An Intersection Theorem for Systems of Finite Sets

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Abstract

For nonnegative reals ω, ψ and natural $t \leq k \leq (n+t-1)/2$, the maximum of

$$\omega \left|\mathcal{A} \cap \tbinom{[n]}{k}\right| + \psi \left|\mathcal{A} \cap \tbinom{[n]}{n+t-1-k}\right|$$

among all t-intersecting set systems $\mathcal{A} \subseteq 2^{[n]}$ is determined.

Introduction and Result

Throughout this note, n, k, t are positive integers with $n \geq k \geq t$. Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, $2^{[n]}$ the powerset of $[n]$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. For every $\mathcal{A} \subseteq 2^{[n]}$ and nonnegative integer i we set $\mathcal{A}_i := \mathcal{A} \cap \binom{[n]}{i}$.

A set system $\mathcal{A} \subseteq 2^{[n]}$ is called t-intersecting if $|A_1 \cap A_2| \geq t$ for all $A_1, A_2 \in \mathcal{A}$. Set

$$I(n,t) := \{\mathcal{A} \subseteq 2^{[n]} : \mathcal{A} \text{ is } t\text{-intersecting}\}.$$

For every nonnegative integer r, let

$$\mathcal{B}(r) := \{A \subseteq [n] : |A \cap [t+2r]| \geq t + r\}.$$

Obviously, $\mathcal{B}(r) \in I(n,t)$.

We recall the following two basic results in extremal set theory.

Theorem 1 (**Katona [5]**).

(i) *Let $\mathcal{A} \in I(n,t)$, $t \leq k \leq \frac{n+t-1}{2}$ and $0 \leq \omega \leq 1 + \frac{t-1}{k-t+1}$. Then*

$$\omega\,|\mathcal{A}_k| + |\mathcal{A}_{n+t-1-k}| \leq \binom{n}{n+t-1-k}.$$

Equality holds in case of $2 \leq t \leq k < \frac{n+t-1}{2}$ iff $\mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ is isomorphic to

· $\mathcal{B}_{n+t-1-k}(k-t+1) = \binom{[n]}{n+t-1-k}$ *if* $\omega < 1 + \frac{t-1}{k-t+1}$

· $\mathcal{B}_{n+t-1-k}(k-t+1)$ *or* $\mathcal{B}_k(k-t) \cup \mathcal{B}_{n+t-1-k}(k-t)$ *if* $\omega = 1 + \frac{t-1}{k-t+1}$,

and in case of $2 \le t \le k = \frac{n+t-1}{2}$ *iff* $\omega = 1 + \frac{t-1}{k-t+1}$ *and* $\mathcal{A}_{\frac{n+t-1}{2}}$ *is isomorphic to* $\mathcal{B}_{\frac{n+t-1}{2}}(k-t) = \binom{[2k-t]}{k}$.

(ii) *Let* $\mathcal{A} \in I(n,t)$. *Then*

$$|\mathcal{A}| \le \left|\mathcal{B}\Big(\Big\lfloor \frac{n-t}{2} \Big\rfloor\Big)\right| = \begin{cases} \sum_{k=\frac{n+t}{2}}^{n} \binom{n}{k} & \text{if } 2 \mid (n+t) \\ 2\sum_{k=\frac{n-1+t}{2}}^{n-1} \binom{n-1}{k} & \text{if } 2 \nmid (n+t). \end{cases}$$

Equality holds in case of $t \ge 2$ *iff* $\mathcal{A}$ *is isomorphic to* $\mathcal{B}\big(\lfloor \frac{n-t}{2} \rfloor\big)$.

Note that (ii) follows from the case $\omega = 1$ in (i) by adding over k all stated inequalities. Note also that the case $t = 1$ is easily dealt with by considering complements.

