

# NOTES ON LATTICE POINTS OF ZONOTOPES AND LATTICE-FACE POLYTOPES

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ABSTRACT. Minkowski's second theorem on successive minima gives an upper bound on the volume of a convex body in terms of its successive minima. We study the problem to generalize Minkowski's bound by replacing the volume by the lattice point enumerator of a convex body. In this context we are interested in bounds on the coefficients of Ehrhart polynomials of lattice polytopes via the successive minima. Our results for lattice zonotopes and lattice-face polytopes imply, in particular, that for 0-symmetric lattice-face polytopes and lattice parallelepipeds the volume can be replaced by the lattice point enumerator.

## 1. INTRODUCTION

Let  $\mathcal{K}^n$  be the set of all convex bodies in  $\mathbb{R}^n$ , i.e., compact convex sets with non-empty interior. The additional subscript in  $\mathcal{K}_0^n$  points out that the considered convex bodies are 0-symmetric. When dealing with polytopes we write  $\mathcal{P}^n$  and  $\mathcal{P}_0^n$ , and for  $P \in \mathcal{P}^n$  we denote by  $\text{vert}(P)$  its set of vertices. The family of  $n$ -dimensional lattices in  $\mathbb{R}^n$  is written as  $\mathcal{L}^n$  and the usual Lebesgue measure with respect to the  $n$ -dimensional space as  $\text{vol}_n(\cdot)$ . If the ambient space is clear from the context we omit the subscript and just write  $\text{vol}(\cdot)$ . For some subset  $K \subset \mathbb{R}^n$  and some lattice  $\Lambda \in \mathcal{L}^n$  the lattice point enumerator is denoted by  $G(K, \Lambda) = \#(K \cap \Lambda)$ . If  $\Lambda = \mathbb{Z}^n$  we shortly write  $G(K) = G(K, \mathbb{Z}^n)$ . In the following we study relations between this quantity and *Minkowski's successive minima* which are defined as

$$\lambda_i(K, \Lambda) = \min\{\lambda > 0 : \dim(\lambda K \cap \Lambda) \geq i\}, \quad 1 \leq i \leq n,$$

for a 0-symmetric convex body  $K \in \mathcal{K}_0^n$  with respect to a lattice  $\Lambda \in \mathcal{L}^n$ . Note that  $\dim(S)$  denotes the dimension of the affine hull of  $S \subset \mathbb{R}^n$ . If  $\Lambda = \mathbb{Z}^n$  we just write  $\lambda_i(K) = \lambda_i(K, \mathbb{Z}^n)$ . These numbers form an increasing sequence, so  $\lambda_1(K, \Lambda) \leq \dots \leq \lambda_n(K, \Lambda)$ , and as functionals on  $\mathcal{K}_0^n \times \mathcal{L}^n$  they are homogeneous of degree  $-1$  in the first and of degree  $1$  in the second argument. An important and deep result in the geometry of numbers is

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the following theorem which is usually referred to as Minkowski's second theorem on convex bodies (cf. [10, pp. 376]).

**Theorem 1.1** (Minkowski, 1896). *Let  $K \in \mathcal{K}_0^n$  and  $\Lambda \in \mathcal{L}^n$  be a lattice. Then,*

$$\lambda_1(K, \Lambda) \cdot \dots \cdot \lambda_n(K, \Lambda) \text{vol}(K) \leq 2^n \det(\Lambda).$$

The relevance of this result is also illustrated by the big number of proofs and generalizations from various contexts (see [15] for a survey report). A discrete version of Minkowski's theorem was proposed, and proved in the planar case, in [4] where the volume is replaced by the lattice point enumerator of  $K \in \mathcal{K}_0^n$ .

**Conjecture 1.1** (Betke, Henk, Wills, 1993). *Let  $K \in \mathcal{K}_0^n$  and  $\Lambda \in \mathcal{L}^n$  be a lattice. Then,*

$$G(K, \Lambda) \leq \prod_{i=1}^n \left\lfloor \frac{2}{\lambda_i(K, \Lambda)} + 1 \right\rfloor.$$

This conjecture would not only generalize Theorem 1.1 but also unify this and other particular results from geometry of numbers, for example  $G(K) \leq 3^n$ , for  $K \in \mathcal{K}_0^n$  whose only interior lattice point is the origin (cf. [20, p. 79]). Recently, Malikiosis [17, 18] settled the three-dimensional case by an inductive approach and obtained the smallest known constant  $c = \sqrt[3]{40/9} \approx 1.64414$  such that, roughly speaking, the conjecture holds up to the factor  $c^n$ . Already proposed in [11, Ch. 2, §9], it is natural to extend the notion of successive minima to general, not necessarily 0-symmetric, convex bodies  $K \in \mathcal{K}^n$  via some symmetrization, e.g., by considering  $\lambda_i(\frac{1}{2}DK, \Lambda)$ , where  $DK = K - K$ . With this notation the above conjecture for  $K \in \mathcal{K}^n$  reads

$$(1.1) \quad G(K, \Lambda) \leq \prod_{i=1}^n \left\lfloor \frac{1}{\lambda_i(DK, \Lambda)} + 1 \right\rfloor,$$

and we will mostly deal with this more general question.

A helpful observation is, that it suffices to prove (1.1) for lattice polytopes  $P \in \mathcal{P}^n$ , i.e.,  $\text{vert}(P) \subset \Lambda$ . Indeed, since the successive minima are monotonic functionals, i.e., if  $K, K' \in \mathcal{K}_0^n$  with  $K \subseteq K'$ , then  $\lambda_i(K, \Lambda) \geq \lambda_i(K', \Lambda)$ , for all  $1 \leq i \leq n$ , we can consider  $P_K = \text{conv}\{K \cap \Lambda\}$ . If  $\dim P_K < n$  then it suffices to consider (1.1) for  $P_K$  and with respect to the lattice  $\Lambda \cap \text{lin}(P_K)$ , where  $\text{lin}(\cdot)$  denotes the linear hull.

