$ is $n$-shifted $s_{i,n}(X) \in (\mathcal{F} \setminus \mathcal{R})_\Omega$. Clearly $j \notin s_{i,n}(X)$ which is a contradiction to $j \in T \subseteq s_{i,n}(X)$. Note that in the case $|T| = t$ the set $T$ is not an element of $\mathcal{R}$ since otherwise $T \subseteq X$ for all $X \in \mathcal{F}$ which is impossible since $\mathcal{F}_\Omega$ is nontrivial $t$-intersecting. By the definition of $T$ and (50) we have $\mathcal{F}_\Omega \subseteq \mathcal{G}_T$. But then, recalling (46) and the optimality of $\mathcal{F}$, $\omega(\mathcal{F}) = \omega(\mathcal{G}_{T'})$ for some $T' \in \binom{n-1}{t}$. Note that in this case necessarily $n-1 \in \Omega$ since otherwise $\mathcal{F}_\Omega$ would not be nontrivial $t$-intersecting. □

Note that for $i \notin T, \ell \in T$ the relation

$$\mathcal{G}_{s_{i,\ell}(T)} = s_{i,\ell}(\mathcal{G}_T)$$

holds. Thus, under Supposition *(i)* from Theorem 15, it is enough to consider sets $T$ from $\binom{[t+2r]}{t}$.

In addition to the candidates $\mathcal{G}_T$ we define for $T \in \binom{[t+2r]}{t}$, $\ell \in [t+2r, n]$, $r \geq 1$

$$\begin{aligned} \mathcal{G}_{T,\ell} := &\{X \subseteq [n] : T \subseteq X, ([\ell] \setminus T) \cap X \neq \emptyset\} \\ &\cup \{X \subseteq [n] : [\ell] \setminus T \subseteq X, |X \cap T| = t-1\}. \end{aligned}$$

We define further $\mathcal{G}_{T,n+1} := \mathcal{G}_{T,n}$.

Let $\omega_{\max} := \max\{i : i \in \Omega\}$. Iterated application of Theorem 16 together with (44) and (45) yields:

**Theorem 17.** *Let $r \in \left\{1, \ldots, \left\lfloor \frac{n-t-2}{2} \right\rfloor\right\}$. We have*

$$\tilde{M}(n,t;\omega) = \begin{cases} \tilde{M}_{t+2r}(n,t;\omega) & \text{if } \omega_{\max} < t+2r \\ \max\left\{\tilde{M}_{t+2r}(n,t;\omega),\, \omega(\mathcal{G}_{T,\ell}) : T \in \binom{[t+2r]}{t},\, \ell \in [t+2r+1, \omega_{\max}+1]\right\} & \\ & \text{if } \omega_{\max} \geq t+2r \end{cases}$$

*if Supposition 4 and the suppositions from Theorem 15 are satisfied.* □

For applications, the most important case is if $r = 1$. Since $\tilde{M}_{t+2}(n,t;\omega) = \omega(\mathcal{S}_1)$ and $\mathcal{S}_1 = \mathcal{G}_{[t],t+2}$ the determination of $\tilde{M}(n,t;\omega)$ reduces then to a purely numerical problem: Find the maximum of all numbers $\omega(\mathcal{G}_{T,\ell})$, $T \in \binom{t+2}{t}$, $\ell \in [t+2, \omega_{\max}+1]$. Note that if $\omega$ is size–dependent then $\omega(\mathcal{G}_{T,\ell}) = \omega_{n\to t}(t) - \omega_{n\to\ell}(t) + t\,\omega_{n\to\ell}(\ell-1)$.

In general, one has often $M(n,t;\omega) = \omega(\mathcal{S}_r)$ for the smallest $r$ for which the conditions of Theorem 15 are satisfied. If $r \geq 1$ then also $\tilde{M}(n,t;\omega) = \omega(\mathcal{S}_r)$ since $\mathcal{S}_r$ is nontrivial $t$-intersecting.

