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An intersection theorem for weighted sets

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Abstract

A weight function $\omega : 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ from the set of all subsets of $[n]=\{1,\ldots,n\}$ to the nonnegative real numbers is called shift-monotone in $\{m+1,\ldots,n\}$ if $\omega(\{a_1,\ldots,a_j\})\geqslant\omega(\{b_1,\ldots,b_j\})$ holds for all $\{a_1,\ldots,a_j\}$, $\{b_1,\ldots,b_j\}\subseteq[n]$ with $a_i\leqslant b_i$, $i=1,\ldots,j$, and if $\omega(A)\geqslant\omega(B)$ holds for all $A,B\subseteq[n]$ with $A\subseteq B$ and $B\setminus A\subseteq\{m+1,\ldots,n\}$. A family $\mathscr{F}\subseteq 2^{[n]}$ is called intersecting in $[m]$ if $F\cap G\cap[m]\neq\emptyset$ for all $F,G\in\mathscr{F}$. Let $\omega(\mathscr{F})=\sum_{F\in\mathscr{F}}\omega(F)$. We show that $\max\{\omega(\mathscr{F})\colon \mathscr{F}\subseteq 2^{[n]},\ \mathscr{F}$ is intersecting in $[n]\} = \max\{\omega(\mathscr{F})\colon \mathscr{F}\subseteq 2^{[n]}, \mathscr{F}$ is intersecting in $[m]\}$ provided that ω is shift-monotone in $\{m+1,\ldots,n\}$. An application to the poset of colored subsets of a finite set is given. © 2001 Elsevier Science B.V. All rights reserved.

1. Introduction

Let $[n]=\{1,2,\ldots,n\}$, $2^{[n]}=\{A\subseteq[n]\}$ and consider a weight function $\omega : 2^{[n]}\to\mathbb{R}_{\geqslant 0}$, where $\mathbb{R}_{\geqslant 0}$ denotes the set of all nonnegative real numbers. Define the weight of a family $\mathscr{F}\subseteq 2^{[n]}$ as the sum of weights of its members,

$$\omega(\mathscr{F})=\sum_{A\in\mathscr{F}}\omega(A).$$

A subset $\mathscr{F}\subseteq 2^{[n]}$ is called an intersecting family if no two sets in $\mathscr{F}$ are disjoint. Let us define

$$M(n,\omega)=\max\{\omega(\mathscr{F})\colon \mathscr{F}\subseteq 2^{[n]} \text{ is intersecting}\}.$$

Theorem 1 (Erdős and Schönheim [9]). *Let $\omega : 2^{[n]}\to\mathbb{R}_{\geqslant 0}$ be increasing*, *i.e.* $\omega(A)\leqslant\omega(B)$ *whenever $A\subseteq B$. Then*

$$M(n,\omega)=\frac{1}{2}\sum_{A\subseteq[n]}\max\{\omega(A),\omega([n]\setminus A)\}.$$

Note that the RHS is always an upper bound for $M(n,\omega)$, but equality need not occur for arbitrary weights.

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For positive numbers i,j let $[i,j]=\{i,i+1,\ldots,j\}$. We call a family $\mathscr{F}\subseteq 2^{[n]}$ intersecting in $[i,j]\subseteq[n]$ if $A\cap B\cap[i,j]\neq\emptyset$ holds for all $A,B\in\mathscr{F}$.

Definition. The weight function $\omega: 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ is called shift-monotone in $[m+1,n]$, $0\leqslant m\leqslant n$, if

$$\omega(a_1,\ldots,a_j)\geqslant\omega(b_1,\ldots,b_j)\quad\text{for all } \{a_1,\ldots,a_j\},\{b_1,\ldots,b_j\}\subseteq[n]$$
$$\text{with } a_1\leqslant b_1, a_2\leqslant b_2,\ldots,a_j\leqslant b_j$$

and

$$\omega(A)\geqslant\omega(B)\quad\text{for all } A,B\subseteq[n] \text{ with } A\subseteq B \text{ and } B\setminus A\subseteq[m+1,n].$$

Theorem 2. *Let $\omega: 2^{[n]}\to\mathbb{R}_{\geqslant 0}$ be shift-monotone in $[m+1,n]$, $1\leqslant m\leqslant n$. Then*

$$M(n,\omega)=\max\{\omega(\mathscr{F})\colon \mathscr{F}\subseteq 2^{[n]} \textit{ is intersecting in } [1,m]\}.$$

In particular, if ω is shift-monotone in $[2,n]$, then

$$M(n,\omega)=\omega(\{A\subseteq[n]\colon 1\in A\}).$$

The proof of Theorem 2 which is based on the concept of generating sets introduced by Ahlswede and Khachatrian [2] is given in the next section.