Furthermore, since any lattice  $\Lambda \in \mathcal{L}^n$  can be written as  $A\mathbb{Z}^n$  for some invertible matrix  $A \in \mathbb{R}^{n \times n}$ , and  $\lambda_i(K, A\mathbb{Z}^n) = \lambda_i(A^{-1}K, \mathbb{Z}^n)$ , we can also restrict to the case  $\Lambda = \mathbb{Z}^n$ . This reduction to lattice polytopes allows us to utilize Ehrhart theory which is a very active research topic in recent years. Its origin goes back to a work of Eugène Ehrhart [8] from 1962 who showed that for a given lattice polytope  $P \in \mathcal{P}^n$  the function  $k \mapsto G(kP)$  is a

polynomial in  $k \in \mathbb{N}$  of degree  $n$ . Thus,

$$G(kP) = \sum_{i=0}^n g_i(P)k^i,$$

where  $g_i(P)$  depends only on  $P$  and is said to be the  $i$ th Ehrhart coefficient of  $P$ . Ehrhart already noticed that  $g_n(P) = \text{vol}(P)$ ,  $g_0(P) = 1$  and  $g_{n-1}(P)$  is the normalized surface area of  $P$  (see [2] for details). Moreover, it can be easily seen that the coefficient  $g_i$  is homogeneous of degree  $i$ . Having this by hand, instead of (1.1), one can consider the somewhat weaker inequality

$$(1.2) \quad G(P) \leq \prod_{i=1}^n \left( \frac{1}{\lambda_i(DP)} + 1 \right).$$

Let  $L(P)$  denote the right hand side of this inequality. Then

$$L(P) = \prod_{i=1}^n \left( \frac{1}{\lambda_i(DP)} + 1 \right) = \sum_{i=0}^n \sigma_i \left( \frac{1}{\lambda_1(DP)}, \dots, \frac{1}{\lambda_n(DP)} \right),$$

where  $\sigma_i$  denotes the  $i$ th elementary symmetric polynomial of  $n$  numbers  $x_j$ , i.e.,  $\sigma_i(x_1, \dots, x_n) = \sum_{I \subseteq [n], \#I=i} \prod_{j \in I} x_j$ , where  $[n] = \{1, \dots, n\}$ , and  $\sigma_0(x_1, \dots, x_n) = 1$ . For short we will just write

$$\sigma_i(P) = \sigma_i \left( \frac{1}{\lambda_1(DP)}, \dots, \frac{1}{\lambda_n(DP)} \right).$$

With this notation inequality (1.2) is equivalent to  $G(P) \leq L(P)$  and we may ask whether the coefficient-wise inequalities

$$(1.3) \quad g_i(P) \leq \sigma_i(P)$$

hold for all  $i = 0, \dots, n$ . The case  $i = 0$  is trivial since in this case both sides are equal to 1. For  $i \geq 1$  the question is supported by two known inequalities in this list. First of all, we have  $g_n(P) \leq \sigma_n(P)$ , which follows from Theorem 1.1 after applying the Brunn-Minkowski inequality (see [10, Thm. 8.1]) to derive  $g_n(P) = \text{vol}(P) \leq \frac{1}{2^n} \text{vol}(DP)$ . And secondly, in [14] it was proved that  $g_{n-1}(P) \leq \sigma_{n-1}(P)$ , for any lattice polytope  $P \in \mathcal{P}_0^n$ .

Unfortunately, for  $i \neq n, n-1$ , the inequalities do not hold in general.

**Proposition 1.1.** *Let  $Q_l^n = \text{conv} \{lC_{n-1} \times \{0\}, \pm e_n\}$ , where  $l \in \mathbb{N}$  and  $C_n = [-1, 1]^n$  is the cube of edge length 2 centered at the origin. Then, for  $n \geq 3$  and any constant  $c$  there exists an  $l \in \mathbb{N}$  such that  $g_{n-2}(Q_l^n) > c \sigma_{n-2}(Q_l^n)$ . If  $n \geq 4$ , we have the same situation for  $g_{n-3}(Q_l^n)$ .*

The proof of this statement is given at the end of the paper. In this work we show that for special classes of lattice polytopes, however, the coefficient-wise approach leads to positive results.

One of these classes is the family of lattice zonotopes. In general, a zonotope  $Z$  is the Minkowski sum of finitely many line segments, that is,

there is a set of vectors  $v_1, \dots, v_m \in \mathbb{R}^n$  and points  $p_1, \dots, p_m \in \mathbb{R}^n$  such that

$$Z = \sum_{i=1}^m [p_i, p_i + v_i] = \left\{ \sum_{i=1}^m (p_i + \alpha_i v_i) : 0 \leq \alpha_i \leq 1 \right\}.$$

Particularly, zonotopes possess a center of symmetry and furthermore are characterized in the class of centrally symmetric polytopes by the property that all two-dimensional faces are themselves centrally symmetric (see [6, Thm. 3.3]). Zonotopes appear in many different contexts, for instance, in the theory of hyperplane arrangements (cf. [24, Lect. 7]) and in problems on approximation of convex bodies (cf. [12, Sect. 15.2]).

Since we are only interested in lattice zonotopes, i.e.,  $p_i, v_i \in \mathbb{Z}^n$ , and since (1.1) is invariant under translations by lattice vectors, we can simply consider lattice zonotopes given as the sum of line segments  $[0, v_i]$ , with  $v_i \in \mathbb{Z}^n$ . Our first result shows that for any lattice parallelepiped  $Z$  the coefficient-wise inequalities hold true and, in particular, we obtain (1.1).

**Theorem 1.2.** *Let  $Z \in \mathcal{P}^n$  be an  $n$ -dimensional lattice parallelepiped. Then*

$$g_i(Z) \leq \sigma_i(Z), \quad i = 0, \dots, n.$$

We note that these inequalities are best possible. For instance, consider the cube  $Z = [0, 1]^n = \sum_{i=1}^n [0, e_i]$ , where  $e_i$  denotes the  $i$ th standard unit vector. We have  $\lambda_i(DZ) = \lambda_i([-1, 1]^n) = 1$ , and  $G(kZ) = (k+1)^n$  for any integer  $k \in \mathbb{N}$ ; thus  $g_i(Z) = \binom{n}{i} = \sigma_i(Z)$ . For general lattice zonotopes  $Z$  we obtain a relation up to a factor depending only on the dimension and not on the number of generators.

**Theorem 1.3.** *Let  $Z \in \mathcal{P}^n$  be an  $n$ -dimensional zonotope. Then*

$$\frac{g_i(Z)}{\text{vol}(Z)} \leq \frac{n!}{i!} \prod_{j=i+1}^n \lambda_j(DZ), \quad i = 0, \dots, n.$$

*In particular, we get  $g_i(Z) \leq \frac{n!}{i!} \sigma_i(Z)$ .*

The second class of polytopes that we consider was introduced by Liu [16], the so called *lattice-face polytopes*. In order to state the definition, let  $\pi^{(n-i)} : \mathbb{R}^n \rightarrow \mathbb{R}^i$  be the projection that forgets the last  $n-i$  coordinates,  $i = 1, \dots, n$ , where  $\pi^{(0)}$  denotes the identity.