**Example 8.** Let $\omega = e_k$, $t \leq k \leq k_1$. Then one can take $r = 1$ in Theorem 17. We have

$$\omega(\mathcal{G}_{[t],\ell}) = \binom{n-t}{k-t} - \binom{n-\ell}{k-t} + t\binom{n-\ell}{k-\ell+1}.$$

A unimodality–argument (see [1]) yields

$$\tilde{M}(n,t;\omega) = \max\left\{\omega(\mathcal{G}_{[t],t+2}),\, \omega(\mathcal{G}_{[t],k+1})\right\}.$$

**Example 9.** Let $\omega_i = \binom{\ell-1}{\ell-i}$, $t \leq \ell \leq \ell_1$. Then one can take $r = 1$ and we have

$$\omega(\mathcal{G}_{[t],s}) = \binom{n-t+\ell-1}{\ell-t} - \binom{n-s+\ell-1}{\ell-t} + t\binom{n-s+\ell-1}{\ell-s+1}.$$

As in Example 8 one can show that

$$\tilde{M}(n,t;\omega) = \max\left\{\omega(\mathcal{G}_{[t],t+2}),\, \omega(\mathcal{G}_{[t],\ell+1})\right\}.$$

**Example 10.** Let $\omega_i = \alpha^{-i}$, $\alpha \geq \frac{t+1}{2}$. Again, one can take $r = 1$. We have

$$\omega(\mathcal{G}_{[t],\ell}) = \alpha^{-t}(1+\alpha^{-1})^{n-t} - \alpha^{-t}(1+\alpha^{-1})^{n-\ell} + t\alpha^{-(\ell-1)}(1+\alpha^{-1})^{n-\ell}.$$

It is not difficult to verify that

$$\tilde{M}(n,t;\omega) = \max\left\{\omega(\mathcal{G}_{[t],t+2}),\, \omega(\mathcal{G}_{[t],n})\right\}.$$

**Example 11.** Let $\alpha := \alpha_1 = \ldots = \alpha_n \geq 2$, $n > k$, $\omega$ as in (40), and let the equivalent conditions of Example 6 be satisfied. Then one can take $r = 1$. It holds

$$\omega(\mathcal{G}_{[t],\ell}) = W_{k-t}\left(\alpha^{n-t}\right) - W_{k-t}\left((\alpha-1)^{\ell-t}, \alpha^{n-\ell}\right) + t\,W_{k-\ell+1}\left(\alpha-1, \alpha^{n-\ell}\right).$$

We have also in this example

$$\tilde{M}(n,t;\omega) = \max\left\{\omega(\mathcal{G}_{[t],t+2}),\, \omega(\mathcal{G}_{[t],k+1})\right\}.$$

Indeed, let us show that $\omega(\mathcal{G}_{[t],\ell}) < \omega(\mathcal{G}_{[t],\ell+1})$ implies $\omega(\mathcal{G}_{[t],\ell+1}) < \omega(\mathcal{G}_{[t],\ell+2})$. Using

$$W_j\left((\alpha-1)^a, \alpha^b\right) + W_{j+1}\left((\alpha-1)^{a+1}, \alpha^b\right) = W_{j+1}\left((\alpha-1)^a, \alpha^{b+1}\right) \qquad (51)$$

our claim reads:

$$t\,W_{k-\ell+1}\left((\alpha-1)^2,\alpha^{n-\ell-1}\right) < W_{k-t-1}\left((\alpha-1)^{\ell-t},\alpha^{n-\ell-1}\right)$$

implies

$$t\,W_{k-\ell}\left((\alpha-1)^2,\alpha^{n-\ell-2}\right) < W_{k-t-1}\left((\alpha-1)^{\ell+1-t},\alpha^{n-\ell-2}\right).$$

But this is true since the map

$$\tau : N_{k-t-1}\left((\alpha-1)^{\ell-t},\alpha^{n-\ell-1}\right) \times N_{k-\ell}\left((\alpha-1)^2,\alpha^{n-\ell-2}\right) \longrightarrow$$
$$N_{k-t-1}\left((\alpha-1)^{\ell+1-t},\alpha^{n-\ell-2}\right) \times N_{k-\ell+1}\left((\alpha-1)^2,\alpha^{n-\ell-1}\right)$$

defined for $\boldsymbol{a} \in N_{k-t-1}\left((\alpha-1)^{\ell-t},\alpha^{n-\ell-1}\right)$, $\boldsymbol{b} \in N_{k-\ell}\left((\alpha-1)^2,\alpha^{n-\ell-2}\right)$ by

$$\tau(\boldsymbol{a},\boldsymbol{b}) := \begin{cases} (a_1,\dots,a_{\ell-t},a_{\ell-t+1},\dots,a_{n-t-1},\boldsymbol{b},1) & \text{if } a_{\ell-t+1} < \alpha \\ (a_1,\dots,a_{\ell-t},1,a_{\ell-t+2},\dots,a_{n-t-1},\boldsymbol{b},2) & \text{if } a_{\ell-t+1} = \alpha \end{cases}$$

is injective.

