On the detection of singularities of a periodic function

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Dedicated to Professor Dr. Charles K. Chui on the occasion of his 60-th birthday

Abstract

We discuss the problem of detecting the location of discontinuities of derivatives of a periodic function, given either finitely many Fourier coefficients of the function, or the samples of the function at uniform or scattered data points. Using the general theory, we develop a class of trigonometric polynomial frames suitable for this purpose. Our methods also help us to analyze the capabilities of periodic spline wavelets, trigonometric polynomial wavelets, and some of the classical summability methods in the theory of Fourier series.

1 Introduction

Let $f : I\!R \longrightarrow \mathbb{C}$ be a 2π-periodic function, $r \geq 0$ be an integer, and $x_0 \in [-\pi, \pi]$. We say that x_0 is a singularity of order r of f if f is r times continuously differentiable in a deleted neighborhood of x_0, but the derivative $f^{(r)}$ has a jump discontinuity at x_0. In many such applications as signal processing and numerical analysis, one needs to discover the location of singularities of f of different orders (cf. [12]). In recent years, wavelet analysis has provided many popular and powerful tools for this purpose (see [14], [15]).

*This research was supported, in part, by the Alexander von Humboldt Foundation and grant no. F-49620-97-1-0211 from the U. S. Air Force Office of Scientific Research.

In order to apply these techniques, one needs to know the wavelet coefficients of f. In practical applications, one starts with a large number of samples of f and computes these coefficients using numerical integration and the refinement equations for the wavelets ([2]).

If the information available about the function consists of its trigonometric Fourier coefficients, then it is natural to explore trigonometric operators similar to the classsical summability methods. A very general class of wavelets based on this idea has been developed and studied in [3], [11], [20] and [19]. In [18], we have constructed "fast decreasing" trigonometric polynomial operators to detect singularities. An interesting aspect of these investigations is that the Fourier coefficients of f represent global information about f. The ability of the various operators to detect singularities of f thus describes the localization properties of these operators. Some other ideas towards the solution of this problem, as well as references to further applications of this research can be found, for example, in the recent papers of Eckhoff [6], [7], Gelb and Tadmor [8], [9].

One objective of this paper is to study in general the abilities of different operators to detect singularities of different orders. Typically, we expect the "large" values of these operators at certain points to indicate the presence of a singularity "near" these points. We seek to provide a precise quantitative description of this property.

In some applications, we may not start with the Fourier coefficients of the function, but may sample the function at several points. It is customary in wavelet analysis to start with a large number of such samples, and then "compress" this data into a small number of large wavelet coefficients. The second objective of this paper is to explore a paradigm in the opposite direction. We start with a "small" number of samples to construct a trigonometric operator. We will modify these operators step-by-step by taking more and more samples as needed until the singularities are detected with a desired accuracy.

In Section 2, we discuss "spectral methods", i.e., operators based on the Fourier coefficients of the function. In Section 3, we discuss the "build-up methods", i.e., operators based on samples of the functions. In Section 4, we apply the general theory to analyze specific examples. In Section 5, we present a few simple numerical examples to illustrate the theorems in Section 4.

2 Spectral Methods

In this section, we are interested in detecting the singularities of a function, given its trigonometric Fourier coefficients. If $f : \mathbb{R} \longrightarrow \mathbb{C}$ is 2π-periodic and Lebesgue integrable on $[-\pi, \pi]$, its Fourier coefficients are defined by

$$\hat{f}(k) := \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt}\, dt, \qquad k \in \mathbb{Z},$$

and for integer $m \geq 1$, the partial sum of its Fourier series is given by

$$s_m(f, x) := \sum_{|k|<m} \hat{f}(k) e^{ikx}.$$

For the “saw-tooth” function defined on $(-\pi, \pi]$ by

$$\Gamma_0(x) := \begin{cases} -\pi - x, & \text{if} \quad -\pi < x < 0, \\ 0, & \text{if} \quad x = 0, \\ \pi - x, & \text{if} \quad 0 < x \le \pi, \end{cases}$$

and extended to $I\!R$ as a 2π-periodic function, one has

$$\Gamma_0(x) = \sum_{k\in\mathbb{Z}}{}' \frac{e^{ikx}}{ik} := \sum_{k\in\mathbb{Z}\setminus\{0\}} \frac{e^{ikx}}{ik} := \lim_{m\to\infty} s_m(\Gamma_0, x), \qquad x \in [-\pi, \pi].$$

We observe that Γ_0 is continuous on $[-\pi, \pi] \setminus \{0\}$, and $\Gamma_0(0+) - \Gamma_0(0-) = 2\pi$. Consequently, for integer $r \ge 0$, the periodic integral

$$\Gamma_r(x) = \sum_{k\in\mathbb{Z}}{}' \frac{e^{ikx}}{(ik)^{r+1}}, \qquad x \in I\!R,$$

is a 2π-periodic function with r continuous derivatives at each point on $[-\pi, \pi] \setminus \{0\}$, and a singularity of order r at 0.

We say that a 2π-periodic function $f : I\!R \to \mathbb{C}$ is *piecewise R-smooth,* if there exist points

$$-\pi = y_0 < y_1 < \cdots < y_m = \pi$$

such that the restriction of f to (y_i, y_{i+1}), $0 \le i \le m-1$, is R times continuously differentiable, and $f^{(r)}(y_i+)$ and $f^{(r)}(y_{i+1}-)$ exist as finite numbers for $0 \le r \le R$. (Here, y_m and y_0 are identified as usual.) Many applications of analysis involve piecewise R-smooth functions; for example, a periodic spline is a piecewise R-smooth function for a suitable integer R. The function Γ_R is not just a typical example of a piecewise R-smooth function, but *any* such function can be expressed in a cannonical form (2.1) as follows:

$$f(x) = \sum_{r=0}^{R} \sum_{j=1}^{m} \omega_{j,r} \Gamma_r(x - y_j) + F(x), \qquad x \in I\!R, \tag{2.1}$$

where $\omega_{j,r} \in \mathbb{C}$, and F is an R times continuously differentiable 2π-periodic function on $I\!R$. In the recent papers of Eckhoff [6], [7], Gelb and Tadmor [8], [9], the authors have described some applications to the theory of shock detection which require the detection of the location of the points y_j in the above representation.

Our research is motivated by the problem of detecting *hidden periodicities* [12], which arises in certain meteorological and astronomical problems and in analysis of tides. Mathematically, the problem is as follows. We are given readings of the form

$$\mu_k = \sum_{\ell=1}^{m} \omega_\ell \exp(-iky_\ell),$$

for $k = 0, \pm 1, \cdots, \pm N$ for some large integer N, and are interested in finding the number m and the points y_ℓ. In [12], the method given to solve this problem is essentially to consider the peaks of $|\sum_{|k|\le N} g_k \mu_k \exp(ikx)|$. We observe that

$$\sum_{k\in\mathbb{Z}}{}' \frac{\mu_k e^{ikx}}{(ik)^{R+1}} = \sum_{\ell=1}^{m} \omega_\ell \Gamma_R(x - y_\ell)$$

is a piecewise R-smooth function with singularities exactly at y_ℓ.

In [5], Z. Divis has studied the asymptotic behavior of $\{s_m(\Gamma_R, \cdot)\}$ as $m \to \infty$. It follows from her work that if f is of the form

$$f(x) = \sum_{j=1}^{k_R} \omega_{j,R}\Gamma_R(x - x_{j,R}) + F(x),$$

where $F^{(R)}$ is a continuous function having bounded variation on $[-\pi, \pi]$, then for odd integer R (and as $n \to \infty$),

$$|s_{2n}(f,x) - s_n(f,x)| = \begin{cases} \dfrac{\omega_{j,R}c_R}{n^R}(1 + o(1)), & \text{if } x = x_{j,R},\ j = 1, \ldots, k_R, \\ o_x\left(\dfrac{1}{n^R}\right), & \text{otherwise.} \end{cases} \tag{2.2}$$

One may think of (2.2) as a method to detect the R-th order singularities of f. The left hand side can be computed using only the coefficients $\{\hat{f}(k)\}_{k=n}^{2n-1}$. It is large at the singularities, and small away from them. We observe that

$$s_{2n}(f,x) - s_n(f,x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)(D_{2n} - D_n)(x - t)\, dt,$$

where D_n is the Dirichlet kernel $\sum_{|k|<n} e^{ikx}$. All of the wavelet transforms in [11], [20] and [19], as well as the construction given in [12] for the detection of hidden periodicity can also be expressed as a convolution of f with some function dependent upon a scale parameter n. Therefore, in this section, we are interested in studying the convolutions

$$(f * \psi_n)(x) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\psi_n(x - t)dt = \sum_{j \in \mathbb{Z}} \hat{f}(j)\hat{\psi}_n(j)e^{ijx},$$

where the series will converge under suitable conditions on ψ_n.

For functions f of the form (2.1), we need to show that $F * \psi_n$ is uniformly small, and study the asymptotic behavior of $(\Gamma_r * \psi_n)(x)$ for *all* x (in contrast to (2.2)) as $n \to \infty$.

Let $1 \le p \le \infty$, and $A \subset \mathbb{R}$ be a Lebesgue measurable set with positive measure. If $f : A \longrightarrow \mathbb{C}$ is Lebesgue measurable, we write

$$\|f\|_{p,A} := \begin{cases} \left(\displaystyle\int_A |f(x)|^p dx\right)^{1/p}, & \text{if } \ 1 \le p < \infty, \\ \displaystyle\operatorname*{ess\,sup}_{x \in A} |f(x)|, & \text{if } \ p = \infty. \end{cases}$$

The class $L^p(A)$ consists of all functions f for which $\|f\|_{p,A} < \infty$, with two functions considered equal if they are equal almost everywhere. The symbol $X^p(A)$ will denote the class $L^p(A)$ if $1 \le p < \infty$, and the class of all continuous functions on A (vanishing at infinity if A is unbounded) if $p = \infty$. If $A = [-\pi, \pi]$, we will omit the mention of A from the notations, and assume that the functions involved are 2π-periodic. In this case, we will also write

$$\|f\|_p := (2\pi)^{-1/p}\|f\|_{p,[-\pi,\pi]}, \qquad 1 \le p < \infty.$$

From the point of view of the detection of singularities, perhaps X^∞ is the only interesting class. However, in light of the connections with wavelet theory, we will obtain estimates also on $\|f * \psi_n\|_p$ for all p, $1 \le p \le \infty$. These estimates will be in terms of the degree of approximation to f by trigonometric polynomials, as well as certain trigonometric polynomials associated with ψ_n.

For $x \ge 0$, we denote the class of all trigonometric polynomials of order at most x by $I\!H_x$. If $f \in L^p$, we write

$$E_{x,p}(f) := \inf_{T \in I\!H_x} \|f - T\|_p.$$

Let $0 < \alpha < \beta$ and $\chi_{\alpha,\beta}$ be an infinitely many times differentiable function on $I\!R$ such that

$$\chi_{\alpha,\beta}(y) := \begin{cases} 1, & \text{if } |y| \le \alpha, \\ 0, & \text{if } |y| \ge \beta. \end{cases}$$

We write

$$\chi_{n,\alpha,\beta}(y) := \chi_{\alpha,\beta}(\frac{y}{n}), \quad y \in I\!R, \quad n > 0,$$

and

$$\chi^\circ_{n,\alpha,\beta}(x) := \sum_{j \in Z\!\!Z} \chi_{n,\alpha,\beta}(j) e^{ijx}. \tag{2.3}$$

The first basic result of this section is the following.

Theorem 2.1 *Let $n, N \ge 1$ be integers, $\psi_n \in L^1$, $1 \le p \le \infty, 0 < \alpha < \beta$ and $f \in L^p$. Then*

$$\|f * \psi_n\|_p \le \|\psi_n\|_1 E_{\alpha N,p}(f) + 2\|\chi^\circ_{N,\alpha,\beta} * \psi_n\|_1 \|f\|_p. \tag{2.4}$$

REMARK. In applications, it is desirable to choose ψ_n and adjust the parameter N so that $\|\chi^\circ_{N,\alpha,\beta} * \psi_n\|_1 / \|\psi_n\|_1 \to 0$ as $n \to \infty$ and $E_{\alpha N,p}(F)$ has a desired decay for smooth functions F. With this choice, and the cannonical representation (2.1), Theorem 2.1 shows that $F * \psi_n$ is uniformly small. In the statement of the above theorem and in most of our discussions in this and the next section, the parameter n is really superfluous. We retain it partly to make the connection between the theory and applications in Section 4 more transparent, and partly to underline the fact that the various constants to appear are independent of n and ψ_n.