We give three applications of Theorem 2, two of them being known results.

Recall that $\mathscr{I}\subseteq 2^{[n]}$ is called an ideal (or downset) if $A\in\mathscr{I}$, $B\subseteq A$ implies $B\in\mathscr{I}$.

Theorem 3 (Chvátal [4]). *Let $\mathscr{I}\subseteq 2^{[n]}$ be an ideal with the following property: $\{a_1,\ldots,a_j\}\in\mathscr{I}$ and $b_i\leqslant a_i$, $i=1,\ldots,j$, implies $\{b_1,\ldots,b_j\}\in\mathscr{I}$. Then the star*

$$\{A\in\mathscr{I}\colon 1\in A\}$$

is a maximum intersecting subfamily of $\mathscr{I}$.

Obviously, Theorem 3 follows from Theorem 2. We remark that the assertion of Theorem 3 holds under the weaker hypothesis that the ideal $\mathscr{I}$ is compressed with respect to 1, i.e. replacing $a\neq 1$ by 1 in an element of $\mathscr{I}$ gives another element of $\mathscr{I}$ (see [13]).

Theorem 4 (Ahlswede and Cai [1]). *Let $\omega_1,\omega_2,\ldots,\omega_n$ be positive real numbers and let $\omega: 2^{[n]}\to\mathbb{R}_{\geqslant 0}$ be defined by*

$$\omega(A)=\prod_{a\in A}\omega_a\quad \textit{for } A\subseteq[n].$$

If $\omega_1\geqslant\cdots\geqslant\omega_m>1\geqslant\omega_{m+1}\geqslant\cdots\geqslant\omega_n$ then

$$M(n,\omega)=\begin{cases}\omega(\{A\subseteq[n]\colon 1\in A\}) & \textit{if } m\leqslant 1,\\ \frac{1}{2}\sum\limits_{A\subseteq[m]}\max\{\omega'(A),\omega'([m]\setminus A)\} & \textit{otherwise,}\end{cases}$$

where $\omega' : 2^{[m]} \to \mathbb{R}_{\geqslant 0}$ *is defined by*

$$\omega'(A) = \omega(A)\omega(2^{[m+1,n]}) \quad \text{for } A \subseteq [m].$$

Theorem 4 follows from Theorems 1 and 2. The original proof in [1] is by induction on n and can be adopted to shift-monotone weights which then yields the special case of Theorem 2.

Our third application is an intersecting theorem for the poset of colored subsets and is given in Section 3. We will need the following theorem which solves the weighted intersection problem for weight functions depending only on the size of the sets.

Theorem 5 (Erdős et al. [11]). *Let* $\omega : 2^{[n]} \to \mathbb{R}_{\geqslant 0}$ *be a weight function satisfying* $\omega(A) = \omega(B)$ *for all* $A, B \subseteq [n]$ *with* $|A| = |B|$. *Then*

$$M(n,\omega) = \max_{1 \leqslant i \leqslant (n+1)/2} \{\omega(\mathscr{F}_i)\},$$

where the families $\mathscr{F}_i \subseteq 2^{[n]}, 1 \leqslant i \leqslant (n+1)/2$, *are given by*

$$\mathscr{F}_i = \{A \subseteq [n]\colon i \leqslant |A| \leqslant n-i, 1 \in A\} \cup \{A \subseteq [n]\colon |A| \geqslant n-i+1\}.$$

2. Proof of Theorem 2

For every family $\mathscr{F} \subseteq 2^{[n]}$ let $\mathscr{M} = \mathscr{M}(\mathscr{F})$ denote the set of all minimal elements of $\mathscr{F}$ (with respect to set inclusion). Define

$$s^+(A) = \max\{i\colon i \in A\} \quad \text{for } A \subseteq [n],$$

$$s^+(\mathscr{F}) = \max\{s^+(A)\colon A \in \mathscr{F}\} \quad \text{for } \mathscr{F} \subseteq 2^{[n]}.$$

Now choose under all optimal families $\mathscr{A} \subseteq 2^{[n]}$ (i.e. $\omega(\mathscr{A}) = M(n,\omega)$) one for which

$$s^+ = s^+(\mathscr{M}(\mathscr{A})) \tag{1}$$

is minimal, and under all such families choose one for which

$$|\{A \in \mathscr{M}(\mathscr{A})\colon s^+ \in A\}| \tag{2}$$

is minimal.

We may assume that $\mathscr{A}$ is a filter (upset), i.e. $A \in \mathscr{A}$, $A \subseteq B$ implies $B \in \mathscr{A}$.