The two previous examples have the following application. Recall the partially ordered set $F_{\boldsymbol{\alpha}}$ defined in the previous section (before Example 6). Here we deal only the case $\alpha := \alpha_1 = \cdots = \alpha_n (\geq 2)$. A family $\mathcal{F} \subseteq F_{\boldsymbol{\alpha}}$ is called *nontrivial t-intersecting* if it is $t$-intersecting and if the infimum of *all* members of $\mathcal{F}$ has rank less than $t$. This means that the set

$$\{i : a_i = b_i > 0 \text{ for all } \boldsymbol{a},\boldsymbol{b} \in \mathcal{F}\}$$

has cardinality at most $t-1$. Define

$$\tilde{M}_k(n,\boldsymbol{\alpha},t) = \max\{|\mathcal{F}| : \mathcal{F} \subseteq N_k(\boldsymbol{\alpha}),\ \mathcal{F} \text{ is nontrivial } t\text{-intersecting}\}.$$

We suppose that $k \geq t+2$ since otherwise there are no such families. For a tuple $\boldsymbol{a} \in F_{\boldsymbol{\alpha}}$ let the *support of* $\boldsymbol{a}$ be given by

$$\mathrm{supp}\,(\boldsymbol{a}) := \{i : a_i = 1\}.$$

Then we have the following nontrivial $t$-intersecting candidate families:

$$\mathcal{F}_r := \{\boldsymbol{a} \in N_k(\boldsymbol{\alpha}) : \mathrm{supp}\,(\boldsymbol{a}) \in \mathcal{S}_r\},\ r \geq 1.$$

Recall that

$$M_k(n,\boldsymbol{\alpha},t) = M(n,t;\boldsymbol{\omega})$$

with $\omega_i = W_{k-i}((\alpha-1)^{n-i})$.

Now Example 10 and Example 11 solve the uniform nontrivial $t$-intersection problem in $\mathcal{F}_{\boldsymbol{\alpha}}$. This is clear with the next Lemma.

**Lemma 18.** *We have* $\tilde{M}_k(n,\boldsymbol{\alpha},t) = \tilde{M}(n,t;\boldsymbol{\omega})$.

**Proof.** Let $\mathcal{F}$ be a maximum $k$-uniform nontrivial $t$-intersecting family in $F_\alpha$. It suffices to show that

$$|\mathcal{F}| \leq \tilde{M}(n,t;\omega). \tag{52}$$

Recall the operation $s_{i,j,c} : 2^{F_\alpha} \to 2^{F_\alpha}$ defined by (38) and (39). We know that $|s_{i,j,c}(\mathcal{F})| = |\mathcal{F}|$ and that $s_{i,j,c}(\mathcal{F})$ is $t$-intersecting for all $i,j,k$. Let

$$I := I(\mathcal{F}) := \{(i,j,c) : 1 \leq i < j \leq \alpha, c \in [n], s_{i,j,c}(\mathcal{F}) \neq \mathcal{F}\}.$$

If $I = \emptyset$ we are done since then the family

$$\{\text{supp}\,(\boldsymbol{a}) : \boldsymbol{a} \in \mathcal{F}\} \subseteq 2^{[n]}$$

is easily seen to be nontrivial $t$-intersecting.

Thus let $I \neq \emptyset$. We may assume that $s_{i,j,c}(\mathcal{F})$ is not nontrivial $t$-intersecting for all $(i,j,c) \in I$ (otherwise keep applying the corresponding operations $s_{i,j,c}$). Then the set

$$T := \{i : a_i = b_i > 0 \text{ for all } \boldsymbol{a}, \boldsymbol{b} \in \mathcal{F}\}$$

has cardinality $t-1$. Let w.l.o.g. $T = [t-1]$ and $a_i = 1$ for all $\boldsymbol{a} \in \mathcal{F}$, $i \in T$. Moreover, for all $(i,j,c) \in I$ and all $\boldsymbol{a} \in \mathcal{F}$ we have $a_c \in \{i,j\}$, and there are $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{F}$ with $a_c = i$ and $b_c = j$. We have $i = 1$ for all $(i,j,c) \in I$ since otherwise also $(1,i,c) \in I$ and $s_{1,i,c}(\mathcal{F})$ would be nontrivial $t$-intersecting. Analogously, we have $j = 2$ for all $(i,j,c) \in I$. Define

$$C := \{c : (1,2,c) \in I\}.$$

W.l.o.g. let $C = [t, t+q]$, $q = 0, \ldots, n-t$.
**Case** $|C| > 1$, i.e. $q > 0$.
We will show that $|\mathcal{F}| \leq |\mathcal{F}_1|$ which implies (52).