PROOF OF THEOREM 2.1. Let $T \in I\!H_{\alpha N}$ be a polynomial of best approximation to f; i.e.,

$$\|f - T\|_p = E_{\alpha N,p}(f).$$

Using the well known fact that

$$\|F * G\|_p \le \|F\|_p \|G\|_1, \qquad F \in L^p,\ G \in L^1,$$

we obtain

$$\begin{aligned} \|f * \psi_n\|_p &\le \|(f - T) * \psi_n\|_p + \|\psi_n * T\|_p \\ &\le \|\psi_n\|_1 E_{\alpha N,p}(f) + \|\psi_n * T\|_p. \end{aligned} \tag{2.5}$$

Since $T \in I\!H_{\alpha N}$, we have $T = \chi^{\circ}_{N,\alpha,\beta} * T$, and hence,

$$\begin{aligned} \|\psi_n * T\|_p &= \|\psi_n * \chi^{\circ}_{N,\alpha,\beta} * T\|_p \\ &\le \|\psi_n * \chi^{\circ}_{N,\alpha,\beta}\|_1 \|T\|_p \\ &\le \|\psi_n * \chi^{\circ}_{N,\alpha,\beta}\|_1 \left(\|f - T\|_p + \|f\|_p\right) \\ &\le 2\|\psi_n * \chi^{\circ}_{N,\alpha,\beta}\|_1 \|f\|_p. \end{aligned} \tag{2.6}$$

The estimates (2.5) and (2.6) lead to (2.4). □

In many examples (cf. [1, 12]), we are interested in the case when

$$\hat{\psi}_n(j) = g_n\left(\frac{j}{n}\right) \tag{2.7}$$

for some function g_n on $I\!R$. In this case, we will say that ψ_n is *generated by* g_n. Of course, any $\psi_n \in L^1$ is always generated by

$$g_n(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} \psi_n(x) e^{-inxt} dx,$$

or by a variety of interpolation schemes such as cardinal spline interpolants of suitable orders. Nevertheless, in many examples, it is more natural to take a function g_n as a starting point to construct ψ_n via (2.7). Under suitable assumptions on g_n, we wish to obtain an analogue of (2.4) and an asymptotic expression for $\Gamma_r * \psi_n$ in terms of the Fourier transform of g_n. First, we review some basic facts which will be needed in the statement and proof of our result.

If A is an open subset of $I\!R$, the class $BV(A)$ consists of functions having a bounded variation on A and vanishing at infinity if A is unbounded. If $g \in BV(A)$, $V(g, A)$ denotes the total variation of g on A. If $q \ge 0$ is an integer, g is a q-times iterated integral of a function $g^{(q)} \in BV(A)$, and g (and hence, $g^{(\ell)}$, $0 \le \ell \le q$) vanishes at infinity, then we say that $g \in BV^q(A)$.

If $g \in L^1(I\!R)$, its Fourier transform is defined by

$$\mathcal{F}g(t) = \int_{-\infty}^{\infty} g(x) e^{-ixt} dx, \qquad t \in I\!R,$$

and the inverse Fourier transform of g is defined by

$$\mathcal{F}^{-1}g(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(x) e^{ixt} dx, \qquad t \in I\!R.$$

If both g and $\mathcal{F}g$ are in $L^1(I\!R)$, then

$$g = \mathcal{F}^{-1}\mathcal{F}g = \mathcal{F}\mathcal{F}^{-1}g.$$

An integrable function $g : I\!R \to \mathbb{C}$ will be called *weakly admissible* if the following Poisson summation formula is valid for almost all $x \in [-\pi, \pi]$ and for all $\lambda > 0$:

$$\sum_{j\in\mathbb{Z}} g\left(\frac{j}{\lambda}\right) e^{ijx} = 2\pi\lambda \sum_{k\in\mathbb{Z}} \mathcal{F}^{-1} g\Big(\lambda(x + 2k\pi)\Big). \tag{2.8}$$

If (2.8) holds for *all* $x \in [-\pi, \pi]$, then the function g will be called *admissible.* Various conditions under which the function g is weakly admissible, resp. admissible, are described, for example, in [1], §5.1.5. In particular, if g is integrable on $I\!R$, and $\mathcal{F}^{-1}g \in L^1(I\!R) \cap BV(I\!R)$, then g is admissible. If g is weakly admissible, then we have for every $\lambda > 0$,

$$\begin{aligned} \left\| \sum_{j\in\mathbb{Z}} g\left(\frac{j}{\lambda}\right) e^{ij\cdot} \right\|_1 &= \lambda \int_0^{2\pi} \left| \sum_{k\in\mathbb{Z}} \mathcal{F}^{-1} g(\lambda(x+2k\pi)) \right| dx \\ &\le \lambda \int_{-\infty}^{\infty} |\mathcal{F}^{-1}g(\lambda x)| dx \\ &= \|\mathcal{F}^{-1}g\|_{1,I\!R}. \end{aligned} \tag{2.9}$$

In the sequel, the symbols $c, c_1, \cdots$ will denote positive constants depending only on the fixed parameters in question, and other explicitly indicated parameters, but independent of n and ψ_n.

Theorem 2.2 *Let $r \ge 0$, $n, q \ge 1$ be integers, and $\psi_n \in L^1$ be generated by a weakly admissible function $g_n : I\!R \longrightarrow \mathbb{C}$. Then*

$$\|f * \psi_n\|_p \le \|\mathcal{F}^{-1}g_n\|_{1,I\!R} E_{\alpha N,p}(f) + 2\|\mathcal{F}^{-1}(\chi_{\alpha,\beta}(\frac{n}{N}\cdot)g_n)\|_{1,I\!R}\|f\|_p. \tag{2.10}$$

If the function $h_n := h_{n,r}$, defined by

$$h_n(t) := \begin{cases} g_n(t)(it)^{-r-1}, & \text{if } t \ne 0, \\ 0, & \text{if } t = 0, \end{cases}$$

is admissible, and is in $BV^q(I\!R)$, then

$$\left| n^r(\psi_n * \Gamma_r)(x) - 2\pi\mathcal{F}^{-1}h_n(nx) \right| \le \frac{\pi^2 V({h_n}^{(q)}, I\!R)}{4(\pi n)^{q+1}}. \tag{2.11}$$

In particular, for $x \ne 0$, $x \in [-\pi, \pi]$,

$$n^r|(\psi_n * \Gamma_r)(x)| \le \frac{cV({h_n}^{(q)}, I\!R)}{(n|x|)^{q+1}}. \tag{2.12}$$

If $g_n =: g$ is independent of n, then there exists an interval $J := J(r, g)$ such that

$$n^r|(\psi_n * \Gamma_r)(x)| \ge c(g), \qquad nx \in J. \tag{2.13}$$

If $g(t) = 0$ for almost all $t \notin [-\tau, \tau]$ for some $\tau > 0$, and $\epsilon > 0$, we may choose the interval J with $|J| \ge 2(1-\epsilon)/\tau$, and obtain

$$n^r|(\psi_n * \Gamma_r)(x)| \ge c(g)\epsilon, \qquad nx \in J, \tag{2.14}$$

where c is independent of ϵ.

REMARK. In the case of the cannonical representation (2.1), Theorem 2.2 can be interpreted as follows. The estimates (2.13), (2.14) make precise the sense that $f * \psi_n$ is "large" near the singularities of f. In view of (2.12), the contribution to $f * \psi_n$ from these singularities decays as rapidly as the parameter q in this estimate indicates. The estimate (2.10) implies that the contribution $F * \psi_n$ is uniformly small. Thus, with a proper choice of g, one expects that the singularities will show up one-by-one as the peaks of $|f * \psi_n|$.

PROOF OF THEOREM 2.2. The estimate (2.10) follows from (2.4) and (2.9).

The Poisson summation formula (2.8) implies that

$$\begin{aligned} n^{r+1}(\psi_n * \Gamma_r)(x) &= \sum_{j\in\mathbb{Z}} h_n\left(\frac{j}{n}\right)e^{ijx} \\ &= 2\pi n \mathcal{F}^{-1}h_n(nx) + 2\pi n {\sum_{k\in\mathbb{Z}}}' \mathcal{F}^{-1}h_n(n(x+2k\pi)). \end{aligned}$$

Hence,

$$n^r(\psi_n * \Gamma_r)(x) = \int_{-\infty}^{\infty} h_n(t)e^{inxt}dt + 2\pi {\sum_{k\in\mathbb{Z}}}' \mathcal{F}^{-1}h_n(n(x+2k\pi)). \tag{2.15}$$

Since $h_n \in BV^q(I\!R)$, we may use integration by parts to obtain for $x \in [-\pi, \pi]$, $k \neq 0$,

$$\begin{aligned} \left|\mathcal{F}^{-1}h_n(n(x+2k\pi))\right| &= \left|\frac{1}{2\pi}\int_{-\infty}^{\infty} h_n(t)e^{in(x+2\pi k)t}dt\right| \\ &\leq \frac{V({h_n}^{(q)}, I\!R)}{2\pi(n|x+2k\pi|)^{q+1}} \\ &\leq \frac{V({h_n}^{(q)}, I\!R)}{2\pi(n\pi(2|k|-1))^{q+1}}. \end{aligned}$$

Therefore, for $x \in [-\pi, \pi]$,

$$\left|2\pi {\sum_{k\in\mathbb{Z}}}' \mathcal{F}^{-1}h_n(n(x+2k\pi))\right| \leq 2\frac{V({h_n}^{(q)}, I\!R)}{(\pi n)^{q+1}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^{q+1}}.$$

Along with (2.15) and

$$\sum_{k=1}^{\infty} \frac{1}{(2k-1)^{q+1}} \leq \sum_{k=1}^{\infty} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8},$$

this leads to (2.11). Since

$$\left|\int_{-\infty}^{\infty} h(t)\exp(inxt)dt\right| \leq \frac{V({h_n}^{(q)}, I\!R)}{(n|x|)^{q+1}}, \qquad x \neq 0,$$

the estimate (2.12) follows from (2.11).

Next, let $g_n = g$ be independent of n, and $h := h_n$. The function

$$G(z) := \int_{-\infty}^{\infty} h(t)\exp(izt)dt, \qquad z \in \mathbb{C},$$

is an entire function. Hence, there exists an interval J that is free of zeros of this function. This proves (2.13). If $h(t) = 0$ for almost all $t \notin [-\tau, \tau]$ for some $\tau > 0$, then G is an entire function of finite exponential type at most τ. Further, since $h \in L^1$, $G \in X^\infty(I\!R)$. Let $\|G\|_{\infty,I\!R} = |G(a)|$. Then the Bernstein inequality for entire functions of finite exponential type ([24], §4.8.2) implies that

$$|G(x) - G(a)| \le |x - a|\tau|G(a)|, \qquad x \in I\!R.$$

Hence, for $x \in [a - (1-\epsilon)/\tau, a + (1-\epsilon)/\tau]$, we have

$$|G(x) - G(a)| \le (1-\epsilon)|G(a)|;$$

i.e., $|G(x)| \ge \epsilon |G(a)|$.

□

Finally, we wish to examine the behavior of $\psi_n * \Gamma_r$ directly in terms of the Fourier coefficients of ψ_n, without refering to a generating function. If $\{a_j\}_{j\in Z\!\!Z}$ is a bi-infinite sequence, its forward differences are defined recursively by

$$\Delta^0 a_j := a_j, \quad \Delta^1 a_j := \Delta a_j := a_{j+1} - a_j,$$

$$\Delta^{q+1} a_j := \Delta(\Delta^q a_j), \qquad q = 1, 2, \cdots.$$

We say that a 2π-periodic, integrable function ψ has a *linear phase* if for some $a, b \in [-\pi, \pi]$,

$$\hat{\psi}(j) = |\hat{\psi}(j)| e^{-i(aj+b)}, \qquad \text{for all } j \in Z\!\!Z.$$

In this case, the number a will be called the *linear phase of* ψ. The following theorem describes the behavior of $\psi_n * \Gamma_r$ near and away from 0. In the case of the cannonical form (2.1), the estimates (2.16), (2.17) in the following theorem make precise the largeness of $f * \psi_n$ near the singularity, while the part (c) (along with Theorem 2.1) describes the decay of $f * \psi_n$ away from the singularities. An interesting feature of the parts (a) and (b) is that in the case of an even value of the order r of the singularity, $|f * \psi_n|$ has two "high peaks" straddling the singularity (cf. Figure 1(b), Figure 4, Figure 5(a)), while there is only one "high peak" near the singularity when r is odd (cf. Figure 2, Figure 3, Figure 5(b)).

Theorem 2.3 *Let $n, N \ge 1$, $r \ge -1$ be integers, and $\psi_n \in L^1$ have a linear phase a_n. If $r = 0$ or $r = -1$, we assume further that $\psi_n \in X^\infty$.*
(a) *If r is odd, then for $|x - a_n| \le \pi/(4N)$,*

$$|\psi_n * \Gamma_r(x)| \ge |\psi_n * \Gamma_r(a_n)| \cos(2N(x - a_n)) - 8E_{N,\infty}(\psi_n * \Gamma_r), \tag{2.16}$$

where $\psi_n * \Gamma_{-1} := \psi_n$.

(b)*Let r be an even integer, and ψ_n be an even function. Then for $|x - a_n| \le \pi/(4N)$,*

$$|\psi_n * \Gamma_r(x)| \ge |\psi_n * \Gamma_{r-1}(a_n)| \frac{|\sin(2N(x - a_n))|}{2N} - 2\pi \frac{E_{N,\infty}(\psi_n * \Gamma_{r-1})}{N}. \tag{2.17}$$

(c) *Let $q \geq 0$ be an integer, and*

$$b_{j,n,r} := \begin{cases} \dfrac{|\hat{\psi}_n(j)|}{(ij)^{r+1}}, & \text{if } j \in \mathbb{Z} \setminus \{0\}, \\ 0, & \text{if } j = 0. \end{cases}$$

If $\sum_{j \in \mathbb{Z}} |b_{j,n,r}| < \infty$, then for $x \neq a_n$,

$$|\psi_n * \Gamma_r(x)| \leq \frac{c}{|x - a_n|^{q+1}} \sum_{j \in \mathbb{Z}} |\Delta^{q+1} b_{j,n,r}|.$$

In order to prove this theorem, we first review some facts from the theory of Fourier series.

