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Note

An upper bound on the sum of squares of degrees in a hypergraph

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Abstract

We give an upper bound on the sum of squares of ℓ-degrees in a k-uniform hypergraph in terms of ℓ, k and the number of vertices and edges of the hypergraph, where a ℓ-degree is the number of edges of the hypergraph containing a fixed ℓ-element subset of the vertices. For ordinary graphs this bound coincides with one given by de Caen. We show that our bound implies the quadratic LYM-inequality for 2-level antichains of subsets of a finite set.

Keywords: Hypergraph; Degree sequence

1. Introduction

Let $G=(V,\mathscr{E})$ be a simple graph with $n=|V|$ vertices. In [3], de Caen proves the following inequality:

$$\sum_{i\in V} d(i)^2 \leqslant \frac{2}{n-1}\,|\mathscr{E}|^2+(n-2)|\mathscr{E}|.$$

Here $d(i)$ denotes the degree of the vertex i.

It is the aim of this note to generalize de Caen's inequality to hypergraphs. Let $(V,\mathscr{E})$ be a simple k-uniform hypergraph, i.e. $\mathscr{E}$ is a subset of $\binom{V}{k}$, the set of all k-element subsets of V. For $I\subseteq V$ let $d(I)=|\{E\in\mathscr{E} : I\subseteq E\}|$ be the degree of I. We have $d(\emptyset)=|\mathscr{E}|$ and $d(E)=1$ or 0 for every k-element set $E\subseteq V$, according to $E\in\mathscr{E}$ or $E\notin\mathscr{E}$.

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We denote the complete k-uniform hypergraph on n vertices by $K_n^{(k)}$, and an "s-star" (i.e. $\mathscr{E}=$ all edges which contain s fixed vertices) on n vertices by $K_{s,n-s}^{(k)}$. The complement of the k-uniform hypergraph $(V,\mathscr{E})$ is the hypergraph $(V,\binom{V}{k}\setminus\mathscr{E})$.

Theorem 1. *Let $G=(V,\mathscr{E})$ be a k-uniform hypergraph, $|V|=n$, and ℓ be an integer satisfying $0\leqslant\ell\leqslant k$. Then*

$$\sum_{I\in\binom{V}{\ell}} d(I)^2 \leqslant \frac{\binom{k}{\ell}\binom{k-1}{\ell}}{\binom{n-1}{\ell}}|\mathscr{E}|^2 + \binom{k-1}{\ell-1}\binom{n-\ell-1}{k-\ell}|\mathscr{E}|.$$

Equality holds for $0<\ell<k<n$ if and only if G is one of the hypergraphs $K_n^{(k)}$, $K_{1,n-1}^{(k)}$, $K_{i,k+1-i}^{(k)}$ for $i=2,\ldots,\lfloor(k+1)/2\rfloor$, or one of the complements of these hypergraphs.

Note that for $k=2$ (and $\ell=1$) this inequality agrees with de Caen's one.

Theorem 1 is proved in Section 2. In Section 3 we show that it implies the quadratic LYM-inequality [2].

Let us give an interpretation of the above inequality in terms of average degrees. A double counting argument gives

$$\sum_{I\in\binom{V}{\ell}} d(I)^2 = \sum_{E\in\mathscr{E}}\sum_{I\in\binom{E}{\ell}} d(I).$$

With this, it is easy to see that the inequality in Theorem 1 is equivalent to

$$\frac{1}{|\mathscr{E}|}\sum_{E\in\mathscr{E}}\frac{1}{\binom{k}{\ell}}\sum_{I\in\binom{E}{\ell}} d(I) \leqslant \frac{n(k-\ell)}{k(n-\ell)}\tilde{d} + \frac{\ell}{k}\binom{n-\ell-1}{k-\ell},$$

where

$$\tilde{d} = \frac{1}{\binom{n}{\ell}}\sum_{I\in\binom{V}{\ell}} d(I) = \frac{\binom{k}{\ell}}{\binom{n}{\ell}}|\mathscr{E}|$$

is the average degree of ℓ-element subsets of V. Note that the Cauchy–Schwarz inequality gives the lower estimation

$$\frac{1}{|\mathscr{E}|}\sum_{E\in\mathscr{E}}\frac{1}{\binom{k}{\ell}}\sum_{I\in\binom{E}{\ell}} d(I) \geqslant \tilde{d}.$$

2. Proof of Theorem 1

De Caen proved his inequality by considering a nonnegative definite matrix in the Bose–Mesner algebra of the triangular association scheme $T(n)$. We do the same in the Johnson scheme $J(n,k)$.