Note that there are not $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{F}$ with $a_t = 1, b_t = 2$ and $a_i = b_i$ for all $i > t$ since otherwise $\boldsymbol{b} \in s_{1,2,t}(\mathcal{F})$ and hence $s_{1,2,t}(\mathcal{F})$ would be nontrivial $t$-intersecting. Consequently, recalling that for all $\boldsymbol{a} \in \mathcal{F}$ we have $a_1 = \cdots = a_{t-1} = 1$ and $a_t, \ldots, a_{t+q} \in \{1,2\}$,

$$|\mathcal{F}| \leq 2^q W_{k-t-q}(\alpha^{n-t-q}) \leq 2^q \alpha^{-(q-1)} W_{k-t-1}(\alpha^{n-t-1}) \leq 2W_{k-t-1}(\alpha^{n-t-1}).$$

Note that

$$|\mathcal{F}_1| = (t+2)W_{k-t-1}(\alpha-1, \alpha^{n-t-2}) + W_{k-t-2}(\alpha^{n-t-2}).$$

Hence, using (51), $|\mathcal{F}| \leq |\mathcal{F}_1|$ follows from

$$\begin{aligned} 2W_{k-t-1}(\alpha^{n-t-1}) &= 2\left(W_{k-t-1}(\alpha-1, \alpha^{n-t-2}) + W_{k-t-2}(\alpha^{n-t-2})\right) \\ &\leq (t+2)W_{k-t-1}(\alpha-1, \alpha^{n-t-2}) + W_{k-t-2}(\alpha^{n-t-2}). \end{aligned}$$

**Case** $|C| = 1$, i.e. $q = 0$.
Define for $\ell = 1,2$ the (nonempty) families

$$\mathcal{H}_\ell := \{\text{supp}\,(\boldsymbol{a}) \setminus [t] : \boldsymbol{a} \in \mathcal{F},\ a_t = \ell\}.$$

Since $(1,j,c) \notin I$ for all $c > t$, $j \in [\alpha]$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are *cross–intersecting*, i.e.

$$H_1 \cap H_2 \neq \emptyset \text{ for all } H_1 \in \mathcal{H}_1, H_2 \in \mathcal{H}_2. \tag{53}$$

Also, since $\mathcal{F}$ is nontrivial $t$-intersecting,

$$\bigcap_{H \in \mathcal{H}_1 \cup \mathcal{H}_2} H = \emptyset. \tag{54}$$

Since $|\mathcal{F}|$ is maximum, we necessarily have

$$\mathcal{F} = \bigcup_{\ell \in \{1,2\}} \{\boldsymbol{a} \in N_k(\boldsymbol{\alpha}) : [t-1] \subseteq \operatorname{supp}(\boldsymbol{a}),\ a_t = \ell,\ \operatorname{supp}(\boldsymbol{a}) \cap [t+1,n] \in \mathcal{H}_\ell\}.$$

We apply the shift–operation $s_{i,j}$, $t < i < j \leq n$, simultaneously to $\mathcal{H}_1$ and $\mathcal{H}_2$. It is easy to see that $s_{i,j}(\mathcal{H}_1)$ and $s_{i,j}(\mathcal{H}_2)$ still satisfy (53). Let $\mathcal{F}_{i,j}$ be the family which corresponds to the pair $s_{i,j}(\mathcal{H}_1)$, $s_{i,j}(\mathcal{H}_2)$, i.e.

$$\mathcal{F}_{i,j} := \bigcup_{\ell \in \{1,2\}} \{\boldsymbol{a} \in N_k(\boldsymbol{\alpha}) : [t-1] \subseteq \operatorname{supp}(\boldsymbol{a}),\ a_t = \ell,$$
$$\operatorname{supp}(\boldsymbol{a}) \cap [t+1,n] \in s_{i,j}(\mathcal{H}_\ell)\}.$$