Proposition 2.1 (a) *Let $1 \leq p \leq \infty$, $f \in L^p$, $N \geq 1$ be an integer, and the de la Vallée Poussin mean of f be defined by*

$$v_N(f, x) := \frac{1}{N} \sum_{m=N+1}^{2N} s_m(f, x) = \sum_{|j| \leq N} \hat{f}(j) e^{ijx} + \sum_{N < |j| < 2N} (2 - \frac{|j|}{N}) \hat{f}(j) e^{ijx}.$$

Then

$$\|f - v_N(f)\|_p \leq 4 E_{N,p}(f).$$

(b) *Let $\{b_j\}_{j \in \mathbb{Z}}$ be a sequence that satisfies*

$$\sum_{j \in \mathbb{Z}} |j|^{2-1/p} |\Delta^2 b_j| < \infty.$$

Then $\sum_{j \in \mathbb{Z}} b_j e^{ijx}$ converges in the L^p norm, and we have

$$\| \sum_{j \in \mathbb{Z}} b_j e^{ij\cdot} \|_p \leq c \sum_{j \in \mathbb{Z}} |j|^{2-1/p} |\Delta^2 b_j|.$$

Proof. Part (a) is well known, for example, see [13], §7.1. Part (b) follows by a summation by parts argument, and the bounds on the norms of the Féjer kernel (cf. [17]). □

We were unable to find a reference for the following proposition.

Proposition 2.2 *Let $q \geq 0$ be an integer, $\{b_j\}$ be a bi-infinite, absolutely summable sequence. Then for $x \in [-\pi, \pi] \setminus \{0\}$,*

$$\left| \sum_{j \in \mathbb{Z}} b_j e^{ijx} \right| \leq \frac{c}{|x|^{q+1}} \sum_{j \in \mathbb{Z}} |\Delta^{q+1} b_j|.$$

We will prove this proposition using the method of the proof of Theorem 2.2. Thus, we will construct a function g such that $b_j = g(j)$, $j \in \mathbb{Z}$ and $V(g^{(q)}, \mathbb{R}) \le c \sum_{j\in\mathbb{Z}} |\Delta^{q+1} b_j|$. The most convenient way to do this is to construct a cardinal spline interpolant to the data $\{b_j\}$. The following discussion about the cardinal splines is based on [2].

Let $m \ge 2$ be an integer. The cardinal B-spline of order m is defined via its Fourier transform as follows:

$$\mathcal{F}N_m(u) := \exp(-imu/2)\Big(\frac{\sin(u/2)}{u/2}\Big)^m, \qquad u \in \mathbb{R}. \tag{2.18}$$

It is known that $N_m(x) = 0$ if $x \notin [0, m]$, $N_m(x) > 0$ if $x \in (0, m)$, and for all $x \in \mathbb{R}$, $\sum_{k\in\mathbb{Z}} N_m(x+k) = 1$ and $N_m(x+m/2) = N_m(m/2-x)$. The derivative of N_m is computed easily for $m \ge 3$:

$$N_m'(x) = N_{m-1}(x) - N_{m-1}(x-1). \tag{2.19}$$

The *Euler-Frobenius function* associated with the B-spline is defined by

$$\Phi_m(z) := \sum_{k\in\mathbb{Z}} N_m(\frac{m}{2} + k) z^k, \qquad z \in \mathbb{C}. \tag{2.20}$$

Obviously, $\Phi_m(z) = \Phi_m(z^{-1})$. It is also known that $\Phi_m(e^{it})$ is bounded above and below by positive constants depending only on m. Hence, the function $1/\Phi_m$ is analytic on the complex unit circle, and has a Laurent expansion

$$\frac{1}{\Phi_m(z)} = \sum_{\ell\in\mathbb{Z}} c_{\ell,m} z^\ell, \tag{2.21}$$

valid in a neighborhood of the unit circle. Clearly, the coefficients $c_{\ell,m}$ converge exponentially fast to 0 as $|\ell| \to \infty$. From (2.20) and (2.21), we conclude that

$$L_m(x) := \sum_{\ell\in\mathbb{Z}} c_{\ell,m} N_m(\frac{m}{2} + x - \ell)$$

is well defined for all $x \in \mathbb{R}$, and

$$L_m(j) = \begin{cases} 1, & \text{if } j = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, if $\{b_j\}$ is an absolutely summable sequence, then

$$J_m(x) := \sum_{\ell\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} b_{\ell-k} c_{k,m}\Big) N_m(\frac{m}{2} + x - \ell) \tag{2.22}$$

is well defined for all $x \in \mathbb{R}$, and satisfies $J_m(j) = b_j$, $j \in \mathbb{Z}$. If $m \ge 2$ then J_m is an admissible function.

Proof of Proposition 2.2. Without loss of generality, we may assume that $q \ge 1$. We consider the function J_{q+3} as defined in (2.22). Repeatedly using (2.19) in the formula (2.22), we get

$$J_{q+3}{}^{(q+1)}(x) = \sum_{\ell\in\mathbb{Z}} \Big(\sum_{k\in\mathbb{Z}} \Delta^{q+1} b_{\ell-q-1-k} c_{k,q+3}\Big) N_2(\frac{q+3}{2} + x - \ell),$$

and hence, that

$$V(J_{q+3}{}^{(q)}, I\!R) = \|J_{q+3}{}^{(q+1)}\|_{1,I\!R} \le \sum_{\ell\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\Delta^{q+1}b_{\ell-q-1-k}||c_{k,q+3}| \le c\sum_{\ell\in\mathbb{Z}}|\Delta^{q+1}b_\ell|.$$

Now, the Poisson summation formula implies that

$$\sum_{j\in\mathbb{Z}} b_j e^{ijx} = 2\pi\sum_{j\in\mathbb{Z}}\mathcal{F}^{-1}J_{q+3}(x+2j\pi), \qquad x\in[-\pi,\pi].$$

Since $J_{q+3}\in BV^q(I\!R)$, we deduce using integration by parts as in the proof of Theorem 2.2 that for $x\in[-\pi,\pi]$,

$$\begin{aligned}\left|\sum_{j\in\mathbb{Z}} b_j e^{ijx}\right| &\le c\Big(\sum_{j\in\mathbb{Z}}|x+2j\pi|^{-q-1}\Big)V(J_{q+3}{}^{(q)}, I\!R)\\ &\le \frac{cV(J_{q+3}{}^{(q)}, I\!R)}{|x|^{q+1}} \le \frac{c}{|x|^{q+1}}\sum_{\ell\in\mathbb{Z}}|\Delta^{q+1}b_\ell|.\end{aligned}$$

□

Finally, we prove a lemma concerning the behavior of a trigonometric polynomial near its absolute maximum on $[-\pi,\pi]$. Although the result is well known, we were unable to find a reference.

Lemma 2.1 *Let $m\ge 0$ be an integer, $T\in I\!H_m$, and $\|T\|_\infty = |T(x_0)|$ for some $x_0\in[-\pi,\pi]$. Then*

$$|T(x)| \ge |T(x_0)|\cos(m(x-x_0)), \qquad |x-x_0|\le\frac{\pi}{2m}. \tag{2.23}$$

PROOF. Without loss of generality, we may assume that $x_0=0$ and $\|T\|_\infty = T(0)$. Suppose

$$T(t_0) < T(0)\cos(mt_0) \tag{2.24}$$

for some $t_0\in[0,\pi/m)$. It is easy to verify that the polynomial $R(t) := T(t)-T(0)\cos(mt)$ has at least one zero in each of the intervals $[k\pi/m,(k+1)\pi/m]$, $k=1,\cdots,2m-2$, and a double zero at 0. If $R(\pi/m)=0$, then also $R'(\pi/m)=0$. Otherwise, $R(\pi/m)>0$, and (2.24) implies the existence of some $t_1\in(t_0,\pi/m)$ with $R(t_1)=0$. In either case, $R\in I\!H_m$ has $2m+1$ zeros on $[0,2\pi)$. Therefore, R is identically equal to zero, a contradiction to (2.24). Along with a similar argument for $[-\pi/m,0]$, we have proved that

$$T(x)\ge T(0)\cdot\cos(mx), \qquad |x|\le\frac{\pi}{m}.$$

Since $\cos(mx)\ge 0$ for $|x|\le\pi/(2m)$, this leads to (2.23). □

We are now in a position to prove Theorem 2.3.

PROOF OF THEOREM 2.3. Without loss of generality, we may assume that $a_n=0$, and that $\hat{\psi}_n(j)\ge 0$ for all $j\in\mathbb{Z}$. First, let r be an odd integer. By our assumption, the

Fourier coefficients of $v_N(\psi_n * \Gamma_r)$ are either all non-negative, or all non-positive. Hence, $\|v_N(\psi_n * \Gamma_r)\|_\infty = |v_N(\psi_n * \Gamma_r, 0)|$. Lemma 2.1 implies that

$$|v_N(\psi_n * \Gamma_r, x)| \geq |v_N(\psi_n * \Gamma_r, 0)| \cos(2Nx), \qquad |x| \leq \frac{\pi}{4N}. \tag{2.25}$$

Hence, using Proposition 2.1 (a) (with $p = \infty$), we obtain

$$\begin{aligned} |\psi_n * \Gamma_r(x)| &\geq |v_N(\psi_n * \Gamma_r, x)| - \|\psi_n * \Gamma_r - v_N(\psi_n * \Gamma_r)\|_\infty \\ &\geq |v_N(\psi_n * \Gamma_r, 0)| \cos(2Nx) - 4E_{N,\infty}(\psi_n * \Gamma_r) \\ &\geq |\psi_n * \Gamma_r(0)| \cos(2Nx) - 8E_{N,\infty}(\psi_n * \Gamma_r). \end{aligned}$$

This proves (2.16).

To prove part (b), we observe that since ψ_n is an even function,

$$\hat{\psi}_n(j) = \hat{\psi}_n(-j), \qquad j \in \mathbb{Z}.$$

Since r is an even integer, this implies that $v_m(\psi_n * \Gamma_r, 0) = 0$ for all integer $m \geq 1$, and hence, that $\psi_n * \Gamma_r(0) = 0$. We use (2.25) with $r-1$ in place of r, observe that $v_N(\psi_n * \Gamma_{r-1})$ does not change sign on $[-\pi/(4N), \pi/(4N)]$, and integrate to obtain

$$\left|\int_0^x v_N(\psi_n * \Gamma_{r-1}, t)dt\right| \geq |v_N(\psi_n * \Gamma_{r-1}, 0)| \frac{|\sin(2Nx)|}{2N}, \qquad |x| \leq \frac{\pi}{4N}.$$

Proposition 2.1 (a) now leads to (2.17).

The part (c) is only a restatement of Proposition 2.2. $\square$

3 Build-up methods

In this section, we study the analogues of the results in Section 2 in the case when the information available about the target function consists of the values of the function at finitely many points. The number of points will increase with the scale parameter n, but the points may be "scattered" in general.

If $Y \subset [-\pi, \pi]$ is a finite set, we define the *mesh norm* of Y by

$$\text{mesh}(Y) := \sup_{x \in [-\pi,\pi]} \min_{y \in Y} |x - y|.$$

Thus, for every $x \in [-\pi, \pi]$, there is some point in Y within a distance mesh(Y). The following proposition [16, Corollary 4.1] states an important fact which forms the basis of this section.

Proposition 3.1 *Let $K \geq 1$ be an integer.*

$$-\pi =: y_0 < y_1 < \cdots < y_K := \pi$$

be points on $[-\pi,\pi]$, $Y := \{y_j\}_{j=0}^K$, *and* $N \le \{4\mathrm{mesh}(Y)\}^{-1}$ *be an integer. Then there exist real numbers* $w_0,\cdots,w_{K-1}$ *such that*

$$|w_j| \le 2(y_{j+1}-y_j), \qquad j=0,\cdots,K-1, \tag{3.1}$$

and for every $T \in I\!H_N$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T(t)dt = \sum_{j=0}^{K-1} w_j T(y_j). \tag{3.2}$$

If $y_j = 2j\pi/K - \pi$, $j = 0,\cdots,K$, *we may choose* $N = K-1$, *and* $w_j = 1/K$, $j = 0,\cdots,K-1$.

Let Y, N, $\{w_j\}$, and K be as in Proposition 3.1. Analogous to the continuous convolution $f * g$, we define the Y-convolution of $f : Y \to \mathbb{C}$ and $g : I\!R \to \mathbb{C}$ by

$$f *_Y g(x) := \sum_{j=0}^{K-1} w_j f(y_j) g(x-y_j). \tag{3.3}$$

In this section, we are interested in the effect of approximating $f * \psi_n$ by the discrete sum $f *_Y \psi_n$ that depends only on the samples $\{f(y_j)\}$. In the applications of classical wavelet theory, one uses such approximation to compute the projection of f in a "high level" scaling space. The decomposition algorithm then helps one to compute the wavelet coefficients. In contrast, the results of this section will show that $f *_Y \psi_n$ itself will have the properties similar to the wavelet transform required for the detection of singularities. In applications, N and K will grow proportional to n. Thus, we may start with a small number of samples to compute $f *_Y \psi_1$, and take more and more samples as needed to calculate the higher transforms $f *_Y \psi_n$ (cf. Theorem 4.3 (b)). In particular, this eliminates the need to start with a large number of samples, or to investigate separately the error in evaluating a wavelet transform by numerical integration. (This error bound is "included" in the statements of the theorems in this section!) Strictly speaking, the parameter n is again superfluous for the discussion in this section, but we retain it for the same reasons as in Section 2.