Theorem 2 (Ahlswede, Khachatrian [1]). *Let* $\mathcal{A} \in I(n,t)$, $t \le k \le \frac{n+t-1}{2}$. *Let* $r \in [0, k-t]$ *be the smallest integer satisfying*

$$\left(2 + \frac{t-1}{r+1}\right)(k-t+1) \le n\,. \tag{1}$$

Then

$$|\mathcal{A}_k| \le |\mathcal{B}_k(r)|\,.$$

Equality holds in case of $(k,t) \ne (\frac{n}{2}, 1)$ *iff* $\mathcal{A}_k$ *is isomorphic to*

· $\mathcal{B}_k(r)$ *if strict inequality holds in (1)*

· $\mathcal{B}_k(r)$ *or* $\mathcal{B}_k(r+1)$ *if equality holds in (1).*

The cases $t = 1$ and n sufficiently large are covered by the classical **Erdős-Ko-Rado Theorem [4]**. See the first section of [1] for further celebrated previously obtained partial results of Theorem 2.

Our contribution in this note is the following intersection theorem, which can be seen as a common extension of Theorem 1 (i) and Theorem 2.

Theorem 3. *Let* $\mathcal{A} \in I(n,t)$, $t \le k \le \frac{n+t-1}{2}$ *and* $\omega, \psi \ge 0$ *(not both 0). Then*

$$\omega\,|\mathcal{A}_k| + \psi\,|\mathcal{A}_{n+t-1-k}| \le \max\{\omega\,|\mathcal{B}_k(r)| + \psi\,|\mathcal{B}_{n+t-1-k}(r)| \,:\, 0 \le r \le k-t+1\}.$$

Equality holds in case of $t \ge 2$ *iff* $\mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ *is isomorphic to one of the systems* $\mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1}(r)$ *which attains the maximum.*

The case $\psi = 0$ is covered by Theorem 2, and the case $\omega/\psi \le 1 + \frac{t-1}{k-t+1}$ is covered by Theorem 1 (i). In the case $\psi \ne 0$ (w.l.o.g. $\psi = 1$) the following more precise theorem holds.

Theorem 3′. *Let* $\mathcal{A} \in I(n,t)$*,* $t \le k < \frac{n+t-1}{2}$ *and* $\omega \ge 0$*. Let* $r \in [0, k-t]$ *be the smallest integer satisfying*

$$(1+\omega)\left(n - \left(2 + \frac{t-1}{r+1}\right)(k-t+1)\right) \ge \left(2 + \frac{t-1}{r+1}\right)(n-2k+t-1) \qquad (2)$$

or, in case of (2) does not hold for $r = k-t$*, let* $r := k-t+1$*. Then*

$$\omega\,|\mathcal{A}_k| + |\mathcal{A}_{n+t-1-k}| \le \omega\,|\mathcal{B}_k(r)| + |\mathcal{B}_{n+t-1-k}(r)|\,.$$

Equality holds in case of $t \ge 2$ *iff* $\mathcal{A}_k \cup \mathcal{A}_{n+t-1}$ *is isomorphic to*

- $\mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1-k}(r)$ *if strict inequality holds in (2)*
- $\mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1-k}(r)$ *or* $\mathcal{B}_k(r+1) \cup \mathcal{B}_{n+t-1-k}(r+1)$ *if equality holds in (2)*
- $\mathcal{B}_k(k-t+1) \cup \mathcal{B}_{n+t-1-k}(k-t+1) = \binom{[n]}{n+t-1-k}$ *if* $r = k-t+1$*.*

Note that for $r = k-t$ the inequality (2) reduces to $\omega \ge 1 + \frac{t-1}{k-t+1}$.

Theorem 3′ has implications for estimations of shadows of intersecting set systems. This will be explored elsewhere.

The proof of Theorem 3′ uses the powerful methods developed by Ahlswede and Khachatrian in [1, 2]. We will not prove the uniqueness statement in this note.