Recall the well-known Shifting-operation $S_{ij} : 2^{2^{[n]}} \to 2^{2^{[n]}}$ (see [8]) defined for $i, j \in [n]$ by

$$S_{ij}(\mathscr{F}) = \{S_{ij}(A)\colon A \in \mathscr{F}\} \quad \text{for } \mathscr{F} \subseteq 2^{[n]},$$

where

$$S_{ij}(A) = \begin{cases} A \setminus \{j\} \cup \{i\} & \text{if this is not an element of } \mathscr{F} \text{ and } A \cap \{i,j\} = \{j\}, \\ A & \text{otherwise.} \end{cases}$$

Note that if $\mathscr{F}$ is intersecting then $S_{ij}(\mathscr{F})$ is intersecting too. Furthermore, applying S_{ij} for $i < j$ to $\mathscr{F}$ does not increase the parameters defined in (1) and (2).

Hence, since ω is shift-monotone, we can assume that our optimal family $\mathscr{A}$ is left-shifted (= stable with respect to left-shifting), i.e.

$$S_{ij}(\mathscr{A}) = \mathscr{A} \quad \text{for all } i < j.$$

We will show that $s^+(\mathscr{M}(\mathscr{A})) \leqslant m$ which finishes the proof. Assume the contrary, i.e.

$$s^+ = s^+(\mathscr{M}(\mathscr{A})) > m.$$

Consider a set $A \in \mathscr{M} = \mathscr{M}(\mathscr{A})$ with $s^+ \in A$. Let $A' = A \setminus \{s^+\}$, $B = [s^+] \setminus A'$ and $B' = B \setminus \{s^+\}$.

First note that no set $C \subsetneq B$ is contained in the intersecting family $\mathscr{A}$. Indeed, for $i \in B \setminus C$ we have $A' \cup \{i\} \in \mathscr{A}$ and $(A' \cup \{i\}) \cap C = \emptyset$.

Next, we claim that $B \in \mathscr{M}$. Otherwise, the previous remark shows $B \notin \mathscr{A}$. But then $\mathscr{A} \cup \{A'\}$ is intersecting which contradicts our choice of $\mathscr{A}$.

Now consider the filter $\mathscr{A}_1$ (resp. $\mathscr{A}_2$) generated by $\mathscr{M}_1 = (\mathscr{M} \setminus \{A, B\}) \cup \{A'\}$ (resp. $\mathscr{M}_2 = (\mathscr{M} \setminus \{A, B\}) \cup \{B'\}$), i.e.

$$\mathscr{A}_i = \{F \subseteq [n]\colon F \supseteq G \text{ for some } G \in \mathscr{M}_i\}, \quad i = 1, 2.$$

Note that both $\mathscr{A}_1$ and $\mathscr{A}_2$ are intersecting. We claim that

$$\max\{\omega(\mathscr{A}_1), \omega(\mathscr{A}_2)\} \geqslant \omega(\mathscr{A}) \tag{3}$$

which contradicts again our choice of $\mathscr{A}$.

Obviously, $\mathscr{A} \setminus \mathscr{A}_1$ consists exactly of those sets of $\mathscr{A}$ which are generated only by B (i.e. which are supersets of B but not of any other set from $\mathscr{M}$). Furthermore, these sets are exactly the sets $B \cup C$ with $C \subseteq [s^+ + 1, n]$. Indeed, every other set $D \supseteq B$, say with $i \in (D \setminus B) \cap [s^+]$, is also generated by those sets from $\mathscr{M}$ which are subsets of $(B \setminus \{s^+\}) \cup \{i\}$. It follows

$$\omega(\mathscr{A} \setminus \mathscr{A}_1) = \sum_{C \subseteq [s^+ + 1, n]} \omega(B \cup C).$$

Similarly, it is easy to see that $\mathscr{A}_1 \setminus \mathscr{A}$ consists exactly of the sets $A' \cup C$ with $C \in [s^+ + 1, n]$. It follows

$$\omega(\mathscr{A}_1 \setminus \mathscr{A}) = \sum_{C \subseteq [s^+ + 1, n]} \omega(A' \cup C).$$

Analogously, it holds

$$\omega(\mathscr{A} \setminus \mathscr{A}_2) = \sum_{C \subseteq [s^+ + 1, n]} \omega(A \cup C),$$

$$\omega(\mathscr{A}_2 \setminus \mathscr{A}) = \sum_{C \subseteq [s^+ + 1, n]} \omega(B' \cup C).$$

Since ω is shift-monotone in $[m + 1, n]$, the last four equalities imply

$$\omega(\mathscr{A} \setminus \mathscr{A}_1) + \omega(\mathscr{A} \setminus \mathscr{A}_2) \leqslant \omega(\mathscr{A}_1 \setminus \mathscr{A}) + \omega(\mathscr{A}_2 \setminus \mathscr{A})$$

which shows (3).