We define the discrete L^p norm of $f : Y \to \mathbb{C}$ by

$$\|f\|_{p,Y} := \begin{cases} \left\{\displaystyle\sum_{j=0}^{K-1} |w_j||f(y_j)|^p\right\}^{1/p}, & \text{if } 1 \le p < \infty, \\ \displaystyle\max_{0\le j<K} |f(y_j)|, & \text{if } p = \infty, \end{cases}$$

and for $x \ge 0$, write

$$E_{x,p,Y}(f) := \min_{T\in I\!H_x} \|f-T\|_{p,Y}.$$

We note that Y being a finite set, there is no conflict of notation with the notation $\|\cdot\|_{p,A}$ for Lebesgue measurable sets A having a positive measure. If $g : I\!R \to \mathbb{C}$ is a 2π-periodic, absolutely continuous function, we define

$$|||g|||_{p,N} := ||g||_1^{1/p}\left\{4\pi||g||_1 + \frac{2\pi||g'||_1}{N}\right\}^{1-1/p}.$$

We note that if $g \in I\!\!H_{\alpha N}$ for some $\alpha > 0$, then the Bernstein inequality for trigonometric polynomials implies that

$$(4\pi)^{1-1/p}\|g\|_1 \le |||g|||_{p,N} \le (4\pi + 2\pi\alpha)^{1-1/p}\|g\|_1. \tag{3.4}$$

The analogue of Theorem 2.1 is the following.

Theorem 3.1 *Let Y, N, $\{w_j\}$, and K be as in Proposition 3.1, $n \ge 1$ be an integer, ψ_n be an absolutely continuous, 2π-periodic function, $1 \le p \le \infty$, $f : Y \to \mathbb{C}$, $0 < \alpha_1 < \beta_1 \le \alpha_2 < \beta_2$, $\alpha_1 + \beta_2 \le 1$. Then*

$$\begin{aligned}\|f *_Y \psi_n\|_p \ &\le\ E_{\alpha_1 N,p,Y}(f)|||\psi_n|||_{p,N} \\ &\quad +2\Big(|||\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n|||_{p,N} + |||\psi_n - \chi^\circ_{N,\alpha_2,\beta_2} * \psi_n|||_{p,N}\Big)\|f\|_{p,Y}.\end{aligned} \tag{3.5}$$

Remark. The utility of this theorem is most apparent in the case when $\psi_n \in I\!\!H_{\alpha_2 N}$. In this case, $\psi_n = \chi^\circ_{N,\alpha_2,\beta_2} * \psi_n$, and (3.4) yields

$$\|f *_Y \psi_n\|_p \le c\Big\{E_{\alpha_1 N,p,Y}(f)\|\psi_n\|_1 + \|\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n\|_1\|f\|_{p,Y}\Big\},$$

which has the same form as (2.4), except for the discrete norms and a constant factor.

In order to prove Theorem 3.1, we first prove two lemmas. In the following lemma, we give an error bound for an approximation of $\displaystyle\int_{-\pi}^{\pi} F(t)dt$ by $2\pi\sum_{j=0}^{K-1} w_j F(y_j)$ for any absolutely continuous function F on $[-\pi,\pi]$.

Lemma 3.1 *Let F be a 2π-periodic, absolutely continuous function, Y, N, K, and $\{w_j\}$ be as in Proposition 3.1. Then*

$$\sum_{j=0}^{K-1} |w_j F(y_j)| \le 2\sum_{j=0}^{K-1}(y_{j+1}-y_j)|F(y_j)| \le 4\pi\Big\{||F||_1 + \frac{\|F'\|_1}{2N}\Big\} = |||F|||_{\infty,N}. \tag{3.6}$$

Further,

$$\left|\frac{1}{2\pi}\int_{-\pi}^{\pi} F(t)dt - \sum_{j=0}^{K-1} w_j F(y_j)\right| \le c\frac{E_{N,1}(F')}{N}. \tag{3.7}$$

Proof. The first inequality in (3.6) follows from (3.1). It is easy to verify that

$$\frac{1}{2}\max_{0\le j<K}|y_{j+1}-y_j| = \max_{0\le j<K}\max_{x\in[y_j,y_{j+1}]}\min(|x-y_j|,|x-y_{j+1}|) \le \mathrm{mesh}(Y) \le \frac{1}{4N}.$$

Let $0 \le j \le K-1$ be an integer. Then

$$\begin{aligned}\left|\int_{y_j}^{y_{j+1}}|F(t)|dt - (y_{j+1}-y_j)|F(y_j)|\right| \ &\le\ \int_{y_j}^{y_{j+1}}|F(t)-F(y_j)|dt \\ &\le\ \int_{y_j}^{y_{j+1}}\int_{y_j}^{y_{j+1}}|F'(u)|dudt \\ &\le\ \frac{1}{2N}\int_{y_j}^{y_{j+1}}|F'(u)|du.\end{aligned}$$

Hence,

$$\begin{aligned}
\left|\int_{-\pi}^{\pi}|F(t)|dt-\sum_{j=0}^{K-1}(y_{j+1}-y_j)|F(y_j)|\right| &\leq \sum_{j=0}^{K-1}\left|\int_{y_j}^{y_{j+1}}|F(t)|dt-(y_{j+1}-y_j)|F(y_j)|\right| \\
&\leq \frac{1}{2N}\sum_{j=0}^{K-1}\int_{y_j}^{y_{j+1}}|F'(u)|du=\frac{2\pi}{2N}\|F'\|_1.
\end{aligned}$$

This leads to the second inequality in (3.6).

In view of (3.6) and (3.2), we obtain for any $T\in I\!H_N$,

$$\begin{aligned}
\left|\frac{1}{2\pi}\int_{-\pi}^{\pi}F(t)dt-\sum_{j=0}^{K-1}w_jF(y_j)\right| &= \left|\frac{1}{2\pi}\int_{-\pi}^{\pi}(F(t)-T(t))dt-\sum_{j=0}^{K-1}w_j(F(y_j)-T(y_j))\right| \\
&\leq c\left\{\|F-T\|_1+\frac{\|F'-T'\|_1}{N}\right\}. \qquad (3.8)
\end{aligned}$$

If we choose T to be the polynomial of best L^1-approximation to F from $I\!H_N$, then by Favard's theorem,

$$\|F-T\|_1\leq c\frac{E_{N,1}(F')}{N}. \qquad (3.9)$$

Also, a result of Czipszer and Freud [4] implies that for this T,

$$\|F'-T'\|_1\leq cE_{N,1}(F').$$

Thus, (3.8) leads to (3.7). □

In the following proposition, we summarize some of the important properties of the Y-convolution, which we will use often in the sequel.

Proposition 3.2 *Let Y, N, $\{w_j\}$, and K be as in Proposition 3.1.*
(a) *If $\alpha,\beta>0$, $\alpha+\beta\leq 1$, $T_1\in I\!H_{\alpha N}$, $T_2\in I\!H_{\beta N}$, then*

$$T_1 *_Y T_2 = T_1 * T_2.$$

(b) *Let $f: Y\to\mathbb{C}$, $g: I\!R\to\mathbb{C}$ be 2π-periodic and absolutely continuous, and $1\leq p\leq\infty$. Then*

$$\|f *_Y g\|_p\leq\|f\|_{p,Y}|||g|||_{p,N}. \qquad (3.10)$$

PROOF. Part (a) is clear from (3.2). From the definition (3.3), it follows that

$$\|f *_Y g\|_1\leq\|f\|_{1,Y}\|g\|_1.$$

From (3.6), we obtain

$$\|f *_Y g\|_\infty\leq\|f\|_{\infty,Y}\left\{4\pi||g||_1+\frac{2\pi\|g'\|_1}{N}\right\}.$$

The estimates (3.10) follow by applying the Riesz-Thorin interpolation theorem with the operator $f\to f *_Y g$. □

We can now give a proof of Theorem 3.1.

Proof of Theorem 3.1. Let $Q \in I\!\!H_{\alpha_1 N}$ be a polynomial of best approximation satisfying

$$\|f - Q\|_{p,Y} = E_{\alpha_1 N,p,Y}(f).$$

Then by (3.10),

$$\|(f - Q) *_Y \psi_n\|_p \le \|f - Q\|_{p,Y}|||\psi_n|||_{p,N} = E_{\alpha_1 N,p,Y}(f)|||\psi_n|||_{p,N}. \tag{3.11}$$

In contrast to the proof of Theorem 2.1, $Q *_Y \psi_n$ may not be equal to $Q *_Y (\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n)$. Nevertheless, since $\alpha_1 + \beta_2 \le 1$, Proposition 3.2 (a) shows that

$$Q *_Y (\chi^\circ_{N,\alpha_2,\beta_2} * \psi_n) = Q * (\chi^\circ_{N,\alpha_2,\beta_2} * \psi_n) = Q * (\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n) = Q *_Y (\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n).$$

Hence, using (3.10) again, we conclude that

$$\begin{aligned}
\|Q *_Y \psi_n\|_p &\le \|Q *_Y (\chi^\circ_{N,\alpha_2,\beta_2} * \psi_n)\|_p + \|Q *_Y (\psi_n - \chi^\circ_{N,\alpha_2,\beta_2} * \psi_n)\|_p \\
&= \|Q *_Y (\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n)\|_p + \|Q *_Y (\psi_n - \chi^\circ_{N,\alpha_2,\beta_2} * \psi_n)\|_p \\
&\le \|Q\|_{p,Y}\Big(|||\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n|||_{p,N} + |||\psi_n - \chi^\circ_{N,\alpha_2,\beta_2} * \psi_n|||_{p,N}\Big) \\
&\le 2\|f\|_{p,Y}\Big(|||\chi^\circ_{N,\alpha_1,\beta_1} * \psi_n|||_{p,N} + |||\psi_n - \chi^\circ_{N,\alpha_2,\beta_2} * \psi_n|||_{p,N}\Big).
\end{aligned}$$

Along with (3.11), this completes the proof. □

The analogue of Theorem 2.3 is the following.

Theorem 3.2 *Let Y, N, $\{w_j\}$, and K be as in Proposition 3.1. Further, let $n \ge 1$ be an integer, ψ_n be an absolutely continuous, 2π-periodic function having a linear phase a_n, and $y \in [-\pi, \pi]$.*

(a) *If $r \ge 1$ is an odd integer, then for $|x - y - a_n| \le \pi/(4N)$,*

$$|\Gamma_r(\cdot - y) *_Y \psi_n(x)| \ge |\psi_n * \Gamma_r(a_n)| \cos(2N(x - y - a_n)) - c\Big\{\frac{|||\psi_n|||_{\infty,N}}{N^r} + \frac{E_{N/4,1}(\psi_n')}{N}\Big\}.$$

(b) *Let $r \ge 0$ be an even integer. We assume that ψ_n is an even function. Then for $|x - y - a_n| \le \pi/(4N)$,*

$$|\Gamma_r(\cdot - y) *_Y \psi_n(x)| \ge |\psi_n * \Gamma_{r-1}(a_n)|\Big|\frac{\sin(2N(x - y - a_n))}{2N}\Big| - c\Big\{\frac{|||\psi_n|||_{\infty,N}}{N^r} + \frac{E_{N/4,1}(\psi_n')}{N}\Big\}.$$

(c) *Let $q \ge 0$, $r \ge 1$ be integers, and $\sum_{j\in\mathbb{Z}} |\hat\psi_n(j)| < \infty$. Then for $x - y \ne a_n$,*

$$\begin{aligned}
|\Gamma_r(\cdot - y) *_Y \psi_n(x)| \;\le\; & \frac{c}{|x - y - a_n|^{q+1}}\Big\{\sum_{j\in\mathbb{Z}} |\Delta^{q+1} b_{j,n,r}| + \frac{|||\psi_n|||_{\infty,N}}{N^{q+r+1}} \\
&+ \frac{1}{N^{r+1}} \sum_{j\in\mathbb{Z}} \big|\Delta^{q+1}\hat\psi_n(j)\big|\Big\} + c\frac{E_{N/4,1}(\psi_n')}{N},
\end{aligned} \tag{3.12}$$

where $b_{j,n,r}$ are as in Theorem 2.3 (c).

In order to prove Theorem 3.2, we first prove a lemma.