**Definition 1.1** (Lattice-face polytopes). *A polytope  $P \in \mathcal{P}^n$  is called a lattice-face polytope, if for any  $0 \leq k \leq n-1$  and any subset  $U \subset \text{vert}(P)$  that spans a  $k$ -dimensional affine space,  $\pi^{(n-k)}(\text{aff}(U) \cap \mathbb{Z}^n) = \mathbb{Z}^k$ .*

For example, any integral cyclic polytope, i.e., the convex hull of finitely many lattice points on the moment curve  $t \mapsto (t, t^2, \dots, t^n)$ , is lattice-face (cf. [1, 16]). In [16] it is also shown that lattice-face polytopes are necessarily lattice polytopes and moreover, that every combinatorial type of a rational polytope has a representative among lattice-face polytopes.

**Theorem 1.4.** *Let  $P \in \mathcal{P}^n$  be a lattice-face polytope.*

i) *If  $P$  is 0-symmetric, then, for  $1 \leq i \leq n$ ,*

$$g_i(P) \leq \sigma_i(P).$$

ii) *If  $0 \in \text{vert}(P)$  and  $SP = \text{conv}(P, -P)$ , then, for  $1 \leq i \leq n$ ,*

$$g_i(P) \leq \sigma_i\left(\frac{2}{\lambda_1(SP)}, \dots, \frac{2}{\lambda_n(SP)}\right).$$

The paper is organized as follows. In Section 2 a geometric description of the Ehrhart coefficients of lattice zonotopes is discussed and the proofs of Theorem 1.2 and 1.3 are given. Also, some further results on coefficient-wise inequalities are described, which are obtained by adding some extra conditions on the generators. In Section 3 we give a brief introduction to lattice-face polytopes and the proof of Theorem 1.4. We close the paper with the proof of Proposition 1.1.

## 2. LATTICE ZONOTOPES

Let  $v_1, \dots, v_m \in \mathbb{Z}^n$  and consider  $Z = \sum_{i=1}^m [0, v_i]$ . Concerning the coefficients  $g_i(Z)$  of the Ehrhart polynomial of  $Z$ , Betke and Gritzmann [3] showed that

$$(2.1) \quad g_i(Z) = \sum_{F \in \mathcal{F}_i(Z)} \gamma(F, P) \frac{\text{vol}_i(F)}{\det(\text{aff}F \cap \mathbb{Z}^n)},$$

where  $\mathcal{F}_i(Z)$  denotes the set of all  $i$ -faces of  $Z$ ,  $\gamma(F, P)$  the external angle of  $F$  at  $P$  (cf. [12, p. 308]), and  $\det(\text{aff}F \cap \mathbb{Z}^n)$  the determinant of the sublattice of  $\mathbb{Z}^n$  contained in the affine hull of  $F$ . Another presentation was given by Stanley [23, Exer. 31, p. 272]

$$(2.2) \quad g_i(Z) = \sum_{X \in \mathcal{X}_i(Z)} \text{gcd}(i\text{-minors of } X),$$

where  $\mathcal{X}_i(Z)$  denotes the set of all linearly independent  $i$ -element subsets of  $\{v_1, \dots, v_m\}$  and  $\text{gcd}(a_1, \dots, a_k)$  is the greatest common divisor of the integers  $a_1, \dots, a_k \in \mathbb{Z}$ . From (2.2) – as well as (2.1) – we can get a slightly more geometric description: To this end we denote for a given  $J \subseteq [m]$ ,  $\#J = i$ , the zonotope generated by the vectors  $v_j$ ,  $j \in J$ , by  $P_J$ , that is,  $P_J = \sum_{j \in J} [0, v_j] = \left\{ \sum_{j \in J} \mu_j v_j : 0 \leq \mu_j \leq 1 \right\}$ .

**Proposition 2.1.** *For  $1 \leq i \leq n$  we have*

$$g_i(Z) = \sum_{J \subseteq [m], \#J=i} \frac{\text{vol}_i(P_J)}{\det(\text{lin}P_J \cap \mathbb{Z}^n)}.$$

*Proof.* If the vectors  $v_j$ ,  $j \in J$ , are linearly dependent, then  $\text{vol}_i(P_J) = 0$  and so any non-trivial contribution in that sum comes from an  $i$ -dimensional parallelepiped. The index of a sublattice  $\Lambda'$  of  $\Lambda \in \mathcal{L}^n$  is defined as  $\text{ind}\Lambda' = \frac{\det \Lambda'}{\det \Lambda}$  (cmp. [19, Sect. 1.1]). Thus, by the definition of the determinant of

a lattice, these non-trivial contributions are just the index of the sublattice generated by  $v_j$ ,  $j \in J$ , with respect to the lattice  $\text{lin}P_J \cap \mathbb{Z}^n$ .

Without loss of generality let  $\{v_j : j \in J\} = \{v_1, \dots, v_i\} = V_J$  and let the vectors be linearly independent. First we observe that

$$(2.3) \quad V_J \text{ is a lattice basis of } \text{lin}V_J \cap \mathbb{Z}^n \Leftrightarrow \gcd(i\text{-minors of } V_J) = 1.$$

For the “if-part” assume that  $V_J$  is not a basis of  $\text{lin}V_J \cap \mathbb{Z}^n$  but let  $\bar{V}$  be an  $n \times i$  matrix whose columns constitute a basis of the lattice. Then there exists a matrix  $D_J \in \mathbb{Z}^{i \times i}$  with  $V_J = \bar{V} D_J$  and so  $|\det D_J|$  is a divisor of each  $i$ -minor of  $V_J$ . Since  $|\det D_J| \geq 2$  we get the desired contradiction. In order to show the “only if-part” we extend the vectors in  $V_J$  to a basis  $\tilde{V}$  of  $\mathbb{Z}^n$  of determinant 1. Developing that determinant with respect to the last  $n - i$  columns yields

$$1 = \det \tilde{V} = \sum_{i\text{-minors } \mu_k \text{ of } V_J} \rho_k \mu_k$$

for some integers  $\rho_k$ . Hence,  $\gcd(i\text{-minors of } V_J) = 1$ .

Next, let  $\Lambda_J$  be the lattice generated by  $v_1, \dots, v_i$ . Then for the index of  $\Lambda_J$  with respect to  $\text{lin}V_J \cap \mathbb{Z}^n$  holds

$$(2.4) \quad \text{ind } \Lambda_J = \gcd(i\text{-minors of } V_J).$$

To see this, we use the same notation as in the “if-part” above and have  $V_J = \bar{V} D_J$ . Since  $\det D_J = \text{ind } \Lambda_J$  we conclude that  $\text{ind } \Lambda_J$  is a divisor of  $\gcd(i\text{-minors of } V_J)$ . On the other hand we conclude from (2.3) that  $\gcd(i\text{-minors of } \bar{V}) = 1$  which implies the reverse divisibility. Obviously, (2.4), (2.2) and the observation at the beginning of the proof imply the assertion.  $\square$

Since  $\text{vol}(Z) = g_n(Z)$ , Proposition 2.1 is for  $i = n$  just the well-known volume formula of a zonotope  $Z = \sum_{i=1}^m [0, w_i]$ ,  $w_i \in \mathbb{R}^n$ , (cf. [22])

$$(2.5) \quad \text{vol}(Z) = \sum_{1 \leq j_1 < j_2 < \dots < j_n \leq m} |\det(w_{j_1}, \dots, w_{j_n})|.$$

In order to prove Theorem 1.2 we need two auxiliary lemmas. In the following, for a set  $M$  and some  $i \in \mathbb{N}$  we denote by  $\binom{M}{i}$  the collection of all  $i$ -element subsets of  $M$ .