3. An application

Consider a finite set C of $k_1+k_2+\cdots+k_n$ elements that is partitioned into n sets (color classes) C_i, $i=1,\dots,n$, each set C_i having size k_i. The family of all subsets of C which intersect each color class C_i in at most one element is denoted by $\mathrm{Col}(k_1,\dots,k_n)$. With respect to set inclusion, $\mathrm{Col}(k_1,\dots,k_n)$ is a ranked poset called poset of colored subsets (see [7, p. 344]). Let $W_k(k_1,\dots,k_n)$ denote the kth Whitney number of $\mathrm{Col}(k_1,\dots,k_n)$, i.e. $W_k(k_1,\dots,k_n)=|\mathrm{Col}(k_1,\dots,k_n)\cap\binom{C}{k}|$. We assume that $k_1\leqslant\cdots\leqslant k_n$.

It is not difficult to see (and follows also from the proof below) that a maximum intersecting subfamily of $\mathrm{Col}(k_1,\dots,k_n)$ has cardinality $1/k_1+1|\mathrm{Col}(k_1,\dots,k_n)|/(k_1+1)$. We study the uniform case.

Theorem 6. *Let* $1=k_1=\cdots=k_m<k_{m+1}\leqslant\cdots\leqslant k_n$. *Then one of the families* $\mathscr{C}_i\cap\binom{C}{k}$, $0\leqslant i\leqslant(m+1)/2$, *is a maximum* k*-uniform intersecting subfamily of* $Col(k_1,\dots,k_n)$, *where*

$$\mathscr{C}_i=\{A\in Col(k_1,\dots,k_n)\colon i\leqslant|\{j\in[m]\colon A\cap C_j\neq\emptyset\}|\leqslant m-i \text{ and } A\cap C_1\neq\emptyset\}$$
$$\cup\{A\in Col(k_1,\dots,k_n)\colon |\{j\in[m]\colon A\cap C_j\neq\emptyset\}|\geqslant m-i+1\}.$$

for $i=1,\dots,\lfloor(m+1)/2\rfloor$, *and* $\mathscr{C}_0:=\mathscr{C}_1$.

The case $k_1=k_2=\cdots=k_n>1$ was treated by Meyer [12], Deza and Frankl [5], Engel [6], Bollobás and Leader [3] and, in a more general context, by Erdős et al. [10].

Proof. W.l.o.g. let $C_i=[k_1+\cdots+k_{i-1}+1,k_1+\cdots+k_i]$. We can assume that an optimal family $\mathscr{A}\subseteq\mathrm{Col}(k_1,\dots,k_n)$ is left-shifted inside each color class C_i, i.e. $S_{jk}(\mathscr{A})=\mathscr{A}$ for all $j,k\in C_i$, $j<k$. Then, it is easy to see that

$$\mathrm{supp}(\mathscr{A})=\{\{i\colon A \text{ intersects } C_i \text{ in its first element}\}\colon A\in\mathscr{A}\}$$

is an intersecting family in $2^{[n]}$. Thus, we have a weighted intersection problem with weight function $\omega\colon 2^{[n]}\to\mathbb{R}_{\geqslant 0}$ given by

$$\omega(\{i_1,\dots,i_r\})=W_{k-r}(\pi_{\{i_1,\dots,i_r\}}(k_1-1,\dots,k_n-1)),$$

where π_I denotes the projection map onto the coordinates that are not contained in I. This weight function is easily seen to be shift-monotone in $[m+1,n]$. Moreover,

$$\omega(\{A\subseteq[n]\colon A\cap[m]=B\})$$

depends only on $|B|$. This allows us to apply Theorems 2 and 5. □

We remark that for $k\leqslant n/2$ it holds $|\mathscr{C}_1\cap\binom{C}{k}|\geqslant\cdots\geqslant|\mathscr{C}_{\lfloor m+1/2\rfloor}\cap\binom{C}{k}|$. This follows easily from

$$\left|(\mathscr{C}_i\setminus\mathscr{C}_{i+1})\cap\binom{C}{k}\right|=\binom{m-1}{i-1}W_{k-i}(k_{m+1},\dots,k_n),$$
$$\left|(\mathscr{C}_{i+1}\setminus\mathscr{C}_i)\cap\binom{C}{k}\right|=\binom{m-1}{m-i}W_{k-(m-i)}(k_{m+1},\dots,k_n)$$

and the fact that the poset $\mathrm{Col}(k_{m+1},\ldots,k_n)$ is 'top heavy', i.e. it holds $W_a(k_{m+1},\ldots,k_n) \leqslant W_b(k_{m+1},\ldots,k_n)$ for all $a \leqslant b,\ a+b \leqslant n-m$.

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