Lemma 3.2 *Let $1 \le p \le \infty$, $f \in L^p$, $m \ge 1$ be an integer, $0 < \alpha < \beta$, and $\chi^\circ_{m,\alpha,\beta}$ be as in (2.3). Then*

$$\|f - \chi^\circ_{m,\alpha,\beta} * f\|_p \le cE_{\alpha m,p}(f). \tag{3.13}$$

If $\ell \ge 1$ is an integer, and f is an ℓ-times iterated integral of a function $f^{(\ell)} \in L^p$, then

$$\|f^{(\ell)} - (\chi^\circ_{m,\alpha,\beta} * f)^{(\ell)}\|_p \le cE_{\alpha m,p}(f^{(\ell)}). \tag{3.14}$$

In particular, if f is absolutely continuous, then

$$|||f - \chi^\circ_{m,\alpha,\beta} * f|||_{p,N} \le c\frac{E_{\alpha m,1}(f')}{m}. \tag{3.15}$$

Further, if $q \ge 0$ is an integer, and $\{\hat f(j)\}$ is absolutely summable, then for $x \ne 0$, $x \in [-\pi, \pi]$, we have

$$|f(x) - \chi^\circ_{m,\alpha,\beta} * f(x)| \le \frac{c}{(m|x|)^{q+1}} \sum_{\ell=0}^{q+1} m^\ell \sum_{|j| \ge c_1 m} |\Delta^\ell \hat f(j)|. \tag{3.16}$$

Proof. It is easy to verify that for any integer $m \ge 1/\alpha$, the sequence of the Fourier coefficients of $\chi^\circ_{m,\alpha,\beta}$ satisfies the conditions of Proposition 2.1(b) with $p = 1$. Hence,

$$\|\chi^\circ_{m,\alpha,\beta}\|_1 \le c.$$

Since $T * \chi^\circ_{m,\alpha,\beta} = T$ for any $T \in I\!H_{\alpha m}$, this leads to (3.13) in a standard way. The estimate (3.14) follows from (3.13) and the fact that $(\chi^\circ_{m,\alpha,\beta} * f)^{(\ell)} = \chi^\circ_{m,\alpha,\beta} * f^{(\ell)}$. In view of (3.9),

$$E_{\alpha m,1}(f) \le c\frac{E_{\alpha m,1}(f')}{m}.$$

Consequently, (3.14) implies (3.15).

In this proof only, let $b_j := \hat f(j)$, and $g_j := 1 - \chi_{m,\alpha,\beta}(j)$. Since $\{b_j\}$ is assumed to be summable, $\sum_{j\in\mathbb{Z}} |b_j g_j| < \infty$. Using the Leibniz formula for forward differences, we obtain for any integer $L \ge 1$,

$$\Delta^L(g_j b_j) = \sum_{\ell=0}^{L} \binom{L}{\ell} \Delta^{L-\ell} g_j \Delta^\ell b_{j+L-\ell}.$$

The mean value theorem implies that $|\Delta^{q+1-\ell} g_j| \le cm^{\ell-q-1}$. Therefore,

$$\sum_{j\in\mathbb{Z}} |\Delta^{q+1}(g_j b_j)| \le cm^{-q-1} \sum_{\ell=0}^{q+1} m^\ell \sum_{|j|\ge c_1 m} |\Delta^\ell b_j|.$$

The estimate (3.16) now follows from Proposition 2.2. □

PROOF OF THEOREM 3.2. We observe that the set of points $\{y_j - y\}$ has the same mesh norm as Y, and for any $T \in I\!\!H_n$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} T(t)dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(t-y)dt = \sum_{j=0}^{K-1} w_j T(y_j - y).$$

Hence, substituting the set $\{y_j - y\}$ in place of Y, we may assume that $y = 0$. Further, by substituting $\psi_n(\cdot + a_n)$ in place of ψ_n, we may assume that $a_n = 0$. We will estimate $|\Gamma_r *_Y \psi_n(x) - \Gamma_r * \psi_n(x)|$. In this proof only, let $T_1 := \chi^\circ_{N,1/4,1/2} * \Gamma_r$, and $T_2 := \chi^\circ_{N,1/4,1/2} * \psi_n$. Both T_1 and T_2 are in $I\!\!H_{N/2}$. Hence, $T_1 *_Y T_2 = T_1 * T_2$, and we have

$$\begin{aligned} |\Gamma_r *_Y \psi_n(x) - \Gamma_r * \psi_n(x)| \ &\leq \ |(\Gamma_r - T_1) *_Y \psi_n(x)| + |T_1 *_Y (\psi_n - T_2)(x)| \\ &\quad + |(\Gamma_r - T_1) * \psi_n(x)| + |T_1 * (\psi_n - T_2)(x)|. \quad (3.17) \end{aligned}$$

Using (3.10), (3.13), and an obvious estimate on the degree of approximation of Γ_r (for example, using its Fourier series if $r \geq 1$), we obtain

$$\|(\Gamma_r - T_1) *_Y \psi_n\|_\infty \leq c\|\Gamma_r - T_1\|_\infty |||\psi_n|||_{\infty,N} \leq \frac{c}{N^r}|||\psi_n|||_{\infty,N}. \quad (3.18)$$

Similarly,

$$\|(\Gamma_r - T_1) * \psi_n\|_\infty \leq c\|\Gamma_r - T_1\|_\infty \|\psi_n\|_1 \leq \frac{c}{N^r}|||\psi_n|||_{\infty,N}. \quad (3.19)$$

Using (3.15) with $p = \infty$, we get

$$|||\psi_n - T_2|||_{\infty,N} \leq c\frac{E_{N/4,1}(\psi_n')}{N}.$$

The estimate (3.10) now implies that

$$\|T_1 *_Y (\psi_n - T_2)\|_\infty \leq c\|T_1\|_\infty |||\psi_n - T_2|||_{\infty,N} \leq c\frac{E_{N/4,1}(\psi_n')}{N}. \quad (3.20)$$

Similarly,

$$\|T_1 * (\psi_n - T_2)\|_\infty \leq c\|T_1\|_\infty \|\psi_n - T_2\|_1 \leq c|||\psi_n - T_2|||_{\infty,N} \leq c\frac{E_{N/4,1}(\psi_n')}{N}. \quad (3.21)$$

These estimates and (3.18), (3.19), and (3.17) imply that

$$\|\Gamma_r *_Y \psi_n - \Gamma_r * \psi_n\|_\infty \leq c\left\{\frac{|||\psi_n|||_{\infty,N}}{N^r} + \frac{E_{N/4,1}(\psi_n')}{N}\right\}.$$

Finally, we note that for any integer $\ell \geq 0$,

$$\begin{aligned} E_{N,\infty}(\Gamma_\ell * \psi_n) \ &\leq \ \|\Gamma_\ell * \psi_n - \chi^\circ_{N,1/2,1} * \Gamma_\ell * \psi_n\|_\infty \leq \|\Gamma_\ell - \chi^\circ_{N,1/2,1} * \Gamma_\ell\|_\infty \|\psi_n\|_1 \\ &\leq \ c\frac{|||\psi_n|||_{\infty,N}}{N^\ell}. \quad (3.22) \end{aligned}$$

Since

$$E_{N,\infty}(\psi_n) \leq \|\psi_n - \psi_n(0)\|_\infty \leq c\|\psi_n'\|_1 \leq cN|||\psi_n|||_{\infty,N},$$

the estimate (3.22) holds also if $\ell = -1$. The first two parts of the theorem now follow from the corresponding parts of Theorem 2.3.

The proof of part (c) is similar, but we need to use the pointwise estimate (3.16) of Lemma 3.2. Since $r \geq 1$, the sequence $\{\hat{\Gamma}_r(j)\}$ is absolutely summable. It is easy to verify that

$$m^\ell \sum_{|j|\geq c_1 m} |\Delta^\ell \hat{\Gamma}_r(j)| \leq c m^{-r}, \qquad \ell = 0, 1, \cdots.$$

Thus, for $|x - y_j| \leq |x|/2$, $|y_j| \geq |x|/2$, and we obtain that

$$|\Gamma_r(y_j) - T_1(y_j)| \leq c N^{-q-1-r} |x|^{-q-1}.$$

Hence, using (3.6), we deduce that

$$\left| \sum_{|x-y_j|\leq|x|/2} w_j(\Gamma_r(y_j) - T_1(y_j))\psi_n(x - y_j) \right| \leq \frac{c|||\psi_n|||_{\infty,N}}{N^{r+q+1}|x|^{q+1}}. \tag{3.23}$$

Next, suppose that y_j is such that $|x - y_j| \geq |x|/2$. Since the sequence $\{\hat{\psi}_n(\nu)\}$ is absolutely summable, Proposition 2.2 implies that

$$|\psi_n(x - y_j)| \leq c|x|^{-q-1} \sum_{\nu\in\mathbb{Z}} \left|\Delta^{q+1}\hat{\psi}_n(\nu)\right|.$$

Further, in view of (3.6), (3.15) (with $p = \infty$), we have

$$\sum_{j=0}^{K-1} |w_j(\Gamma_r(y_j) - T_1(y_j))| \leq |||\Gamma_r - T_1|||_{\infty,N} \leq c N^{-r-1}.$$

Therefore,

$$\left| \sum_{|x-y_j|\geq|x|/2} w_j(\Gamma_r(y_j) - T_1(y_j))\psi_n(x - y_j) \right| \leq c N^{-r-1} |x|^{-q-1} \sum_{j\in\mathbb{Z}} \left|\Delta^{q+1}\hat{\psi}_n(j)\right|.$$

Together with (3.23), this gives

$$|(\Gamma_r - T_1) *_Y \psi_n(x)| \leq \frac{c}{N^r |x|^{q+1}} \left\{ \frac{|||\psi_n|||_{\infty,N}}{N^{q+1}} + \frac{1}{N} \sum_{j\in\mathbb{Z}} \left|\Delta^{q+1}\hat{\psi}_n(j)\right| \right\}.$$

A similar estimate holds also for $|(\Gamma_r - T_1) * \psi_n(x)|$. In view of Theorem 2.3, these estimates and (3.20), (3.21), (3.17) lead to (3.12).

□

In the important case when $Y = \{y_{j,K} := 2\pi j/K - \pi\}_{j=0}^K$, we can obtain an asymptotic expression for $\Gamma_r *_K \psi_n := \Gamma_r *_Y \psi_n$, analogous to Theorem 2.2. For integer $r \geq 1$, we define the function

$$\mathcal{E}_{r,K}(x) := \frac{1}{i^{r+1}} \sum_{\nu\in\mathbb{Z}} \frac{(-1)^{\nu K}}{(x + \nu)^{r+1}}, \qquad x \in \mathbb{R} \setminus \mathbb{Z}.$$

It can be shown (cf. [2], pp. 195–196) using contour integration that

$$\mathcal{E}_{r,K}(x) = c(r,K)\frac{d^{r-1}}{dx^{r-1}}\cot(\pi x) = \frac{T_{r,K}(\pi x)}{\sin^{r+1}(\pi x)}, \qquad x \in I\!R \setminus Z\!\!Z,$$

where $T_{r,K}$ is an even trigonometric polynomial in $I\!H_r$. The analogue of Theorem 2.2 is the following.

Theorem 3.3 *Let $r, K, n, q \geq 1$ be integers, ψ_n be generated by g_n, and the function $h_{n,K,r}$, defined by*

$$h_{n,K,r}(t) := \begin{cases} g_n(t)\mathcal{E}_{r,K}(\frac{n}{K}t), & \text{if } t \neq K\ell/n,\ \ell \in Z\!\!Z, \\ 0, & \text{otherwise}, \end{cases}$$

be admissible and in $BV^q(I\!R)$. Further assume that

$$\hat{\psi}_n(\ell K) = g_n(\frac{K\ell}{n}) = 0, \qquad \ell \in Z\!\!Z. \tag{3.24}$$

Then

$$\left|\frac{K^{r+1}}{n}(\Gamma_r *_K \psi_n)(x) - 2\pi\mathcal{F}^{-1}h_{n,K,r}(nx)\right| \leq \frac{\pi^2 V((h_{n,K,r})^{(q)}, I\!R)}{8(\pi n)^{q+1}}. \tag{3.25}$$

In particular, for $x \neq 0$, $x \in [-\pi,\pi]$, and $K = \nu n$ for some integer $\nu \geq 1$, we have

$$n^r|(\Gamma_r *_K \psi_n)(x)| \leq \frac{c(\nu)V((h_{n,K,r})^{(q)}, I\!R)}{(n|x|)^{q+1}}.$$

If $g_n =: g$ is independent of n, and $K = \nu n$ for some integer $\nu \geq 1$, then there exists an interval $J := J(r,g,\nu)$ such that

$$n^r|(\Gamma_r *_K \psi_n)(x)| \geq c(\nu, g), \qquad nx \in J,$$

and if $g(t) = 0$ for almost all $t \notin [-\tau,\tau]$ for some $\tau > 0$, and $\epsilon > 0$, we may choose the interval J with $|J| \geq 2(1-\epsilon)/\tau$, and obtain

$$n^r|(\Gamma_r *_K \psi_n)(x)| \geq c(\nu, g)\epsilon, \qquad nx \in J,$$

where c is independent of ϵ.