**Lemma 2.1.** *Let  $\{b_1, \dots, b_n\}$  and  $\{a_1, \dots, a_n\}$  be two bases of an  $n$ -dimensional vector space  $V$ , and let  $i \in \{1, \dots, n-1\}$ . Then there exists a bijection  $\phi : \binom{[n]}{i} \rightarrow \binom{[n]}{n-i}$  such that  $\{b_k : k \in I\} \cup \{a_j : j \in \phi(I)\}$  is a basis of  $V$ , for all  $I \in \binom{[n]}{i}$ .*

*Proof.* We use a standard linear algebra argument involving the exterior algebra  $\Lambda(V) = \bigoplus_{i=0}^n \Lambda_i(V)$  of  $V$  for which we refer to [5, Ch. XVI]. For all  $I \in \binom{[n]}{i}$  and  $J \in \binom{[n]}{n-i}$  let  $b_I = \wedge_{k \in I} b_k \in \Lambda_i(V)$  and  $a_J = \wedge_{j \in J} a_j \in \Lambda_{n-i}(V)$ , respectively. Consider the square matrix  $M$  with row index set

$\binom{[n]}{i}$  and column index set  $\binom{[n]}{n-i}$ , whose  $(I, J)$ -entry is  $b_I \wedge a_J$ . First we note that

$$(2.6) \quad \det M \neq 0.$$

Assume the contrary and suppose that some non-trivial linear combination of the rows of  $M$  is zero, say

$$\sum_{I \in \binom{[n]}{i}} c_I (b_I \wedge a_J) = \left( \sum_{I \in \binom{[n]}{i}} c_I b_I \right) \wedge a_J = 0,$$

for all  $J \in \binom{[n]}{n-i}$ , with scalars  $c_I$ , not all zero. Expanding the nonzero vector  $\sum_{I \in \binom{[n]}{i}} c_I b_I \in \Lambda_i(V)$  in terms of the basis  $\{a_I : I \in \binom{[n]}{i}\}$  of  $\Lambda_i(V)$  yields

$$\left( \sum_{I \in \binom{[n]}{i}} d_I a_I \right) \wedge a_J = \sum_{I \in \binom{[n]}{i}} d_I (a_I \wedge a_J) = 0,$$

for all  $J \in \binom{[n]}{n-i}$ , with scalars  $d_I$ , not all zero. But in view of  $a_I \wedge a_J \neq 0$  if and only if  $I = [n] \setminus J$  we conclude that  $d_I = 0$ , for all  $I \in \binom{[n]}{i}$ , a contradiction.

So  $\det M \neq 0$ , and by Leibniz' formula there exists a bijection  $\phi : \binom{[n]}{i} \rightarrow \binom{[n]}{n-i}$  with  $b_I \wedge a_{\phi(I)} \neq 0$ , for  $I \in \binom{[n]}{i}$ . This is equivalent to  $\{b_k : k \in I\} \cup \{a_j : j \in \phi(I)\}$  being a basis of  $V$ , for  $I \in \binom{[n]}{i}$  (cf. [5, Thm. XVI.13]), which we wanted to show.  $\square$

**Lemma 2.2.** *Let  $K \in \mathcal{K}_0^n$ , and let  $a_1, \dots, a_n \in \mathbb{Z}^n$  be linearly independent such that  $a_j \in \lambda_j(K)K$ ,  $1 \leq j \leq n$ . Let  $\bar{L}$  be an  $i$ -dimensional linear subspace,  $i \in \{1, \dots, n-1\}$ , containing  $i$  linearly independent points of  $\mathbb{Z}^n$ , and assume that  $\text{lin}\{a_{j_1}, \dots, a_{j_{n-i}}\} \cap \bar{L} = \{0\}$ . Then*

$$\prod_{j=1}^i \lambda_j(K \cap \bar{L}, \mathbb{Z}^n \cap \bar{L}) \geq \prod_{k \notin \{j_1, \dots, j_{n-i}\}} \lambda_k(K).$$

*Proof.* For abbreviation we set  $\bar{\Lambda} = \mathbb{Z}^n \cap \bar{L}$ ,  $\bar{K} = K \cap \bar{L}$ ,  $\bar{\lambda}_j = \lambda_j(K \cap \bar{L}, \mathbb{Z}^n \cap \bar{L})$ ,  $1 \leq j \leq i$ , and  $\lambda_j = \lambda_j(K)$ ,  $1 \leq j \leq n$ . Moreover, let  $\bar{w}_1, \dots, \bar{w}_i \in \bar{\Lambda}$  be linearly independent such that  $\bar{w}_j \in \bar{\lambda}_j \bar{K}$ . Let  $j_1 < j_2 < \dots < j_{n-i}$  and let  $k_1 < k_2 < \dots < k_i$  be the indices in  $[n] \setminus \{j_1, \dots, j_{n-i}\}$ . Suppose there exists an index  $l \in \{1, \dots, i\}$  with

$$(2.7) \quad \bar{\lambda}_l < \lambda_{k_l},$$

and let  $m$  be the smallest index such that  $\lambda_m = \lambda_{k_l}$ . Since  $\bar{K} \subset K$ ,  $\bar{\Lambda} \subset \mathbb{Z}^n$ , we get by (2.7), the choice of  $m$  and the definition of the successive minima that

$$\{\bar{w}_1, \dots, \bar{w}_l\} \cup \{a_j : 1 \leq j \leq m-1, j \in \{j_1, \dots, j_{n-i}\}\} \subseteq \text{int}(\lambda_m K) \cap \mathbb{Z}^n.$$

Since there are at most  $l - 1$  indices in the set  $\{1, \dots, m - 1\}$  belonging to  $\{k_1, \dots, k_i\}$ , we conclude that  $\#\{j : j \in \{j_1, \dots, j_{n-i}\} \text{ and } 1 \leq j \leq m - 1\} \geq m - l$ . Hence, on the left hand side of the inclusion above we have at least  $m$  lattice vectors which by the assumption  $\text{lin}\{a_{j_1}, \dots, a_{j_{n-i}}\} \cap \bar{L} = \{0\}$  are linearly independent. This, however, contradicts the definition of  $\lambda_m$ , and so we have shown  $\bar{\lambda}_l \geq \lambda_{k_l}$ ,  $l = 1, \dots, i$ , which implies the assertion.  $\square$