Note that $|\mathcal{F}_{i,j}| = |\mathcal{F}|$. If

$$\bigcap_{H \in s_{i,j}(\mathcal{H}_1) \cup s_{i,j}(\mathcal{H}_2)} H \neq \emptyset$$

then we have $a_i = 1$ for all $\boldsymbol{a} \in \mathcal{F}_{i,j}$. Consequently,

$$|\mathcal{F}| = |\mathcal{F}_{i,j}| \leq 2W_{k-t-1}(\alpha^{n-t-1}) \leq |\mathcal{F}_1|$$

which again implies (52). Hence we may assume that $s_{i,j}(\mathcal{H}_1)$ and $s_{i,j}(\mathcal{H}_2)$ also satisfy (54). We now continue the shifting until we obtain a family (also named $\mathcal{F}$) for which the corresponding families $\mathcal{H}_1$ and $\mathcal{H}_2$ are *left–shifted in* $[t+1,n]$, i.e. $s_{i,j}(\mathcal{H}_\ell) = \mathcal{H}_\ell$ for all $t+1 \leq i < j \leq n$, $\ell = 1,2$. But then there are obviously $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{F}$ with $a_t = 1$, $b_t = 2$, $a_{t+1} = b_{t+1} = \cdots = a_k = b_k$, $a_{k+1} = b_{k+1} = \cdots = a_n = b_n = 0$. Now (52) follows since $s_{1,2,t}(\mathcal{F})$ is nontrivial $t$-intersecting and $I(s_{1,2,t}(\mathcal{F})) = \emptyset$. □

## PUSHING–PULLING

Beside the method of generating sets [2, 4] Ahlswede and Khachatrian developed another proof method, called pushing–pulling, which was used in [3] to give a (new) proof of their Theorem 2, and a proof of Katona's Theorem 6. Since it seems difficult to find general suppositions on $\omega$ under which the pushing–pulling method works, we shall stay quite closely to the original arguments in [3]. This section finishes the proof of Theorem 3a. We will also deduce Theorem 5.

Recall that a family $\mathcal{F} \in 2^{[n]}$ is called left–shifted if

$$s_{i,j}(\mathcal{F}) = \mathcal{F} \text{ for all } i, j \in [n], i < j.$$

Let $\ell \in [n]$. A family $\mathcal{F} \in 2^{[n]}$ is called *invariant in* $[\ell]$ if

$$s_{i,j}(\mathcal{F}) = \mathcal{F} \text{ for all } i, j \in [\ell].$$

**Lemma 19.** *Let $t \geq 2$. Suppose that for some $r \in \{0, \ldots, \lfloor \frac{n-t}{2} \rfloor\}$*

$$\omega_i = 0 \text{ unless } k_{r-1} \leq i.$$

*Then there is a left–shifted optimal family which is invariant in $[t + 2r]$.*

**Proof.** First we deal with an arbitrary (nonnegative) weight vector $\boldsymbol{\omega}$. Among all left–shifted optimal families $\mathcal{F}$ choose one for which

$$\ell := \ell(\mathcal{F}) := \max\{i : \mathcal{F} \text{ is invariant in } [i]\}$$

is maximum. We may assume that $\omega_i = 0$ implies $f_i = 0$.

Now we assume that $\ell < t + 2r$ and look for a contradiction. Let

$$\begin{aligned}
\mathcal{L} &:= \mathcal{L}(\mathcal{F}) := \{X \in \mathcal{F} : s_{\ell+1,i}(X) \notin \mathcal{F} \text{ for some } 1 \leq i \leq \ell\}, \\
\mathcal{L}_i &:= \{X \in \mathcal{L} : |X \cap [\ell]| = i\}, \\
\mathcal{L}' &:= \{X : \ell + 1 \in X \text{ and } s_{i,\ell+1}(X) \in \mathcal{L} \text{ for some } i \in [\ell]\}, \\
\mathcal{L}'_i &:= \{X \in \mathcal{L}' : |X \cap [\ell]| = i - 1\}
\end{aligned}$$

The set $\mathcal{L}$ (and hence also $\mathcal{L}'$) is not empty and invariant in $[\ell]$. Hence,

$$\omega(\mathcal{L}_i) = \binom{\ell}{i} \sum_{X \in \mathcal{L}_i^*} \omega_{|X|+i} \text{ and } \omega(\mathcal{L}'_i) = \binom{\ell}{i-1} \sum_{X \in \mathcal{L}_i^*} \omega_{|X|+i} \tag{55}$$

where

$$\mathcal{L}_i^* := \{X \cap [\ell + 2, n] : X \in \mathcal{L}_i\}.$$

It is not too difficult to verify that $|X \cap Y| \geq t$ for all $X \in \mathcal{L}'_i$ and $Y \in \mathcal{F} \backslash \mathcal{L}_{\ell+t-i}$ (use that $\mathcal{F}$ is left–shifted, invariant in $[\ell]$, and $t$-intersecting). Hence, for any $i \in [t, \frac{\ell+t}{2})$ the two families

$$\begin{aligned}
\mathcal{F}_{1,i} &:= (\mathcal{F} \setminus \mathcal{L}_{\ell+t-i}) \cup \mathcal{L}'_i, \\
\mathcal{F}_{2,i} &:= (\mathcal{F} \setminus \mathcal{L}_i) \cup \mathcal{L}'_{\ell+t-i}
\end{aligned}$$

are $t$-intersecting and since $\mathcal{F}$ is optimal we have

$$\omega(\mathcal{L}'_i) \leq \omega(\mathcal{L}_{\ell+t-i}) \text{ and } \omega(\mathcal{L}'_{\ell+t-i}) \leq \omega(\mathcal{L}_i). \tag{56}$$