Proof. It is easy to verify that

$$\frac{1}{K}\sum_{j=0}^{K-1}\exp(i\nu y_{j,K}) = \begin{cases} (-1)^\nu, & \text{if } \nu = \ell K,\ \ell \in Z\!\!Z, \\ 0, & \text{otherwise}. \end{cases} \tag{3.26}$$

Since $r \geq 1$, the Fourier series for Γ_r converges absolutely. Using (3.26), we obtain for $k \in Z\!\!Z$,

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}(\Gamma_r *_K \psi_n)(t)e^{-ikt}dt &= \frac{\hat{\psi}_n(k)}{K}\sum_{\nu\in Z\!\!Z}{}' \frac{1}{(i\nu)^{r+1}}\sum_{j=0}^{K-1}\exp(i(\nu-k)y_{j,K}) \\ &= \frac{\hat{\psi}_n(k)}{i^{r+1}}\sum_{\substack{\ell\in Z \\ k+\ell K\neq 0}}\frac{(-1)^{\ell K}}{(k+\ell K)^{r+1}}. \end{aligned}$$

Thus, (3.24) implies that $K^{r+1}(\Gamma_r *_K \psi_n)$ is generated by $h_{n,K,r}$. The proof of this theorem is now the same as that of Theorem 2.2, where we write $h_{n,K,r}$ in place of h_n in that proof. □

4 Applications

In this section, we discuss the detection of singularities using periodic spline wavelets constructed by Plonka and Tasche in [21], the trigonometric polynomial wavelets described in [23], and the classical Euler summability method (cf. [18]). We show that the spline wavelets are able to detect the singularities up to a certain order. The trigonometric polynomial wavelets (which are discussed here) can detect, in principle, the singularities of all orders, but have limited localization properties. The Euler summability method can also be used to detect singularities of all orders, but our analysis requires the "smooth part" F in (2.1) to possess a Hölder continuous derivative of the order indicated by the order of the singularity. We will describe the construction of a class of trigonometric polynomial frames which are free of any of these limitations. We will also describe a construction for the detection of singularities using the translates of a general "kernel function", similar to the ones described by Butzer and Nessel in [1].

4.1 The Euler summability kernel

In this section, we are interested in studying the function

$$\psi_n^E(x) := 2^{-n}\Big\{1 + \sum_{|k|\le n} \binom{n}{|k|} e^{ikx}\Big\} = 2\cos\frac{nx}{2}\cos^n\frac{x}{2}. \qquad (4.1)$$

In order to use Theorems 2.1 and 2.3, we first prove the following lemma.

Lemma 4.1 *Let $n \ge 1$, $r \ge -1$ be integers. We have*

$$cn^{-1/2} \le \|\psi_n^E\|_1 \le c_1 n^{-1/2}. \qquad (4.2)$$

Further,

$$c_1 n^{-r-1} \le 2^{-n}\sum_{k=1}^{n}\binom{n}{k} k^{-r-1} \le c_2 n^{-r-1}. \qquad (4.3)$$

PROOF. We observe that

$$\|\psi_n^E\|_1 \le \frac{4}{\pi}\int_0^{\pi/2} \cos^n y\,dy = \frac{4\Gamma(\frac{n+1}{2})}{\sqrt{\pi}n\Gamma(\frac{n}{2})} \le cn^{-1/2}.$$

To obtain the lower estimate, we need only estimate $\int_0^{\pi/4} \cos^n y|\cos ny|dy$ from below. Towards this end, we observe that $|\cos ny| + |\sin ny| \ge \cos^2 ny + \sin^2 ny = 1$ for all $y \in I\!R$, and hence,

$$\begin{aligned}\int_0^{\pi/4}\cos^n y|\cos ny|dy+\int_0^{\pi/4}\cos^n y|\sin ny|dy &\ge \int_0^{\pi/4}\cos^n y dy\\ \ge \int_0^{\pi/2}\cos^n ydy-c_1e^{-cn} &= \frac{4\Gamma(\frac{n+1}{2})}{\sqrt{\pi}n\Gamma(\frac{n}{2})}-c_1e^{-cn}\ge cn^{-1/2}.\end{aligned} \tag{4.4}$$

Now,

$$\begin{aligned}\int_0^{\pi/4}\cos^n y|\sin ny|dy &= \int_{\pi/(2n)}^{\pi/4+\pi/(2n)}\cos^n(y-\frac{\pi}{2n})|\cos ny|dy\\ &\le \int_0^{\pi/4}\cos^n(y-\frac{\pi}{2n})|\cos ny|dy+\frac{c}{n}.\end{aligned}$$

Further, using the mean value theorem, we obtain for $y\in[0,\pi/4]$ that

$$|\log\cos y-\log\cos(y-\frac{\pi}{2n})|\le\frac{c}{n},$$

and hence, that

$$\cos^n(y-\frac{\pi}{2n})\le c\cos^n y.$$

Thus,

$$\int_0^{\pi/4}\cos^n y|\sin ny|dy\le c\Big\{\int_0^{\pi/4}\cos^n y|\cos ny|dy+\frac{1}{n}\Big\},$$

and (4.4) leads to

$$\int_0^{\pi/4}\cos^n y|\cos ny|dy\ge cn^{-1/2}.$$

In turn, this implies the lower bound in (4.2).

The estimate (4.3) is proved in [18]. □

Theorems 2.1 and 2.3 now lead to the following.

Theorem 4.1 *Let $n\ge1$, $r\ge0$ be integers, $1\le p\le\infty$, $f\in L^p$. Then*

$$\|\psi_n^E*f\|_p\le c_1n^{-1/2}E_{n/3,p}(f)+c_2e^{-cn}\|f\|_p, \tag{4.5}$$

$$|\psi_n^E*\Gamma_r(x)|\ge\frac{c}{n^{r+1}}|\cos(2nx+\frac{r+1}{2}\pi)|,\qquad |x|\le\frac{\pi}{4n}, \tag{4.6}$$

and

$$|\psi_n^E*\Gamma_r(x)|\le c\cos^n\frac{x}{2}\le c\exp\Big(-\frac{nx^2}{8}\Big). \tag{4.7}$$

Proof. In Theorem 2.1, we take $\alpha=1/3$, $\beta=5/12$, $N=n$. Since (cf. [10], p. 201)

$$2^{-n}\sum_{|k|\le\frac{5n}{12}}\binom{n}{|k|}\le c_1e^{-cn},$$

we deduce that

$$\|\chi^{\circ}_{n,1/3,5/12} * \psi^E_n\|_1 \le c_1 e^{-cn}.$$

The estimates (4.2) and (2.4) now lead to (4.5). We observe that ψ^E_n is an even function, with linear phase equal to 0. Also, $\psi^E_n * \Gamma_\ell \in I\!H_n$ for every $\ell \ge -1$. Hence, $E_{n,\infty}(\psi^E_n * \Gamma_\ell) = 0$ for $\ell \ge -1$. In view of (4.3), Theorem 2.3 (with $N = n$) leads to (4.6). The first estimate in (4.7) is clear from the definition; the second follows from the fact that $\log\cos y + y^2/2$ is a concave function on $(-\pi/2, \pi/2)$. □

In the simple example where $f = \Gamma_r + F$, where r is an odd integer, and F is an r times continuously differentiable, 2π-periodic function, the estimates (4.5) and (4.7) show that

$$|\psi^E_n * f(x)| \le c_1\{n^{-1/2}E_{n/3,\infty}(F) + e^{-c_2 n}\|F\|_\infty + e^{-c_3 n x^2}\}, \qquad x \ne 0,$$

and

$$|\psi^E_n * f(x)| \ge c n^{-r-1}, \qquad |x| \le \frac{c}{n}.$$

Thus, the Euler method is expected to detect the singularity at 0 only if $E_{n/3,\infty}(F) = o(n^{-r-1/2})$; in particular, if $F^{(r)}$ is Hölder continuous with exponent greater than 1/2.

4.2 Periodic spline wavelets

In this section, we discuss the periodic spline wavelets introduced by Plonka and Tasche in [21]. Although not all of our theorems are applicable in this case, the methods yield quantitative estimates on the capacity of these wavelets to detect singularities.

Plonka and Tasche [21] define the periodic spline wavelets as follows. (In our exposition, we have adopted a slightly different notation, to make the various functions 2π-periodic, and have different normalizations.) Let M, m be fixed positive integers, $n = M_j = 2^j M$, and

$$\phi^S_n(x) := \sum_{k\in\mathbb{Z}} N_m\left(\frac{n}{2\pi}(x + 2\pi k)\right), \qquad x \in I\!R,\ j = 0, 1, \cdots. \tag{4.8}$$

The *scaling space* is defined by

$$V_n := \mathrm{span}\{\phi^S_n\left(\cdot - \frac{2\pi k}{n}\right) \ :\ k = 0, \cdots, n-1\}.$$

Plonka and Tasche proved in [21] that the spaces $\{V_n\ :\ n = 2^j M,\ j = 0, 1, 2, \cdots\}$ provide a "periodic multiresolution analysis", and that the wavelet space W_n, defined to be the orthogonal complement of V_n with respect to V_{2n}, is spanned by the translates of the function ψ^S_n defined as follows.

Let

$$\hat{\psi^S_n}(k) := Q\left(\exp\left(\frac{-2\pi i k}{M_{j+1}}\right)\right)\mathcal{F}N_m\left(\frac{2\pi k}{M_{j+1}}\right),$$

where, with the Euler-Frobenius function Φ_{2m} defined as in (2.20),

$$Q(z) := (-z)^{m-1}\left(\frac{1-z}{2}\right)^m \Phi_{2m}(-z).$$

The mother spline wavelet is defined by

$$\psi_n^S(x) := \sum_{k\in\mathbb{Z}} \hat{\psi}_n^S(k)e^{ikx}.$$

It is proved in [21] that

$$W_n = \text{span}\{\psi_n^S\Big(\cdot - \frac{2\pi k}{n}\Big) \ : \ k = 0, \cdots, n-1\}.$$

For a function $f \in L^2$, we may think of $(f * \psi_n^S)(2\pi k/n)$ as a wavelet coefficient in the expansion with respect to the dual wavelet basis for ψ_n^S. The following theorem gives the analogue of Theorem 2.2.

Theorem 4.2 *Let $m \geq 2$, $M \geq 1$, and $0 \leq r \leq m-2$ be integers. If F is an r times continuously differentiable, 2π-periodic function, then*

$$|(F * \psi_n^S)(\frac{2\pi k}{n})| = o(n^{-r}), \qquad k = 0, \cdots, n-1, \ n = 2^j M, \ j \to \infty, \tag{4.9}$$

where the constant involved may depend upon F, r, and m, but is independent of n and k. For $n \geq c(m,r)$ and $x \in [-\pi, \pi]$, we have

$$n^r(\psi_n * \Gamma_r)(x) = 2\pi \mathcal{F}^{-1}h(nx), \tag{4.10}$$

where

$$h(t) := \frac{(-1)^{m-1}i^{m-r-1}}{t^{r+1}} \exp(-(2m-1)i\pi t)\Big(\frac{\sin^2(\pi t/2)}{\pi t/2}\Big)^m \Phi_{2m}(-e^{-i\pi t}), \ t \neq 0,$$

and $h(0) := 0$. In particular, $\psi_n * \Gamma_r(x) = 0$ if $|x| \geq c/n$.

Proof. Plonka and Tasche have proved in [21] that

$$\psi_n^S(x) = n \sum_{k\in\mathbb{Z}} q_k \phi_{2n}^S\Big(x - \frac{\pi k}{n}\Big),$$

where

$$q_k := \begin{cases} (-1)^k 2^{1-m} \sum_{\ell=0}^{m} \binom{m}{\ell} N_{2m}(k+1-\ell), & \text{if } 0 \leq k \leq 3m-2, \\ 0, & \text{otherwise.} \end{cases}$$

Using this expression and the definition (4.8), it is not difficult to verify that

$$\|\psi_n^S\|_1 \leq c.$$

Hence, (2.4) leads to

$$\|f * \psi_n^S\|_\infty \leq c\|f\|_\infty, \qquad f \in L^\infty.$$

If $P \in V_n$, then $(P * \psi_n^S)(2\pi k/n) = 0$. Hence, the above estimate shows that

$$|(F * \psi_n^S)(\frac{2\pi k}{n})| \leq c \min_{P\in V_n} \|F - P\|_\infty.$$

The estimate (4.9) now follows from the known approximation properties of the spaces V_n (cf. [21]).

Next, we observe that

$$\hat{\psi_n^S}(k) = g(\frac{k}{n}), \qquad k \in \mathbb{Z},$$

where

$$g(t) := (-1)^{m-1} i^m \exp(-(2m-1)i\pi t)\Big(\frac{\sin^2(\pi t/2)}{\pi t/2}\Big)^m \Phi_{2m}(-e^{-i\pi t}).$$

The function h in the statement of the theorem is now seen to be $g/(i\cdot)^{r+1}$. As in the proof of Theorem 2.2, we have

$$n^r(\psi_n^S * \Gamma_r)(x) = \int_{-\infty}^{\infty} h(t) e^{inxt} dt + 2\pi {\sum_{k\in\mathbb{Z}}}' \mathcal{F}^{-1} h(n(x+2k\pi)).$$

Since $\Phi_{2m}(-e^{-i\pi t})$ is a trigonometric polynomial in t, we deduce from (2.18) that $\mathcal{F}^{-1}h$ is a finite linear combination of certain B-splines. In particular, it is a compactly supported function. Therefore, for sufficiently large n, $\mathcal{F}^{-1}h(n(x+2k\pi)) = 0$ for all $k \neq 0$ and $x \in [-\pi, \pi]$. This leads to (4.10). □

We observe that in the context of wavelet theory, we are interested only in the values of $\psi_n^S * f$ at the points $2\pi k/n$. Therefore, one cannot rule out the possibility that all these values may be small, in spite of (4.10). Our analysis requires the restriction $r \le m-2$. The slightly weaker restriction, that $r \le m-1$, is expected in analogy with the spline wavelets on the whole line.