*Proof of Theorem 1.2.* Let  $Z$  be the parallelepiped generated by  $v_1, \dots, v_n \in \mathbb{Z}^n$ . Abbreviate  $\lambda_j(DZ)$  by  $\lambda_j$  and for  $J \subseteq [n]$  with  $\#J = i$ , let  $DP_J = P_J - P_J = \left\{ \sum_{j \in J} \mu_j v_j : -1 \leq \mu_j \leq 1 \right\}$  and write  $\Lambda_J = \text{lin}\{v_j : j \in J\} \cap \mathbb{Z}^n$ . In view of Proposition 2.1 and the fact that  $\text{vol}_i(P_J) = \frac{1}{2^i} \text{vol}_i(DP_J)$  we have to show

$$g_i(Z) = \frac{1}{2^i} \sum_{J \subseteq [n], \#J=i} \frac{\text{vol}_i(DP_J)}{\det \Lambda_J} \leq \sum_{I \subseteq [n], \#I=i} \frac{1}{\prod_{k \in I} \lambda_k}.$$

By the second theorem of Minkowski (Theorem 1.1) we can estimate each summand on the left and get

$$g_i(Z) = \frac{1}{2^i} \sum_{J \subseteq [n], \#J=i} \frac{\text{vol}_i(DP_J)}{\det \Lambda_J} \leq \sum_{J \subseteq [n], \#J=i} \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)}.$$

Hence it suffices to show

$$(2.8) \quad \sum_{J \subseteq [n], \#J=i} \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)} \leq \sum_{I \subseteq [n], \#I=i} \frac{1}{\prod_{k \in I} \lambda_k}.$$

Now, let  $a_1, \dots, a_n \in \mathbb{Z}^n$  be linearly independent with  $a_j \in \lambda_j DZ$ ,  $1 \leq j \leq n$ . Furthermore  $v_1, \dots, v_n \in \mathbb{Z}^n$  are linearly independent as well. Thus by Lemma 2.1 there is a bijection  $\phi : \binom{[n]}{i} \rightarrow \binom{[n]}{n-i}$  such that for all  $J \in \binom{[n]}{i}$

$$\text{lin}\{v_j : j \in J\} \cap \text{lin}\{a_k : k \in \phi(J)\} = \{0\}.$$

Thus together with Lemma 2.2 we get

$$\prod_{j=1}^i \lambda_j(DZ \cap \text{lin}\{v_l : l \in J\}, \mathbb{Z}^n \cap \text{lin}\{v_l : l \in J\}) \geq \prod_{k \notin \phi(J)} \lambda_k,$$

and on account of  $\lambda_j(DP_J, \Lambda_J) \geq \lambda_j(DZ \cap \text{lin}\{v_l : l \in J\}, \mathbb{Z}^n \cap \text{lin}\{v_l : l \in J\})$  we obtain

$$(2.9) \quad \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)} \leq \frac{1}{\prod_{k \notin \phi(J)} \lambda_k}.$$

Since  $\phi$  is a bijection we get (2.8).  $\square$

For the proof of Theorem 1.3 we need the following counterpart to Minkowski's Theorem 1.1 (e.g. see [15, Thm. 1.2])

$$(2.10) \quad \frac{2^n}{n!} \det(\Lambda) \leq \lambda_1(K, \Lambda) \cdot \dots \cdot \lambda_n(K, \Lambda) \text{vol}(K),$$

where  $K \in \mathcal{K}_0^n$  and  $\Lambda \in \mathcal{L}^n$ .



*Proof of Theorem 1.3.* Let  $Z$  be generated by  $v_1, \dots, v_m \in \mathbb{Z}^n$  and let  $\dim Z = n$ . For short we write  $\lambda_i$  instead of  $\lambda_i(DZ)$  and for  $I \subseteq [m]$ ,  $\#I = i$ , let  $P_I = \left\{ \sum_{j \in I} \mu_j v_j : 0 \leq \mu_j \leq 1 \right\}$ ,  $L_I = \text{lin}\{v_j : j \in I\}$  and  $L_I^\perp$  be its orthogonal complement. The orthogonal projection of a set  $S \subseteq \mathbb{R}^n$  onto a linear subspace  $L$  is denoted by  $S|L$ .

For  $J \subseteq [m]$ ,  $\#J = n$ , and  $i \in [n]$ , let  $I \subseteq J$  with  $\#I = i$ . Then

$$\text{vol}(P_J) = \text{vol}_i(P_I) \text{vol}_{n-i}(P_J|L_I^\perp),$$

which, e.g., can easily be seen by Gram-Schmidt orthogonalization. Hence, by Proposition 2.1 or (2.5) we can write

$$\begin{aligned} \text{vol}(Z) &= \sum_{J \subseteq [m], \#J=n} \text{vol}(P_J) \\ &= \sum_{J \subseteq [m], \#J=n} \frac{1}{\binom{n}{i}} \sum_{I \subseteq J, \#I=i} \text{vol}_i(P_I) \text{vol}_{n-i}(P_J|L_I^\perp) \\ &= \frac{1}{\binom{n}{i}} \sum_{I \subseteq [m], \#I=i} \text{vol}_i(P_I) \sum_{I \subseteq J \subseteq [m], \#J=n} \text{vol}_{n-i}(P_J|L_I^\perp). \end{aligned}$$

Furthermore, for  $I \subseteq [m]$  with  $\#I = i$ , we have

$$\sum_{I \subseteq J \subseteq [m], \#J=n} \text{vol}_{n-i}(P_J|L_I^\perp) = \text{vol}_{n-i}(Z|L_I^\perp),$$

because the sum on the left hand side covers all volumes of  $(n-i)$ -dimensional parallelepipeds that are spanned by generators of  $Z|L_I^\perp$  (cf. (2.5)). This implies

$$\begin{aligned} \text{vol}(Z) &= \frac{1}{\binom{n}{i}} \sum_{I \subseteq [m], \#I=i} \text{vol}_i(P_I) \text{vol}_{n-i}(Z|L_I^\perp) \\ &= \frac{1}{\binom{n}{i}} \sum_{I \subseteq [m], \#I=i} \frac{\text{vol}_i(P_I)}{\det(\mathbb{Z}^n \cap L_I)} \frac{\text{vol}_{n-i}(Z|L_I^\perp)}{\det(\mathbb{Z}^n|L_I^\perp)}, \end{aligned}$$

where for the last step we refer to [19, Corollary 1.3.5]. Together with the identity  $\text{vol}_{n-i}(Z|L_I^\perp) = \frac{1}{2^{n-i}} \text{vol}_{n-i}(DZ|L_I^\perp)$  and (2.10) we get

$$\text{vol}(Z) \geq \frac{1}{\binom{n}{i}} \sum_{I \subseteq [m], \#I=i} \frac{\text{vol}_i(P_I)}{\det(\mathbb{Z}^n \cap L_I)} \left( \frac{1}{(n-i)!} \prod_{j=1}^{n-i} \frac{1}{\lambda_j(DZ|L_I^\perp, \mathbb{Z}^n|L_I^\perp)} \right).$$