Proof

The case $t = 1$ is easily dealt with by using complements and applying the Erdős-Ko-Rado Theorem. Also, as in Theorem 1 (i), it suffices to consider the case $\omega \ge 1$.

Given a set system $\mathcal{A} \subseteq 2^{[n]}$ and a vector $\nu = (\nu_0, \nu_1, \ldots, \nu_n)$ of nonnegative real numbers (weights), put

$$\nu(\mathcal{A}) := \sum_{A \in \mathcal{A}} \nu_{|A|} = \sum_{i=0}^{n} |\mathcal{A}_i|\,\nu_i$$

and

$$M(n,t,\nu) := \max\{\nu(\mathcal{A}) : \mathcal{A} \in I(n,t)\}.$$

We consider the weights

$$\nu_i := \begin{cases} \omega & \text{if } i = k \\ 1 & \text{if } i = n+t-1-k \\ 0 & \text{otherwise,} \end{cases} \qquad (3)$$

so that $\nu(\mathcal{A}) = \omega|\mathcal{A}_k| + |\mathcal{A}_{n+t-1-k}|$ for all $\mathcal{A} \subseteq 2^{[n]}$.

Unimodality of the sequence $\omega|\mathcal{B}_k(r)| + |\mathcal{B}_{n+t-1-k}(r)|$, $r \geq 0$

Lemma 4. *The sequence $\nu(\mathcal{B}(0)), \nu(\mathcal{B}(1)), \ldots, \nu(\mathcal{B}(k-t+1))$ is unimodal. More precisely, for every $r \in [0, k-t]$, $\nu(\mathcal{B}(r)) \geq \nu(\mathcal{B}(r+1))$ holds iff (2) is satisfied.*

Proof. A comparison of the numbers

$$\nu(\mathcal{B}_r \setminus \mathcal{B}_{r+1}) = \binom{t+2r}{t+r}\left(\binom{n-t-2r-2}{k-t-r}\omega + \binom{n-t-2r-2}{k-t-r-1}\right),$$
$$\nu(\mathcal{B}_{r+1} \setminus \mathcal{B}_r) = \binom{t+2r}{t+r-1}\left(\binom{n-t-2r-2}{k-t-r-1}\omega + \binom{n-t-2r-2}{k-t-r}\right)$$

shows that

$$\nu(\mathcal{B}(r)) \geq \nu(\mathcal{B}(r+1))$$

is equivalent to

$$\binom{n-t-2r-2}{k-t-r}\left(\omega - \left(1 + \frac{t-1}{r+1}\right)\right) \geq \binom{n-t-2r-2}{k-t-r-1}\left(\left(1+\frac{t-1}{r+1}\right)\omega - 1\right). \quad (4)$$

In particular, for $r = k-t$,

$$\nu(\mathcal{B}(k-t)) \geq \nu(\mathcal{B}(k-t+1)) \text{ iff } \omega \geq 1 + \frac{t-1}{k-t+1} \text{ iff (2) holds with } r = k-t.$$

For $r < k-t$ inequality (4) is equivalent to

$$\left(1 + \frac{n-2k+t-1}{k-t-r}\right)\left(\omega - \left(1+\frac{t-1}{r+1}\right)\right) \geq \left(\left(1+\frac{t-1}{r+1}\right)\omega - 1\right),$$

which in turn is equivalent to (2). □

Generating Sets [1]

For an arbitrary weight vector $\nu = (\nu_0, \ldots, \nu_n)$ and $m \in [t, n]$, let $\nu^{(m)} = (\nu_0^{(m)}, \ldots, \nu_m^{(m)})$ be given by

$$\nu_i^{(m)} := \sum_{j=0}^{n-m} \binom{n-m}{j} \nu_{i+j}, \quad i = 0, \ldots, m.$$

Based on the arguments in [1], the following proposition has been proved in [3].