4.3 Trigonometric polynomial frames

Let $q \ge 0$ be an integer, $g \in BV^q(\mathbb{R})$, $g(x) > 0$ if $x \in (1,2)$, and $g(x) = 0$ if $x \notin [1,2]$. We define the trigonometric polynomials

$$\psi_n^F(x) := \sum_{n \le |k| < 2n} g\Big(\frac{2|k|+1}{2n}\Big) e^{ikx}. \tag{4.11}$$

These polynomials can be used as frames as follows. Let $V_n := \mathbb{H}_{n-1}$, $W_n := V_{2n} \ominus V_n$ be the orthogonal complement of V_n in V_{2n}. We write

$$\psi_n^{F*}(x) := \sum_{n \le |k| < 2n} \Big(g\Big(\frac{2|k|+1}{2n}\Big)\Big)^{-1} e^{ikx}.$$

Then for any $T \in W_n$, and integer $N \ge 4n$, we have the representation (cf. [25], Chapter X, formula (2.5))

$$T(x) = T * \psi_n^F * \psi_n^{F*}(x) = \frac{1}{N}\sum_{\ell=0}^{N-1} (T * \psi_n^F)\Big(\frac{2\pi\ell}{N} - \pi\Big)\psi_n^{F*}\Big(x - \frac{2\pi\ell}{N} + \pi\Big). \tag{4.12}$$

We also have the following frame bounds which follow easily from (4.12):

$$\|\psi_n^{F*}\|_1^{-1}\|T\|_p \le \|T * \psi_n^F\|_p \le \|\psi_n^F\|_1 \|T\|_p, \qquad T \in W_n.$$

Further, in view of the well known Marcinkiewicz-Zygmund type inequalities for trigonometric polynomials, (cf. [25], Chapter X, Theorems 7.5, 7.28), we have

$$c_1\|T*\psi_n^F\|_p \le \left\{\frac{1}{N}\sum_{\ell=0}^{N-1}\left|(T*\psi_n^F)\left(\frac{2\pi\ell}{N}-\pi\right)\right|^p\right\}^{1/p} \le c_2\|T*\psi_n^F\|_p \qquad (4.13)$$

for $1<p<\infty$, and also for $p=1,\infty$ (with obvious modifications for the case $p=\infty$) if $N\ge 4(1+\delta)n$ for some $\delta>0$. (Since $T*\psi_n\in I\!H_{2n-1}$, the second estimate in (4.13) follows for all p from the remark on p. 30, Vol. 2 of [25]. The lower estimate is given only for $N=4n-1$, but in view of the formulas (2.5), (7.9) in Chapter X of [25], it is obvious that this lower estimate holds also with any $N\ge 2n$ in the case $1<p<\infty$. The cases $p=1,\infty$ are given explicitly in [25], Chapter X, Theorem 7.28.) We observe that for any $f\in L^1$,

$$f*\psi_n^F=(s_{2n}(f)-s_n(f))*\psi_n^F.$$

For $j=1,2,\cdots$, let $J=2^j$, $M_j\ge 4J$ be an integer,

$$C_{J,M_j,\ell}(f):=(f*\psi_J^F)\left(\frac{2\pi\ell}{M_j}-\pi\right),$$

and

$$\Psi^*_{J,M_j,\ell}(x):=\psi_J^{F*}\left(x-\frac{2\pi\ell}{M_j}+\pi\right).$$

For every $f\in L^2$, we thus have the frame expansion

$$f=\hat{f}(0)+\sum_{j=1}^{\infty}\frac{1}{M_j}\sum_{\ell=0}^{M_j-1}C_{J,M_j,\ell}(f)\Psi^*_{J,M_j,\ell},$$

where the convergence is understood in the sense of L^2.

In this section, we are interested in studying the behavior of $f*\psi_n^F$. If the data for the function is available only in the form of its values at a discrete set Y, we may approximate $f*\psi_n^F$ by $f*_Y\psi_n^F$ as in Section 3. The frame expansion will no longer be valid, but our theorems in Section 3 will enable us to examine the singularity detection also with this discrete Y-convolution.

Theorem 4.3 *Let $q\ge 1$ be an integer, $g\in BV^q(I\!R)$, $g(x)>0$ if $x\in(1,2)$, $g(x)=0$ if $x\notin(1,2)$, and ψ_n^F be defined as in (4.11). Let $r\ge 0$ be an integer, and $1\le p\le\infty$. In the following statements, all constants will depend upon g.*
(a) *For $f\in L^p$,*

$$\|f*\psi_n^F\|_p\le cE_{n-1,p}(f). \qquad (4.14)$$

We have

$$|\Gamma_r*\psi_n^F(x)|\ge cn^{-r}|\cos(4nx+\frac{r+1}{2}\pi)|, \qquad |x|\le\frac{\pi}{8n}, \qquad (4.15)$$

and the estimate (2.12) holds.
(b) *Let Y, N, $\{w_j\}$, and K be as in Proposition 3.1, and $8n\le N\le Ln$ for some $L\ge 8$. Then for $f\in L^p$,*

$$\|f*_Y\psi_n^F\|_p\le cE_{n-1,p}(f). \qquad (4.16)$$

There exists a constant $C > 0$ such that if $L > C$, $Cn \le N \le Ln$, then for $y \in [-\pi, \pi]$, we have

$$|\Gamma_r(\cdot - y) *_Y \psi_n^F(x)| \ge cn^{-r}|\cos(2N(x-y))|, \qquad |x-y| \le \frac{c}{n},\ r \text{ odd}\ , \tag{4.17}$$

and

$$|\Gamma_r(\cdot - y) *_Y \psi_n^F(x)| \ge cn^{-r}|\sin(2N(x-y))|, \qquad \frac{\pi}{4N} \le |x-y| \le \frac{c}{n},\ r \text{ even}. \tag{4.18}$$

Moreover, for $x \neq y$, and $r \ge 1$,

$$|\Gamma_r(\cdot - y) *_Y \psi_n^F(x)| \le cn^{-r-q-1}|x-y|^{-q-1}. \tag{4.19}$$

In the case when Y is as in Theorem 3.3, and $r \ge 1$, the asymptotic expression (3.25) holds with $K = 8n$.

Proof. Since g has a derivative having a bounded variation, and g is compactly supported, the mean value theorem implies that

$$\sum_{k\in\mathbb{Z}} |k||\Delta^2 g\Big(\frac{2|k|+1}{2n}\Big)| \le cV(g', I\!R).$$

Hence, Proposition 2.1(b) implies that $\|\psi_n^F\|_1 \le c$. In Theorem 2.1, we take $N = n$, $\alpha = 1 - 1/n$, and $\beta = 1$, and observe that $\chi^\circ_{n,1-1/n,1} * \psi_n^F = 0$ to arrive at (4.14). Since $g(t) = 0$ for $t \in [0,1] \cup [2,\infty)$, the function h_n as defined in Theorem 2.2 has a derivative of bounded variation, and hence, $\mathcal{F}^{-1}h_n$ is an integrable, entire function of exponential type at most 2. The Bernstein inequality for such functions implies, in particular, that $\mathcal{F}^{-1}h_n \in BV(I\!R)$. Thus, the hypotheses of Theorem 2.2 are satisfied. Hence, in particular, (2.12) holds. Further, in view of (2.11),

$$n^r(\Gamma_r * \psi_n^F)(0) \sim \int_1^2 \frac{g(t)}{\left(t - \frac{1}{2n}\right)^{r+1}} dt \sim c. \tag{4.20}$$

Using the Poisson summation formula, it is not difficult to see that (4.20) is valid also for $r = -1$. Now, In Theorem 2.3, we take $N = 2n$. Since $\psi_n^F * \Gamma_\ell \in I\!H_{2n-1}$ for all ℓ, $E_{2n,\infty}(\psi_n^F * \Gamma_\ell) = 0$ for $\ell = r, r-1$. In view of (4.20), the parts (a) and (b) of Theorem 2.3 now lead to (4.15). This completes the proof of part (a).

Since $\psi_n^F \in I\!H_{2n-1}$, and $\|\psi_n^F\|_1 \le c$, (3.4) implies that $|||\psi_n^F|||_{p,N} \le c$. Now, in Theorem 3.1, we take $N = 8n$, $\alpha_1 = (1/8)(1 - 1/n)$, $\alpha_2 = 1/4$, $\beta_1 = 1/8$, and $\beta_2 = 1/2$. Since $\hat{\psi_n^F}(k) = 0$ if $|k| \le n-1$, we have $\chi^\circ_{8n,\alpha_1,\beta_1} * \psi_n^F = 0$, and similarly, $\chi^\circ_{8n,\alpha_2,\beta_2} * \psi_n^F = \psi_n^F$. Hence, (3.5) leads to (4.16). Again, we observe that $E_{N/4,1}((\psi_n^F)') = 0$. In view of (4.20), the parts (a) and (b) of Theorem 3.2 now lead to (4.17) and (4.18) by taking $N \ge Cn$ for a sufficiently large value of C. Since $g^{(q)}$ has bounded variation, an application of the mean value theorem shows that with the notation as in Theorem 2.3(c),

$$\sum_{j=1}^{\infty} |\Delta^{q+1} b_{j,n,r}| \le cn^{-r-q-1}$$

and

$$\sum_{j\in\mathbb{Z}} \left|\Delta^{q+1}\hat{\psi}_n^F(j)\right| \le cn^{-q}.$$

The estimate (3.12) now leads to (4.19). Finally, since $g(t) = 0$ in the neighborhood of every point of the form $8j$, $j \in \mathbb{Z}$, the hypotheses of Theorem 3.3 are satisfied. □

Remark. In [23], a class of trigonometric polynomial wavelets is defined as follows. Let $d \in (0, 1/3)$, and

$$g(t) := e^{-it\pi/2} \times \begin{cases} 0, & \text{if } |t| \le 1-d, \\ \dfrac{d+|t|-1}{2d}, & \text{if } 1-d < |t| < 1+d, \\ 1, & \text{if } 1+d \le |t| \le 2-2d, \\ \dfrac{2d-|t|+2}{4d}, & \text{if } 2-2d < |t| < 2+2d, \\ 0, & \text{if } |t| \ge 2+2d. \end{cases}$$

The wavelets are now defined by

$$\psi_n^T(x) := \sum_{j\in\mathbb{Z}} g\left(\frac{j}{n}\right)e^{ijx}.$$

In particular, for any $f \in L^2$, and $n = N_j = 2^jM$, the numbers $f * \psi_n^T(k\pi/n)$, $k = -n+1, \cdots, n$, are the wavelet coefficients with respect to an expansion described in detail in [23]. Since $g \in BV^1(\mathbb{R})$ and is supported on $[-2d-2, -1+d] \cup [1-d, 2+2d]$, one can obtain a theorem similar to Theorem 4.3 describing the behavior of $f * \psi_n^T$ and $f *_Y \psi_n^T$, using exactly the same arguments. We omit the details, but observe again that in the context of wavelet theory, one is only interested in the wavelet coefficients $f * \psi_n^T(k\pi/n)$, $k = -n+1, \cdots, n$. Using (2.4), we conclude as before that for an r times continuously differentiable, 2π-periodic function F,

$$n^r|(F * \psi_n^T)(\frac{k\pi}{n})| = o(1), \qquad k = -n+1, \ldots, n, \quad n \to \infty.$$

Furthermore, the conditions of Theorem 2.2 are satisfied. In particular, we obtain from (2.12) that

$$n^r|(\Gamma_r * \psi_n^T)(\frac{k\pi}{n})| \le \frac{c}{k^2}, \qquad k = -n+1, \ldots, n, \ k \ne 0.$$

4.4 A general construction

Let $q \ge 1$ be an integer, $G : \mathbb{R} \to \mathbb{C}$ be an integrable function in $BV^q(\mathbb{R} \setminus [-c, c])$ for some $c > 0$, and $\mathcal{F}G \in BV(\mathbb{R}) \cap L^1(\mathbb{R})$. Then G is admissible and the function

$$\phi_n(x) := \sum_{k\in\mathbb{Z}} G(\frac{k}{n})e^{ikx} := \lim_{m\to\infty} \sum_{|k|<m} G(\frac{k}{n})e^{ikx} \tag{4.21}$$

has a bounded variation on $[-\pi, \pi]$ (cf. [1], Proposition 3.1.11, Proposition 5.1.29). In this section, we construct linear combinations of ϕ_n which can be used for the detection of singularities. In particular, our discussion will be applicable to the case when ϕ_n is the Poisson kernel

$$\frac{1-\rho^{2/n}}{1+\rho^{2/n}-2\rho^{1/n}\cos x},$$

for some $\rho \in (0,1)$. Many more examples of such kernels can be found in [1].