Since  $\lambda_j(DZ|L_I^\perp, \mathbb{Z}^n|L_I^\perp) \leq \lambda_{i+j}(DZ)$ , for  $j = 1, \dots, n-i$ , we obtain

$$\text{vol}(Z) \geq \frac{i!}{n!} \sum_{I \subseteq [m], \#I=i} \frac{\text{vol}_i(P_I)}{\det(\mathbb{Z}^n \cap L_I)} \prod_{j=i+1}^n \frac{1}{\lambda_j}.$$

With Proposition 2.1 we finally obtain

$$(2.11) \quad \text{vol}(Z) \geq \frac{i!}{n!} g_i(Z) \prod_{j=i+1}^n \frac{1}{\lambda_j},$$

as desired. The second part of the theorem can now be derived with the help of  $\text{vol}(DZ) = 2^n \text{vol}(Z)$  and Theorem 1.1.  $\square$

We remark that Henk, Linke and Wills [13, Cor. 1.1] improved the bound (2.10) for the class of zonotopes by, roughly speaking, a factor of order  $(\sqrt{n})^{n+1}$ , which leads to the better inequalities

$$g_i(Z) \leq \binom{n}{i} (n-i)^{\frac{n-i}{2}} \sigma_i(Z), \text{ for } 1 \leq i \leq n.$$

The remaining part of this section will be devoted to some partial results concerning the coefficient-wise approach to Conjecture 1.1 in the case when one imposes additional assumptions on the generators of a lattice zonotope.

The first one is an extension of Theorem 1.2 and depending on the number of generators it improves upon Theorem 1.3.

**Theorem 2.1.** *Let  $\{v_1, \dots, v_m\} \subset \mathbb{Z}^n$  be in general position, i.e., every  $n$  of them are linearly independent, and let  $Z \in \mathcal{P}^n$  be the zonotope generated by these vectors. Then, for  $1 \leq i \leq n$ ,*

$$g_i(Z) \leq \frac{\binom{m}{i}}{\binom{n}{i}} \sigma_i(Z).$$

*Proof.* We follow the outline of the proof of Theorem 1.2 and also use its notation. Based on Proposition 2.1 and Minkowski's second theorem (Theorem 1.1) here it suffices to show (cf. (2.8))

$$(2.12) \quad \sum_{J \subseteq [m], \#J=i} \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)} \leq \frac{\binom{m}{i}}{\binom{n}{i}} \sum_{I \subseteq [n], \#I=i} \frac{1}{\prod_{k \in I} \lambda_k}.$$

Now, since every set  $J \subseteq [m]$  with  $\#J = i$  is contained in  $\binom{m-i}{n-i}$  sets  $I \subseteq [m]$  of size  $\#I = n$ , we can replace the left hand side by

$$\frac{1}{\binom{m-i}{n-i}} \sum_{I \subseteq [m], \#I=n} \sum_{J \subseteq I, \#J=i} \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)}.$$

and (2.12) becomes

$$(2.13) \quad \sum_{I \subseteq [m], \#I=n} \sum_{J \subseteq I, \#J=i} \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)} \leq \binom{m}{n} \sum_{I \subseteq [n], \#I=i} \frac{1}{\prod_{k \in I} \lambda_k}.$$

Now let  $a_1, \dots, a_n \in \mathbb{Z}^n$  be linearly independent with  $a_j \in \lambda_j DZ$ ,  $1 \leq j \leq n$ . By our assumption, any choice of  $n$  generators  $v_{i_1}, \dots, v_{i_n} \in \mathbb{Z}^n$  is linearly independent and so we may apply Lemma 2.1 to any  $n$ -subset

$I = \{i_1, \dots, i_n\} \subseteq [m]$ . Hence, as in the proof of Theorem 1.2 we find that there is a bijection  $\phi : \binom{I}{i} \rightarrow \binom{[n]}{n-i}$  such that for all  $J \in \binom{I}{i}$  (cf. (2.9))

$$\frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)} \leq \frac{1}{\prod_{k \notin \phi(J)} \lambda_k}.$$

Since  $\phi$  is a bijection we get

$$\sum_{J \subseteq I, \#J=i} \frac{1}{\prod_{j=1}^i \lambda_j(DP_J, \Lambda_J)} \leq \sum_{T \subseteq [n], \#T=i} \frac{1}{\prod_{t \in T} \lambda_t},$$

which implies (2.13).  $\square$

As an immediate consequence of Theorem 1.1 one can prove (1.3) for  $i = 1$  and lattice zonotopes with primitive generators in general position. Here a non-trivial lattice vector  $z \in \mathbb{Z}^n$  is said to be *primitive*, if the greatest common divisor of its entries equals one.

**Corollary 2.1.** *Let  $\{v_1, \dots, v_m\} \subset \mathbb{Z}^n$  be primitive vectors in general position, and let  $Z \in \mathcal{P}^n$  be the zonotope generated by these vectors. Then*

$$g_1(Z) = m \leq \sum_{i=1}^n \frac{1}{\lambda_i(DZ)} = \sigma_1(Z).$$

*Proof.* First, by (2.2) it holds  $g_1(Z) = \sum_{i=1}^m \gcd(v_i)$ , which equals  $m$  because the  $v_i$  are chosen to be primitive. Moreover, the generators are in general position and any parallelepiped with integer vertices has volume at least one, which yields – using also Proposition 2.1 – that  $\text{vol}(Z) = g_n(Z) \geq \binom{m}{n}$  and together with  $\text{vol}(Z) = \frac{1}{2^n} \text{vol}(DZ)$  and Theorem 1.1 we conclude that

$$\begin{aligned} 2^n &\geq \lambda_1(DZ) \cdot \dots \cdot \lambda_n(DZ) \text{vol}(DZ) \\ &= 2^n \lambda_1(DZ) \cdot \dots \cdot \lambda_n(DZ) \text{vol}(Z) \geq 2^n \lambda_1(DZ) \cdot \dots \cdot \lambda_n(DZ) \binom{m}{n}. \end{aligned}$$

Thus,

$$\frac{1}{\lambda_1(DZ)} \cdot \dots \cdot \frac{1}{\lambda_n(DZ)} \geq \binom{m}{n}$$

and the inequality of the arithmetic and geometric mean finally yields

$$\frac{1}{\lambda_1(DZ)} + \dots + \frac{1}{\lambda_n(DZ)} \geq n \left( \frac{1}{\lambda_1(DZ) \cdot \dots \cdot \lambda_n(DZ)} \right)^{1/n} \geq n \binom{m}{n}^{1/n} \geq m.$$