It follows that $\mathcal{L}_i = \emptyset$ for all $i \in [\ell] \setminus \{\frac{\ell+t}{2}\}$ because otherwise (56) together with (55) yields

$$i(\ell + t - i) \leq (\ell - i + 1)(i + 1 - t),$$

which is easily seen to be false since $t \geq 2$.

It follows $2 \mid \ell + t$ since otherwise we have $\mathcal{L} = \emptyset$. The assumption $\ell < t + 2r$ then implies

$$\ell \leq t + 2r - 2. \tag{57}$$

Suppose that we find an *intersecting* subfamily $\mathcal{T}^*$ of $\mathcal{L}^*_{\frac{\ell+t}{2}}$ which satisfies

$$\frac{\sum_{X \in \mathcal{T}^*} \omega_{|X| + \frac{\ell+t}{2}}}{\sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}}} \omega_{|X| + \frac{\ell+t}{2}}} > \frac{\ell - t + 2}{2(\ell + 1)} = \frac{\binom{\ell}{(\ell+t)/2}}{\binom{\ell+1}{(\ell+t)/2}}. \tag{58}$$

Let

$$\begin{aligned} \mathcal{T} &:= \{X \in \mathcal{L}_{\frac{\ell+t}{2}} : X \cap [\ell+2, n] \in \mathcal{T}^*\}, \\ \mathcal{T}' &:= \{X \in \mathcal{L}'_{\frac{\ell+t}{2}} : X \cap [\ell+2, n] \in \mathcal{T}^*\}. \end{aligned}$$

Then, as in (55),

$$\omega(\mathcal{T}) = \binom{\ell}{(\ell+t)/2} \sum_{X \in \mathcal{T}^*} \omega_{|X| + \frac{\ell+t}{2}}, \tag{59}$$

$$\omega(\mathcal{T}') = \binom{\ell}{(\ell+t)/2 - 1} \sum_{X \in \mathcal{T}^*} \omega_{|X| + \frac{\ell+t}{2}}. \tag{60}$$

It is easy to see that

$$\mathcal{F}_1 := \left(\mathcal{F} \setminus \mathcal{L}_{\frac{\ell+t}{2}}\right) \cup \mathcal{T} \cup \mathcal{T}'$$

is $t$-intersecting. But (58) together with (55), (59), and (60) yields

$$\omega(\mathcal{T}) + \omega(\mathcal{T}') > \omega(\mathcal{L}_{\frac{\ell+t}{2}}),$$

hence $\omega(\mathcal{F}_1) > \omega(\mathcal{F})$, a contradiction.

Now let $\omega$ satisfy the hypothesis of Lemma 19. We have by double counting

$$\sum_{i \in [\ell+2, n]} \sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}} : i \in X} \omega_{|X| + \frac{\ell+t}{2}} = \sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}}} \left(\omega_{|X| + \frac{\ell+t}{2}}\right) |X|.$$

Hence there is some $i \in [\ell + 2, n]$ such that

$$\begin{aligned} \sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}} : i \in X} \omega_{|X| + \frac{\ell+t}{2}} &\geq \frac{1}{n - \ell - 1} \sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}}} \left(\omega_{|X| + \frac{\ell+t}{2}}\right) |X| \\ &\geq \frac{k_{r-1} - \frac{\ell+t}{2}}{n - \ell - 1} \sum_{X \in \mathcal{L}^*_{\frac{\ell+t}{2}}} \left(\omega_{|X| + \frac{\ell+t}{2}}\right), \end{aligned} \tag{61}$$

where the last inequality follows from

$$|X| < k_{r-1} - (\ell + t)/2 \text{ implies } \omega_{|X| + \frac{\ell+t}{2}} = 0.$$

Using the definition of $k_r$ it is easy to show that

$$\frac{k_{r-1}-(\ell+t)/2}{n-\ell-1} \geq \frac{\ell-t+2}{2(\ell+1)} \text{ if } \ell \leq t+2r-2.$$