In this section only, let

$$\chi(x) := 1-\chi_{\pi/4,\pi/2}(x), \qquad x \in [-\pi,\pi],$$

and extended to $I\!R$ as a 2π-periodic function. Let

$$T_n(x) = \sum_{|k|<n} b_k e^{ikx} + b_n \cos nx$$

be chosen so as to satisfy

$$T_n(\frac{\pi k}{n}) = \chi(\frac{\pi k}{n}), \qquad k = 0, \cdots, 2n-1.$$

It is well known that

$$\|T_n - \chi\|_\infty \le c \log n E_{n,\infty}(\chi).$$

The theorem of Czipszer and Freud [4] now implies that

$$\|{T_n}^{(\ell)} - \chi^{(\ell)}\|_\infty \le c \log n E_{n,\infty}(\chi^{(\ell)}), \qquad \ell = 0, 1, \cdots.$$

Since χ is infinitely many times differentiable, this implies that $\|{T_n}^{(\ell)}\|_\infty \le c(\ell)$ for $\ell = 0, 1, \cdots$. We write

$$R_n(x) := |T_n(x)|^2.$$

Then $R_n \in I\!H_{2n}$, $R_n \ge 0$, $\|{R_n}^{(\ell)}\|_\infty \le c(\ell)$ for $\ell = 0, 1, \cdots$, and

$$R_n(\frac{\pi k}{n}) = \begin{cases} 0, & \text{if } |k| \le n/4, \\ 1, & \text{if } n/2 \le |k| \le n. \end{cases}$$

Finally, we let

$$\psi_n^G(x) := \sum_{k \in Z\!\!Z} R_n(\frac{\pi k}{n}) G(\frac{k}{n}) e^{ikx}. \tag{4.22}$$

It is easy to see that ψ_n^G is a finite linear combination of translates of ϕ_n.

Theorem 4.4 *Let $q \ge 1$, $G : I\!R \to \mathbb{C}$ be an integrable function in $BV^q(I\!R \setminus [-c,c])$ for some $c > 0$, and $\mathcal{F}G \in BV(I\!R) \cap L^1(I\!R)$. Let ϕ_n be defined by (4.21), and ψ_n^G be defined by (4.22). If $1 \le p \le \infty$ and $f \in L^p$, we have*

$$\|f * \psi_n^G\|_p \le c E_{n/5,p}(f). \tag{4.23}$$

With

$$g_n(t) := R_n(t\pi) G(t),$$

the hypotheses of Theorem 2.2 are satisfied.

Proof. We observe that ψ_n^G is generated by g_n. Since $g_n(t) = 0$ for $|t| \le 1/4$, we see that $\chi^\circ_{n,1/5,1/4} * \psi_n^G = 0$. The estimate (4.23) now follows from (2.4) with $\alpha = 1/5$, $\beta = 1/4$, and $N = n$. The fact, that the hypotheses of Theorem 2.2 are satisfied, is easy to verify. □

5 Numerical Experiments

The theoretical results similar to those discussed in Section 4 have been illustrated numerically in various contexts in [22], [8], [17], [18], [21], among others. For the convenience of the reader, we present a few simple results related to the trigonometric frames introduced in Section 4.3 and the Euler means discussed in Section 4.1. We restrict our attention to the spectral methods. Experiments suggest that the same results are obtained by discrete convolution methods based on equidistant data, provided that the number of sampling points is $4n$, where n is the scale parameter in ψ_n.

We illustrate our theory in Section 4.3 by analyzing the function

$$f(x) = 10\Gamma_2(x-1) + \Gamma_3(x+2), \tag{5.1}$$

having a singularity of an even order (2) at 1 and an odd order (3) at -2 (cf. Figure 1(a)).

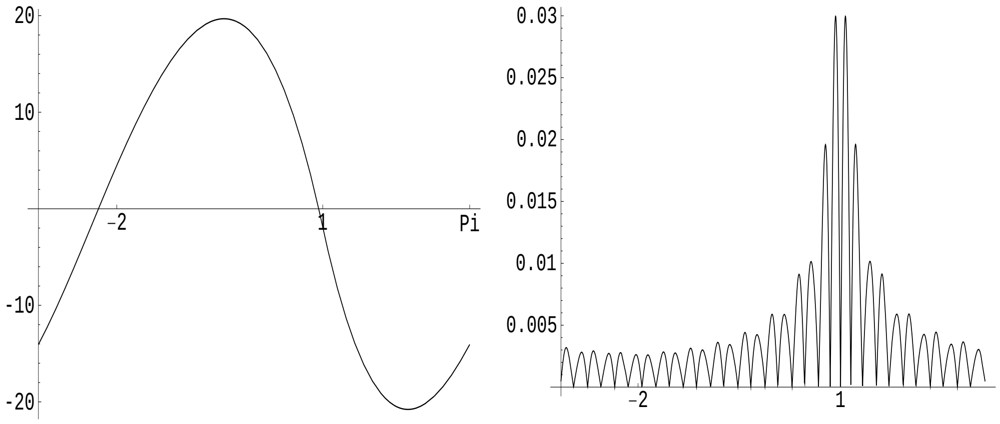

Figure 1: (a) $f(x) = 10\Gamma_2(x-1) + \Gamma_3(x+2)$, (b) $|f * \psi_{16}^{F,1}|$.

We shall compare $f * \psi_n^F$ with three different choices of the function g as in Section 4.3. The function g_1 is defined by

$$g_1(t) := \begin{cases} 50(|t|-1), & \text{if } 1 \le |t| < 1.02, \\ 1, & \text{if } 1.02 \le |t| \le 1.98, \\ 50(2-|t|), & \text{if } 1.98 \le |t| \le 2, \\ 0, & \text{otherwise.} \end{cases} \tag{5.2}$$

This is a piecewise differentiable function, with the derivative having a high total variation. The functions g_2 and g_3 are defined by

$$g_2(t) := N_2(2t-2), \qquad g_3(t) := N_5(5t-5), \tag{5.3}$$

where the B-spline N_m is defined in (2.18). The function g_2 is also piecewise differentiable, but with the derivative having a substantially smaller total variation than g_1'. The function g_3 is three times continuously differentiable. In this section, the function $\psi_n^{F,k}$ denotes the function ψ_n^F as in (4.11) with g_k in place of g.

As Figure 1(b) shows, the discontinuity in the second derivative, f'', at 1 is detected by two prominent peaks straddling 1, already with $f * \psi_{16}^{F,1}$, in spite of the fact that g_1 is not even once continuously differentiable, and the high total variation of g_1'. The main problem is the *simultaneous detection* of the singularity in the third derivative at 2. Figure 2 and Figure 3(a) show that all the frames are capable of such simultaneous detection for sufficiently large degrees of the trigonometric frames, and also that the smoother the function g, the smaller the required degree of the trigonometric frame. (Actually, the singularity at -2 could be detected already with $f * \psi_{32}^{F,3}$.) We notice also that because the singularity is of an odd order, there is only one peak in these figures. Figure 4 shows

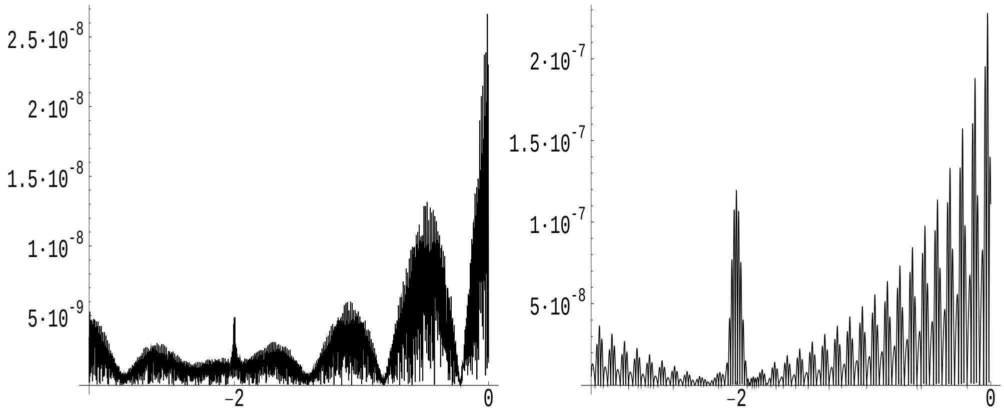

Figure 2: (a) $|f * \psi_{512}^{F,1}|$, (b) $|f * \psi_{128}^{F,2}|$.

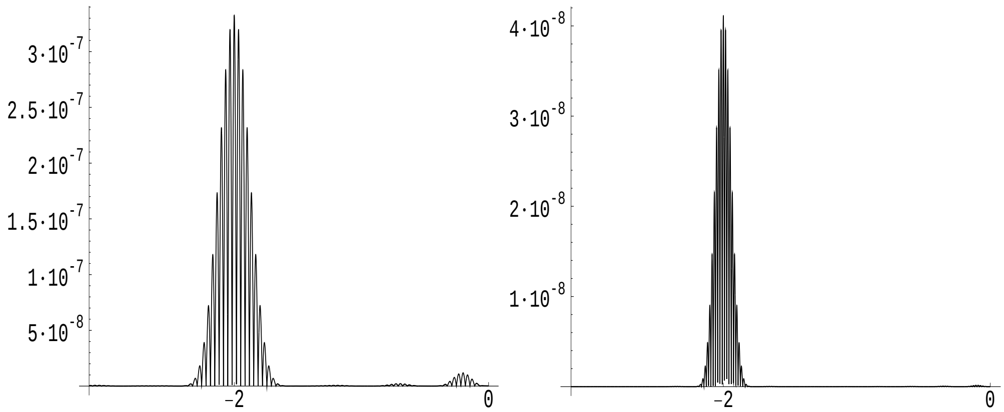

Figure 3: (a) $|f * \psi_{64}^{F,3}|$, (b) $|f * \psi_{128}^{F,3}|$.

the behavior of $|f * \psi_{16}^{F,k}|$ for $k = 2, 3$. Both of these have two prominent peaks, indicating the even order singularity at 1. It may be suprizing that the smoother choice of g has led to an apparent loss of localization. However, Figure 2(b) shows that $|f * \psi_{128}^{F,2}|$ remains relatively large even near 0 due to the singularity at 1, while Figure 3(a) shows that the trigonometric frame $|f * \psi_{64}^{F,3}|$ (having order equal to half of that of $|f * \psi_{128}^{F,2}|$) has very little of the residual oscillations near 0, and the peak at -2 is far more prominent than in Figure 2(b). Even these small residual oscillations near 0 are invisible in Figure 3(b).

Finally, in Figure 5, we demonstrate the simultaneous detection of both the singularities using the Euler means $f * \psi_{32}^{E}$, where ψ_{32}^{E} is defined by (4.1). We notice that the

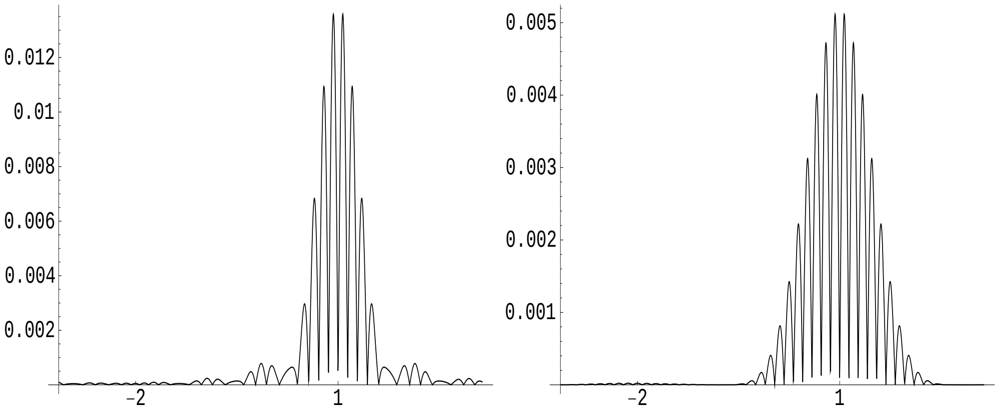

Figure 4: (a) $|f * \psi_{16}^{F,2}|$, (b) $|f * \psi_{16}^{F,3}|$.

exponential decay given in (4.7) allows us the detect both the singularities simultaneously even with a frame having a small order.

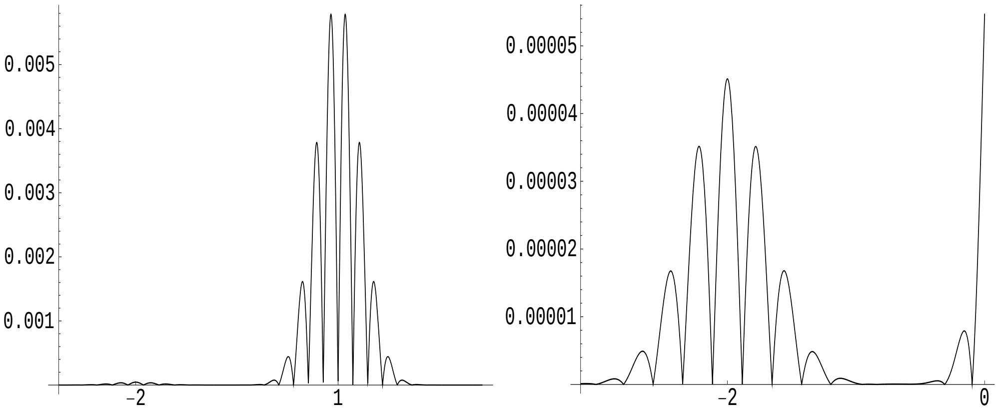

Figure 5: (a) $|f * \psi_{32}^{E}|$ on $[-\pi, \pi]$, (b) $|f * \psi_{32}^{E}|$ on $[-\pi, 0]$.

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