$\square$

In the context of  $g_1(Z)$  it might be also of interest to have a look at the so called *Davenport constant*  $s(G)$  of a finite Abelian group  $G$ : it is the minimal  $d$  such that every sequence of  $d$  elements of  $G$  contains a nonempty subsequence with zero-sum. For a survey on this and related zero-sum problems see [9] and the references therein. It is conjectured that

$$s(\mathbb{Z}_k^n) = n(k-1) + 1,$$

where  $\mathbb{Z}_k^n$  is the  $n$ -fold product of the cyclic group  $\mathbb{Z}_k$  of order  $k$ . The conjecture is known to be true if  $k$  is a prime power (cf. [21]), and so we get, for instance,

**Proposition 2.2.** *Let  $k \in \mathbb{N}$  be a prime power, and let  $m \in \mathbb{N}$  such that  $n(k-1) + 1 \leq m \leq kn$ . Let  $Z \in \mathcal{P}^n$  be a zonotope generated by  $m$  primitive lattice vectors. Then*

$$g_1(Z) \leq n \frac{1}{\lambda_1(DZ)}.$$

*Proof.* As in the proof of Corollary 2.1 we have  $g_1(Z) = m$  and so we have to show that  $\lambda_1(DZ) \leq \frac{n}{m}$ . Let  $H = \{x \in \mathbb{R}^n : a^\top x = 0\}$  be a hyperplane such that the half-space  $\{x \in \mathbb{R}^n : a^\top x > 0\}$  contains, without loss of generality, all the vectors  $v_1, \dots, v_m$  (if not replace  $v_i$  by  $-v_i$ , which does not change  $DZ$ ). This implies, that any sum of the generators is non-zero. Since  $s(\mathbb{Z}_k^n) = n(k-1) + 1 \leq m$ , there exists a subset  $v_{i_1}, \dots, v_{i_l}$  of the generators whose sum is divisible by  $k$  and so  $\lambda_1(DZ) \leq \frac{1}{k} \leq \frac{n}{m}$  as desired.  $\square$

### 3. LATTICE-FACE POLYTOPES

In this section, we study Conjecture 1.1 on the class of lattice-face polytopes which were already defined in the introduction (see Definition 1.1). First of all, we state some properties of these polytopes being relevant for our further discussion. Recall that  $\pi^{(n-i)}$  denotes the projection that forgets the last  $n-i$  coordinates,  $i = 1, \dots, n$ . For sake of brevity we write  $\pi = \pi^{(1)}$ .

**Lemma 3.1** (cf. [16]). *Let  $P \in \mathcal{P}^n$  be a lattice-face polytope. Then,*

- i)  $\pi(P) \in \mathcal{P}^{n-1}$  is a lattice-face polytope.
- ii)  $mP$  is a lattice-face polytope, for any integer  $m$ .
- iii) Let  $H$  be an  $(n-1)$ -dimensional affine space spanned by some subset of  $\text{vert}(P)$ . Then, for any lattice point  $y \in \mathbb{Z}^{n-1}$ , the preimage  $\pi^{-1}(y) \cap H$  is also a lattice point.
- iv)  $P$  is a lattice polytope.

As Liu [16, Thm. 1.1] showed, the coefficients of the Ehrhart polynomial of lattice-face polytopes have a nice geometric meaning.

**Theorem 3.1** (Liu, 2009). *Let  $P \in \mathcal{P}^n$  be a lattice-face polytope. Then*

$$G(P, k) = \sum_{i=0}^n \text{vol}_i(\pi^{(n-i)}(P)) k^i,$$

where  $\text{vol}_0(\pi^{(n)}(P)) := 1$ .

This will be our starting point to prove Theorem 1.4. But first, we need an auxiliary lemma that relates the successive minima of lattice-face polytopes to those of their projections.

**Lemma 3.2.** *Let  $P \in \mathcal{P}^n$  be a lattice-face polytope.*

i) If  $P$  is 0-symmetric, then, for  $1 \leq j \leq i \leq n$ ,

$$\lambda_j(\pi^{(n-i)}(P), \mathbb{Z}^i) \geq \lambda_j(P).$$

ii) If  $0 \in \text{vert}(P)$  and  $SP = \text{conv}(P, -P)$ , then, for  $1 \leq j \leq i \leq n$ ,

$$\lambda_j(\pi^{(n-i)}(SP), \mathbb{Z}^i) \geq \lambda_j(SP).$$

*Proof.* i): It suffices to show that  $\lambda_j := \lambda_j(\pi(P), \mathbb{Z}^{n-1}) \geq \lambda_j(P)$ , for all  $j = 1, \dots, n-1$ . To this end, let  $\{z_1, \dots, z_j\} \subset \mathbb{Z}^{n-1}$  be linearly independent lattice points in  $\lambda_j \pi(P)$ . Our first observation is that any set of vectors  $\{\bar{z}_1, \dots, \bar{z}_j\} \subset \mathbb{R}^n$  with  $z_i = \pi(\bar{z}_i), i = 1, \dots, j$ , is also linearly independent, because any linear dependence would be preserved by the projection  $\pi$ . Therefore, we need to show that, for all  $i = 1, \dots, j$ , there is always a lattice point  $\bar{z}_i \in \lambda_j P$  such that  $z_i = \pi(\bar{z}_i)$ .

In order to see this, we fix an  $i$  and set  $z = z_i$  and  $\mu = \lambda_i > 0$ . In particular, we have  $z \in \mu\pi(P) \cap \mathbb{Z}^{n-1}$ . Since,  $0 \in \mu\pi(P)$ , there are linearly independent  $v_1, \dots, v_{n-1} \in \text{vert}(\pi(P))$  and  $\gamma_1, \dots, \gamma_{n-1} \in [0, 1]$  with  $\sum_{i=1}^{n-1} \gamma_i \leq 1$ , such that  $z = \mu \sum_{i=1}^{n-1} \gamma_i v_i$ . For any  $v_i$  there is a vertex  $\bar{v}_i$  of  $P$  in the preimage of  $v_i$  under  $\pi$ , and these  $\bar{v}_1, \dots, \bar{v}_{n-1}$  are linearly independent. This means, that the hyperplane  $H = \text{aff}\{0, \bar{v}_1, \dots, \bar{v}_{n-1}\} = \text{aff}\{\pm \bar{v}_1, \dots, \pm \bar{v}_{n-1}\}$  is  $(n-1)$ -dimensional and spanned by vertices of  $P$ , because  $P = -P$ . Therefore, since  $P$  is a lattice-face polytope we have by Lemma 3.1 iii) that the point  $\bar{z} = \pi^{-1}(z) \cap H$  has integral coordinates. It remains to show that  $\bar{z}$  lies in  $\mu P$ . The containment of  $\bar{z}$  in  $H$  gives us  $\beta_1, \dots, \beta_{n-1} \in \mathbb{R}$  such that  $\bar{z} = \sum_{i=1}^{n-1} \beta_i \bar{v}_i$ . Furthermore, it is