Proposition 5 ([3, Theorem 15]). *Let $0 \leq r \leq \frac{n-t-1}{2}$. If*

$$\nu(\mathcal{B}(r)) \geq \nu(\mathcal{B}(r+1)) \geq \cdots \geq \nu(\mathcal{B}(\lfloor \tfrac{n-t}{2} \rfloor)) \quad (5)$$

and

$$\nu_{i-1}^{(m)}\nu_{m+t-i-1}^{(m)} \geq \nu_i^{(m)}\nu_{m+t-i}^{(m)} \text{ for all } m \in [t+2r+1, n],\ i \in [t, \tfrac{m+t}{2}), \quad (6)$$

then there exists an $\mathcal{A} \in I(n,t)$ with $\nu(\mathcal{A}) = M(n,t,\nu)$ which is t-intersecting in $[t+2r]$, i.e. $|A \cap B \cap [t+2r]| \geq t$ for all $A, B \in \mathcal{A}$.

Proposition 5 is applicable in our situation:

Lemma 6. *Let ν be the weight vector given by (3), and let $r \in [0, k-t+1]$ be determined as in Theorem 3′. Then there exists an $\mathcal{A} \in I(n,t)$ with $\nu(\mathcal{A}) = M(n,t,\nu)$ which is t-intersecting in $[t+2r]$.*

Proof. The monotonicity (5) is clear by Lemma 4 and $\nu(\mathcal{B}(k-t+1)) = \cdots = \nu(\mathcal{B}(\lfloor \frac{n-t}{2} \rfloor))$. Condition (6), i.e.

$$\left(\binom{n-m}{k-i+1}\omega + \binom{n-m}{k-m-t+i}\right)\left(\binom{n-m}{k-m-t+i+1}\omega + \binom{n-m}{k-i}\right) \\ \geq \left(\binom{n-m}{k-i}\omega + \binom{n-m}{k-m-t+i+1}\right)\left(\binom{n-m}{k-m-t+i}\omega + \binom{n-m}{k-i+1}\right),$$

or equivalently,

$$(\omega^2-1)\left(\binom{n-m}{k-i+1}\binom{n-m}{n+t-k-i-1} - \binom{n-m}{k-i}\binom{n-m}{n+t-k-i}\right) \geq 0,$$

is satisfied due to $\omega \geq 1$, $k-i+1 \leq n+t-k-i$ and the log-concavity of the binomial coefficients. □

Pushing-Pulling [2]

We need the following well-known notion. A set system $\mathcal{A} \subseteq 2^{[n]}$ is called left-compressed if $(A \setminus \{j\}) \cup \{i\} \in \mathcal{A}$ for all $A \in \mathcal{A}$, $i,j \in [n]$ with $i < j$, $j \in A$, $i \notin A$. Put

$$LI(n,t) := \{\mathcal{A} \in I(n,t) : \mathcal{A} \text{ is left-compressed}\}.$$

Lemma 7. *Let ν be the weight vector given by (3), and let $r \in [0, k-t+1]$ be determined as in Theorem 3′. Then every $\mathcal{A} \in LI(n,t)$ with $\nu(\mathcal{A}) = M(n,t,\nu)$ and $\mathcal{A} = \mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ is invariant in $[t+2r]$, i.e. $(A \setminus \{j\}) \cup \{i\} \in \mathcal{A}$ for all $A \in \mathcal{A}$, $i,j \in [t+2r]$ with $j \in A$, $i \notin A$.*

Proof. Assume the contrary. Let

$$\begin{aligned} \ell &:= \max\{i \in [n] : \mathcal{A} \text{ is invariant in } [i]\} \\ \mathcal{L} &:= \{A \in \mathcal{A} : \ell+1 \notin A, (A \setminus \{i\}) \cup \{\ell+1\} \notin \mathcal{A} \text{ for some } i \in A \cap [\ell]\} \\ \mathcal{L}^* &:= \{A \cap [\ell+2, n] : A \in \mathcal{L}\}. \end{aligned}$$

Clearly, $\mathcal{L} \neq \emptyset$ and hence $\mathcal{L}^* \neq \emptyset$. By our assumption we have $\ell < t+2r$. The following facts (i) - (iii) follow from the pushing-pulling method [2] (see also [3], which treats an arbitrary weight vector ν):

(i) $\ell \geq t,\ 2 \mid \ell + t,\ \mathcal{L} = \left\{B \cup C : B \in \binom{[\ell]}{\frac{\ell+t}{2}},\ C \in \mathcal{L}^*\right\}$.