Let A_j be the $\binom{n}{k}\times\binom{n}{k}$ matrix with rows and columns indexed by the k-element subsets of V, whose entry in row R and column C is 1 if $|R\cap C| = k-j$ and 0 otherwise. It is known that for $2k\leqslant n$ resp. $2k\geqslant n$ the linear space $\mathbb{R}^{\binom{n}{k}}$ is a direct sum of $k+1$ resp. $n-k+1$ pairwise orthogonal subspaces each of which is an eigenspace of all matrices $A_j,\ j\geqslant 0$. The eigenvalues of the matrix A_j on these eigenspaces are given by

$$P_j(i)=\sum_{s=0}^{j}(-1)^{j-s}\binom{k-s}{j-s}\binom{k-i}{s}\binom{n-k+s-i}{s},$$

where the eigenspaces are indexed by $i=0,\dots,k$ resp. $i=0,\dots,n-k$ (cf. [4, p. 48]). The eigenspace for $i=0$ is generated by the all-ones vector $(1,\dots,1)$.

We will use the following identity for the eigenvalues, which follows from the above formula by an easy inversion: For each $\ell=0,\dots,k$,

$$\sum_{j=0}^{k-\ell}\binom{k-j}{\ell}P_j(i)=\binom{k-i}{k-\ell}\binom{n-\ell-i}{k-\ell}. \tag{1}$$

Now consider the matrix

$$M=\frac{\binom{k}{\ell}\binom{k-1}{\ell}}{\binom{n-1}{\ell}}J+\binom{k-1}{\ell-1}\binom{n-\ell-1}{k-\ell}I-\sum_{j=0}^{k-\ell}\binom{k-j}{\ell}A_j,$$

where J resp. I is the $\binom{n}{k}\times\binom{n}{k}$ all-ones resp. identity matrix. We show that the matrix M is nonnegative definite by examining its eigenvalues. The all-ones vector is an eigenvector of M with eigenvalue

$$\frac{\binom{k}{\ell}\binom{n-\ell-1}{k-\ell-1}}{\binom{n-1}{k-1}}\binom{n}{k}+\binom{k-1}{\ell-1}\binom{n-\ell-1}{k-\ell}-\sum_{j=0}^{k-\ell}\binom{k-j}{\ell}P_j(0),$$

which, using (1) with $i=0$, is equal to

$$\binom{k}{\ell}\binom{n-\ell}{k-\ell}\left(\frac{n(k-\ell)}{k(n-\ell)}+\frac{\ell(n-k)}{k(n-\ell)}-1\right)=0.$$

Since all other common eigenvectors of the matrices A_j, $j \geqslant 0$, are orthogonal to the all-ones vector, the remaining eigenvalues of M are given by (again using (1))

$$\binom{k-j}{\ell-1}\binom{n-\ell-1}{k-\ell} - \sum_{j=0}^{k-\ell}\binom{k-j}{\ell}P_j(i)$$
$$= \binom{k-1}{\ell-1}\binom{n-\ell-1}{k-\ell} - \binom{k-i}{k-\ell}\binom{n-\ell-i}{k-\ell},$$

for $i = 1,\dots,k$ resp. $i = 1,\dots,n-k$. They are obviously nonnegative. Hence M is nonnegative definite.

Let δ denote the restriction of the degree function d to all k-element subsets of V. Obviously, δ can be considered as the characteristic (column-) vector of the edge set $\mathscr{E}$. Since M is nonnegative definite, we have

$$\sum_{j=0}^{k-\ell}\binom{k-j}{\ell}\delta^\top A_j\delta \leqslant \frac{\binom{k}{\ell}\binom{k-1}{\ell}}{\binom{n-1}{\ell}}\delta^\top J\delta + \binom{k-1}{\ell-1}\binom{n-\ell-1}{k-\ell}\delta^\top\delta.$$

This inequality proves the one in the theorem. Indeed, the LHS counts the number of triples $(I, E_1, E_2) \in \binom{V}{\ell} \times \mathscr{E} \times \mathscr{E}$ with $I \subseteq E_1 \cap E_2$, which is $\sum_{I\in\binom{V}{\ell}} d(I)^2$. For the RHS, note that $\delta^\top\delta = |\mathscr{E}|$ and $\delta^\top J\delta = |\mathscr{E}|^2$.