$$\mu \sum_{i=1}^{n-1} \gamma_i v_i = z = \pi(\bar{z}) = \sum_{i=1}^{n-1} \beta_i \pi(\bar{v}_i) = \sum_{i=1}^{n-1} \beta_i v_i,$$

which yields  $\beta_i = \mu\gamma_i$ , for all  $i = 1, \dots, n-1$ , because the  $v_i$ 's were chosen to be linearly independent. So, with  $\sum_{i=1}^{n-1} \gamma_i \leq 1$ , we get  $\bar{z} = \mu \sum_{i=1}^{n-1} \gamma_i \bar{v}_i \in \mu P$  as claimed.

In conclusion, we found the point  $\bar{z} \in \mu P \cap \mathbb{Z}^n$  for which  $z = \pi(\bar{z})$  and we are done.

The proof of ii) follows the same lines as above. We only note, that  $\text{vert}(SP) \subseteq \{\pm v : v \in \text{vert}(P)\}$  and the assumption  $0 \in \text{vert}(P)$  is used to simultaneously control the signs of the vertices which span  $H$ .  $\square$

**Remark 3.1.** *The above lemma does not hold for general polytopes. For example, consider  $P_t = \text{conv}\{\pm \binom{t-1}{1}, \pm \binom{t}{1}\}$ ,  $t \in \mathbb{N}$ . We have  $\lambda_1(P_t, \mathbb{Z}^2) = 1$  and  $\lambda_1(P_t|e_2^\perp, \mathbb{Z}) = \frac{1}{t}$ . Therefore, there does not even exist a constant depending on the dimension such that the successive minima of the projection could be bounded from below, up to this constant, by those of the original polytope.*

*Proof of Theorem 1.4.* i): By Theorems 3.1 and 1.1 we obtain, for all  $i = 1, \dots, n$ ,

$$g_i(P) = \text{vol}_i(\pi^{(n-i)}(P)) \leq \prod_{j=1}^i \frac{2}{\lambda_j(\pi^{(n-i)}(P), \mathbb{Z}^i)}.$$

Using Lemma 3.2 i), we continue this inequality to get

$$g_i(P) \leq \prod_{j=1}^i \frac{2}{\lambda_j(P)} \leq \sigma_i(P).$$

Note, that for  $i \neq n$  the last inequality sign is actually a strict one.

ii): By definition it is  $P \subset SP$  and so  $\text{vol}_i(\pi^{(n-i)}(P)) \leq \text{vol}_i(\pi^{(n-i)}(SP))$ . Thus, using Lemma 3.2 ii) we can argue in the same way as in the first part.  $\square$

#### 4. PROOF OF PROPOSITION 1.1

Recall  $Q_l^n = \text{conv}\{lC_{n-1} \times \{0\}, \pm e_n\}$  as the polytope under consideration. By cutting  $kQ_l^n$  into lattice slices orthogonal to  $e_n$ , we find that the Ehrhart polynomial of  $Q_l^n$  is given by

$$\begin{aligned} G(kQ_l^n) &= (2kl + 1)^{n-1} + 2 \sum_{j=0}^{k-1} (2jl + 1)^{n-1} \\ &= (2kl + 1)^{n-1} + 2 \sum_{j=0}^{k-1} \sum_{i=0}^{n-1} \binom{n-1}{i} (2jl)^i \\ &= \sum_{i=0}^{n-1} \binom{n-1}{i} (2l)^i k^i + 2 \sum_{i=0}^{n-1} \binom{n-1}{i} (2l)^i \left( \sum_{j=0}^{k-1} j^i \right). \end{aligned}$$

Faulhaber's formula (see [7, p. 106]) expresses the sum  $\sum_{j=0}^{k-1} j^i$  as a polynomial in  $k$ . Plugging this into the above identity and collecting for powers of  $k$  yields

$$g_i(Q_l^n) = 2(2l)^{i-1} \left( \binom{n-1}{i} l + \sum_{j=i-1}^{n-1} P(i, j) \binom{n-1}{j} (2l)^{j-i+1} \right),$$

where  $P(i, j) = \sum_{t=i}^{j+1} \frac{(-1)^{t-i} \binom{j+1}{t} \binom{t}{i}}{j+1} B_{j+1-t}$  and  $B_m$  are the Bernoulli numbers, with  $B_1 = \frac{1}{2}$  (see [7, p. 107]). Therefore, via  $P(n, n-1) = \frac{1}{n}$ ,  $P(n, n) = -\frac{1}{2}$ ,  $P(n-1, n) = \frac{n}{12}$  and  $P(n-2, n) = 0$ , we obtain

$$\begin{aligned} g_{n-2}(Q_l^n) &= (n-1)(2l)^{n-3} \left( \frac{2}{3} l^2 + 1 \right) \quad \text{and} \\ g_{n-3}(Q_l^n) &= \frac{2}{3} \binom{n-1}{2} (2l)^{n-4} (2l^2 + 1). \end{aligned}$$

The successive minima are  $\lambda_1(Q_l^n) = \dots = \lambda_{n-1}(Q_l^n) = \frac{1}{l}$  and  $\lambda_n(Q_l^n) = 1$ , from which we get

$$\sigma_i(Q_l^n) = \binom{n-1}{i} (2l)^i + 2 \binom{n-1}{i-1} (2l)^{i-1}, \text{ for } 1 \leq i \leq n-1.$$

Seen as polynomials in  $l$ , the  $\sigma_i(Q_l^n)$  have degree  $i$ , whereas  $g_{n-2}(Q_l^n)$  and  $g_{n-3}(Q_l^n)$  have degree  $n-1$  and  $n-2$ , respectively. Thus, for  $i \in \{n-2, n-3\}$  and any fixed constant  $c$ , there exists an  $l \in \mathbb{N}$  such that  $g_i(Q_l^n) > c \sigma_i(Q_l^n)$ .

Note, that Conjecture 1.1 nevertheless holds for all the polytopes  $Q_l^n$ . As a final remark, we consider the special case  $n = 3$ . Here, we get

$$G(kQ_l^3) = \frac{8}{3}l^2k^3 + 4lk^2 + \left(\frac{4}{3}l^2 + 2\right)k + 1,$$

i.e., all Ehrhart coefficients of  $Q_l^3$  are positive, and

$$L(kQ_l^3) = 8l^2k^3 + (4l^2 + 8l)k^2 + (4l + 2)k + 1.$$

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