In particular, by our assumption on ℓ, we have $\ell \le t + 2r - 2$ and hence $r \ge 1$. Also, the structure of $\mathcal{L}$ implies

$$\mathcal{L}^* = \mathcal{L}^*_{k-\frac{\ell+t}{2}} \cup \mathcal{L}^*_{n+t-1-k-\frac{\ell+t}{2}}. \tag{7}$$

(ii) $\mathcal{L}^*$ is complement-closed, i.e. $C \in \mathcal{L}^*$ implies $[\ell+2, n] \setminus C \in \mathcal{L}^*$.

In particular,

$$\left|\mathcal{L}^*_{k-\frac{\ell+t}{2}}\right| = \left|\mathcal{L}^*_{n+t-1-k-\frac{\ell+t}{2}}\right|. \tag{8}$$

(iii) For every intersecting subsystem $\mathcal{T}^*$ of $\mathcal{L}^*$,

$$\frac{\sum_{C\in\mathcal{T}^*} \nu_{|C|+\frac{\ell+t}{2}}}{\sum_{C\in\mathcal{L}^*} \nu_{|C|+\frac{\ell+t}{2}}} \le \frac{\ell-t+2}{2(\ell+1)}.$$

We will show that one of the systems $\mathcal{L}^*(j) := \{C \in \mathcal{L}^* : j \in C\}$, $j = \ell, \dots n$, contradicts fact (iii). Indeed, by averaging and by (7) and (8), there exists a j such that

$$\frac{\sum_{C\in\mathcal{L}^*(j)} \nu_{|C|+\frac{\ell+t}{2}}}{\sum_{C\in\mathcal{L}^*} \nu_{|C|+\frac{\ell+t}{2}}} \ge \frac{1}{(n-\ell-1)} \frac{\sum_i i\,|\mathcal{L}^*_i|\,\nu_{i+\frac{\ell+t}{2}}}{\sum_i |\mathcal{L}^*_i|\,\nu_{i+\frac{\ell+t}{2}}} = \frac{(k-\frac{\ell+t}{2})\,\omega + (n+t-1-k-\frac{\ell+t}{2})}{(n-\ell-1)(\omega+1)}.$$

Now, the desired inequality

$$\left(k - \frac{\ell+t}{2}\right)\omega + \left(n+t-1-k-\frac{\ell+t}{2}\right) > \frac{(\ell-t+2)}{2(\ell+1)}\,(n-\ell-1)(\omega+1),$$

or equivalently

$$(1+\omega)\left(n - \left(2 + \frac{t-1}{\frac{\ell-t}{2}+1}\right)(k-t+1)\right) < \left(2 + \frac{t-1}{\frac{\ell-t}{2}+1}\right)(n-2k+t-1),$$

holds in view of (2) by our choice of r and $\frac{\ell-t}{2} \le r-1$. □

Conclusion of Proof

Consider an optimal system $\mathcal{A} = \mathcal{A}_k \cup \mathcal{A}_{n+t-1-k}$ from Lemma 6. By the well-known shifting technique of [4] we may assume that $\mathcal{A} \in LI(n,t)$, so that Lemma 7 applies. Then, as $\mathcal{A}$ is both t-intersecting and invariant in $[t+2r]$, necessarily $\mathcal{A} = \mathcal{B}_k(r) \cup \mathcal{B}_{n+t-1-k}(r)$. □

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