Hence, recalling (57), strict inequality in (61) gives an intersecting family $\mathcal{T}^* \subseteq \mathcal{L}^*_{\frac{\ell+t}{2}}$ satisfying (58). If we have equality in (61) for all $i \in [\ell+2, n]$ then take $i := \ell+2$. This gives a $\mathcal{T}^*$ for which the corresponing family $\mathcal{F}_1$ is left–shifted and (obviously) invariant in $[\ell+1]$, a contradiction to our choice of $\mathcal{F}$. □

Note that if in Lemma 19

$$\omega_i = 0 \text{ unless } k_{r-1} < i,$$

then the above proof yields that all left–shifted optimal families are invariant in $[t+2r]$.

Now we are ready to prove Theorem 3a.

**Proof of Theorem 3a.** The case $t=1$ is trivial. Let $t>1$. By Example 4 we know $M(n,t;\omega) = M_{t+2r+2}(n,t;\omega)$. As in Lemma 19, choose among all left–shifted optimal families $\mathcal{F} \in I_{t+2r+2}(n,t)$ one for which $\ell(\mathcal{F})$ is maximum. Then the proof of Lemma 19 shows that also this family $\mathcal{F}$ is invariant in $[t+2r]$. (Note that if we take $i := \ell+2$ in (61) then the corresponding family $\mathcal{F}_1$ is still in $I_{t+2r+2}(n,t)$.) Let $\mathcal{F}'_i := \{X \in \mathcal{F} : |X \cap [t+2r]| = i\}$. Then the following facts are easy consequences of the $(t+2r+2)$-$t$-intersection and the $[t+2r]$-invariance property of $\mathcal{F}$ :

1) $\mathcal{F}'_i = \emptyset$ for all $i < t+r-1$,

2) $\{t+2r+1, t+2r+2\} \in X$ for all $X \in \mathcal{F}'_{t+r-1}$,

3) if $\mathcal{F}'_{t+r-1} \neq \emptyset$ then $|\{t+2r+1, t+2r+2\} \cap X| \geq 1$ for all $X \in \mathcal{F}'_{t+2r}$.

It follows that $\mathcal{F} = \mathcal{S}_r$ or $\mathcal{F} = \mathcal{S}_{r+1}$. □

Let $\mathcal{F}$ be $t$-intersecting. Note that if $2 \mid n+t$ and $\mathcal{F}$ is invariant in $[n]$ or if $2 \nmid n+t$ and $\mathcal{F}$ is invariant in $[n-1]$ then $\mathcal{F} \subseteq \mathcal{S}_{\lfloor \frac{n-t}{2} \rfloor}$. Hence the pushing–pulling method can be used to prove the optimality of the last candidate family.

**Proof of Theorem 5.** Again, the case $t=1$ is trivial. Let $t>1$. It suffices to show the existence of an optimal family which is invariant in $[n]$ resp. $[n-1]$ if $2 \mid n+t$ resp. if $2 \nmid n+t$. We proceed as in the proof of Lemma 19. Hence we assume $\ell < n$ if $2 \mid n+t$ and $\ell < n-1$ if $2 \nmid n+t$. Then (57) becomes

$$\ell \leq n-2 \text{ if } 2 \mid n+t,\ \ell \leq n-3 \text{ if } 2 \mid n+t. \tag{62}$$

Now we claim that $\mathcal{L}^*_{\frac{\ell+t}{2}}$ is *self-complementary* (in $2^{[\ell+2,n]}$), i.e. $X \in \mathcal{L}^*_{\frac{\ell+t}{2}}$ implies $[\ell+2,n] \setminus X \in \mathcal{L}^*_{\frac{\ell+t}{2}}$. Indeed, for every set $X \in \mathcal{L}^*_{\frac{\ell+t}{2}}$, $X \neq \emptyset$, there is a set $Y \in \mathcal{L}^*_{\frac{\ell+t}{2}}$ with $X \cap Y = \emptyset$. Otherwise one could add any set $s_{\ell+1,i}(Z)$, with $Z \in \mathcal{L}_{\frac{\ell+t}{2}}$, $Z \cap [\ell+2,n] = X$, $i \in Z \cap [\ell]$, to the family $\mathcal{F}$ without violating the