We note that for $0 < \ell < k < n$ the hypergraph $(V, \mathscr{E})$ attains equality if and only if its characteristic vector lies in the sum of the first two eigenspaces (corresponding to $i = 0$ and 1). It is known that this sum is the span of the characteristic vectors of all stars $K_{1,n-1}^{(k)}$. Using this, it is easy to see that the only hypergraphs whose characteristic vectors belong to this space are the ones given in the theorem. We omit the details.

3. An application

Let $[n] = \{1,\dots,n\}$. Consider $\mathscr{A} \subseteq \binom{[n]}{\ell}$, $\mathscr{B} \subseteq \binom{[n]}{k}$ with $\ell < k$ such that $A \not\subseteq B$ for all $A \in \mathscr{A}$, $B \in \mathscr{B}$. Thus, $\mathscr{A} \cup \mathscr{B}$ is an antichain of subsets of $[n]$, each subset consisting of ℓ or k elements. An easy double counting argument gives

$$\frac{|\mathscr{A}|}{\binom{n}{\ell}} + \frac{|\mathscr{B}|}{\binom{n}{k}} \leqslant 1,$$

the so called LYM-inequality for the antichain $\mathscr{A} \cup \mathscr{B}$ (cf. [1, p. 3]). We show that Theorem 1 implies a sharpening of this inequality.

Theorem 2. *Let $\mathscr{A} \cup \mathscr{B}$ be an antichain, $\mathscr{A} \subseteq \binom{[n]}{\ell}$, $\mathscr{B} \subseteq \binom{[n]}{k}$, $0 < \ell < k < n$. Then*

$$\frac{|\mathscr{A}|}{\binom{n}{\ell}} + \frac{|\mathscr{B}|}{\binom{n}{k}} + \frac{n(k-\ell)}{\ell(n-k)}\frac{|\mathscr{A}|}{\binom{n}{\ell}}\frac{|\mathscr{B}|}{\binom{n}{k}} \leqslant 1.$$

Equality holds if and only if $\mathscr{A}\cup\mathscr{B}$ *is one of the families* $\binom{[n]}{\ell}$, $\binom{[n]}{k}$, *or, up to permutations*, $\{A\in\binom{[n]}{\ell} : n\in A\}\cup\binom{[n-1]}{k}$.

A corresponding inequality for antichains with more than two cardinalities of their sets is proved in [2].

Proof. Consider the hypergraph $(V,\mathscr{E})=([n],\mathscr{B})$. Let us give a lower estimate for the sum $\sum_{I\in\binom{[n]}{\ell}} d(I)^2$. Since $\mathscr{A}\cup\mathscr{B}$ is an antichain, we have $d(I)=0$ for all $I\in\mathscr{A}$. Hence, there are at most $\binom{n}{\ell}-|\mathscr{A}|$ nonzero terms in this sum. Applying the Cauchy–Schwarz inequality therefore yields

$$\left(\binom{n}{\ell}-|\mathscr{A}|\right)\sum_{I\in\binom{[n]}{\ell}} d(I)^2 \geqslant \left(\sum_{I\in\binom{[n]}{\ell}} d(I)\right)^2 = \binom{k}{\ell}^2|\mathscr{B}|^2.$$

Combining this inequality with the one in Theorem 1 gives

$$\left(\binom{n}{\ell}-|\mathscr{A}|\right)\left(\frac{\binom{k}{\ell}\binom{n-\ell-1}{k-\ell-1}}{\binom{n-1}{k-1}}|\mathscr{B}|+\binom{k-1}{\ell-1}\binom{n-\ell-1}{k-\ell}\right)$$

$$\geqslant \binom{k}{\ell}^2|\mathscr{B}|,$$

from which the inequality of the theorem follows by elementary manipulations. The case of equality is easily dealt with. □

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