$t$-intersection property, but this contradicts the optimality of $\mathcal{F}$ since we have assumed that $\omega_{|X|} > 0$ for all $X \in \mathcal{F}$. Using

$$0 < \omega_{|X|+\frac{\ell+t}{2}} \leq \omega_{|[\ell+2,n]\setminus X|+\frac{\ell+t}{2}} \text{ for } X \in \mathcal{L}^*_{\frac{\ell+t}{2}},\ |X| + (\ell+t)/2 \leq (n+t-1)/2$$

we deduce that

$$Z \in \mathcal{L}_{\frac{\ell+t}{2}}, |Z \cap [\ell+2,n]| \leq \frac{n-\ell-1}{2} \text{ implies } (Z \cap [\ell]) \cup ([\ell+2,n] \setminus Z) \in \mathcal{F},$$

and hence implies (using the $[\ell]$-invariance of $\mathcal{F}$)

$$(Z \cap [\ell]) \cup ([\ell+2,n] \setminus Z) \in \mathcal{L}_{\frac{\ell+t}{2}}.$$

This establishes that $\mathcal{L}^*_{\frac{\ell+t}{2}}$ is self-complementary.

Now let $\mathcal{T}^*$ be the intersecting family of all sets $X \in \mathcal{L}^*_{\frac{\ell+t}{2}}$ with $|X| > \frac{n-\ell-1}{2}$ and (in the case $2 \mid n-\ell-1$) all sets $X \in \mathcal{L}^*_{\frac{\ell+t}{2}}$ with $|X| = \frac{n-\ell-1}{2}$ and $n \notin X$. Then, using the hypothesis on $\omega$ and the fact that $\mathcal{L}^*_{\frac{\ell+t}{2}}$ is self-complementary, it is easy to deduce that this family $\mathcal{T}^*$ satisfies (58):

$$\frac{\sum_{X\in\mathcal{T}^*} \omega_{|X|+\frac{\ell+t}{2}}}{\sum_{X\in\mathcal{L}^*_{\frac{\ell+t}{2}}} \omega_{|X|+\frac{\ell+t}{2}}} \geq \frac{1}{2} > \frac{\ell-t+2}{2(\ell+1)}.$$

This finishes the proof. □

## References


[1] R. Ahlswede and L.H. Khachatrian. "The complete nontrivial–intersection theorem for systems of finite sets". *J. Combin. Theory Ser. A*, **76**:121–138, (1996).

[2] R. Ahlswede and L.H. Khachatrian. "The complete intersection theorem for systems of finite sets". *European J. Combin.*, **18**:125–136, (1997).

[3] R. Ahlswede and L.H. Khachatrian. "A pushing–pulling method: New proofs of intersection theorems". *Combinatorica*, **19**:1–15, (1999).

[4] R. Ahlswede and L.H. Khachatrian. "The diametric theorem in Hamming space – optimal anticodes". *Adv. in Appl. Math.*, **20**:429–449, (1998).

[5] C. Bey. "Durchschnittsprobleme im Booleschen Verband". Ph. D. Thesis. Universität Rostock, (1999).

[6] C. Bey. "The Erdős–Ko–Rado bound for the function lattice". *Discrete Appl. Math.*, **95**:115–125, (1999).

[7] C. Bey. "An intersection theorem for weighted sets". *Discrete Math.*, to appear.

[8] C. Bey and K. Engel. "An asymptotic complete intersection theorem for chain products". *European J. Combin.*, **20**:321–327, (1999).

[9] K. Engel. *Sperner Theory.* Cambridge University Press, Cambridge, (1997).

[10] K. Engel and P. Frankl. "An Erdős–Ko–Rado theorem for integer sequences of given rank". *European J. Combin.*, **7**:215–220, (1986).

[11] P. Erdős, C. Ko, and R. Rado. "Intersection theorems for systems of finite sets". *Quart. J. Math. Oxford Ser.*, **12**:313–320, (1961).

[12] P.L. Erdős, P. Frankl, and G.O.H. Katona. "Extremal hypergraph problems and convex hulls". *Combinatorica*, **5**:11–26, (1985).

[13] P. Frankl. "The shifting technique in extremal set theory". In C. Whitehead, editor, *Surveys in Combinatorics*, volume **123** of *Lond. Math. Soc. Lect. Note Ser.*, pages 81–110, Cambridge, (1987). Cambridge University Press.

[14] P. Frankl and N. Tokushige. "The Erdős–Ko–Rado theorem for integer sequences". *Combinatorica*, **19**:55–63, (1999).

[15] G.O.H. Katona. "Intersection theorems for systems of finite sets". *Acta Math. Acad. Sci. Hung.*, **15**:329–337, (1964).