Accurate sampling formula for approximating the partial derivatives of bivariate analytic functions

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Abstract

The bivariate sinc-Gauss sampling formula is introduced in [8] to approximate analytic functions of two variables which satisfy certain growth condition. In this paper, we apply this formula to approximate partial derivatives of any order for entire and holomorphic functions on an infinite horizontal strip domain using only finitely many samples of the function itself. The rigorous error analysis is carried out with sharp estimates based on a complex-analytic approach. The convergence rate of this technique will be of exponential type and it has a high accuracy in comparison with the accuracy of the bivariate classical sampling formula. Several computational examples are exhibited, demonstrating the exactness of the obtained results.

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1. Introduction

A function $f : \mathbb{C}^2 \to \mathbb{C}$ is entire if it has an absolutely convergent Taylor series, cf. e.g. [18]. An entire function $f(z)$, $z \in \mathbb{C}^2$ is said to have an exponential type $\sigma$, $\sigma > 0$, if for each $\varepsilon > 0$ we can find a constant $A_{\varepsilon}$, independent of $z$, such that for all $z \in \mathbb{C}^2$,

$$|f(z)| \leq A_{\varepsilon} \mathrm{e}^{(\sigma + \varepsilon) \sum_{j=1}^{2} |z_j|},$$

where $z := (z_1, z_2)$, cf. e.g. [10, p. 98]. A two variable Paley-Wiener theorem, [10, p. 109], states that a function $f(z)$ is entire of exponential type $\sigma$, which belongs to $L^2(\mathbb{R}^2)$ when $z$ is restricted to $\mathbb{R}^2$ if its two variable Fourier transform vanishes outside $\Delta_{\sigma} := \{ x \in \mathbb{R}^2 : |x_j| < \sigma, j = 1, 2 \}$, where $x := (x_1, x_2)$. Such a space is called a Paley-Wiener space and will be denoted by $B^2_\sigma(\mathbb{R}^2)$. Many authors have investigated multivariate analogues of the Whittaker-Kotel’nikov-Shannon (WKS) sampling theorem, see e.g. [2, 9, 11, 12]. In particular, it is proved that if $f \in B^2_\sigma(\mathbb{R}^2)$, then

$$f(z) = \sum_{k \in \mathbb{Z}^2} f \left( \frac{k\pi}{\sigma} \right) \prod_{j=1}^{2} \mathrm{sinc} (\sigma z_j - k_j \pi),$$

(1.1)
where \( k := (k_1, k_2) \) and the series (1.1) converges uniformly on \( \mathbb{R}^2 \) and uniformly on any compact subset of \( \mathbb{C}^2 \). Here the sinc function is defined by

\[
sinc(t) := \begin{cases} 
\frac{\sin t}{t}, & t \neq 0, \\
1, & t = 0.
\end{cases}
\]

In [6], the authors established a derivative sampling theorem of the type (1.1) on \( \mathbb{R}^2 \). It is easy to check that this theorem also holds on \( \mathbb{C}^2 \). Therefore, we can rewrite it on the complex domain as follows. Let, here and throughout the paper, \( \alpha := (\alpha_1, \alpha_2), \alpha_1, \alpha_2 \in \mathbb{N}_0 \) and \( D^\alpha := \frac{\partial^{\alpha_1+\alpha_2}}{\partial z_1^{\alpha_1} \partial z_2^{\alpha_2}} \). If \( f \in B_2^2(\mathbb{R}^2) \), then for \( z \in \mathbb{C}^2 \) we have

\[
D^\alpha f(z) = \sum_{k \in \mathbb{Z}^2} f \left( \frac{k\pi}{\sigma} \right) D^\alpha \prod_{j=1}^{2} \text{sinc}(\sigma z_j - k_j \pi),
\]

which converges uniformly on \( \mathbb{R}^2 \) and uniformly on any compact subset of \( \mathbb{C}^2 \). The authors of [6] studied (1.2) to approximate partial derivatives of real-valued functions \( f \in B_2^2(\mathbb{R}^2) \), using only a finite number of samples from the function itself. Unfortunately, the convergence of the series (1.2) is slow unless \( |f(\mathbb{R}z_1, \mathbb{R}z_2)| \) decays rapidly as \( \mathbb{R}z_1, \mathbb{R}z_2 \to \pm \infty \). To increase the rate of convergence, a modification of series (1.1) is established in [8] using a Gaussian multiplier. Let us briefly describe this modification since it helps us to figure out the general series (1.2). Denote \( E_2^2(\varphi), \sigma > 0, \) to be the class of entire functions of two variables satisfying the following growth condition

\[
|f(z)| \leq \varphi(|\Re z_1|, |\Re z_2|) \exp \left( \sigma \sum_{j=1}^{2} |\Im z_j| \right), \quad z := (z_1, z_2) \in \mathbb{C}^2,
\]

where \( \varphi \) is a non-negative function on \( \mathbb{R}^2_+ \) and non-decreasing for each of the variables \( |\Re z_j|, j = 1, 2 \). For \( h \in (0, \pi/\sigma) \), set \( \beta := (\pi - \sigma h)/2 \) and let \( E^2 \) be the class of all entire functions on \( \mathbb{C}^2 \). For the class \( E_2^2(\varphi) \), in [8] we defined a localization operator \( \mathcal{G}_{h,N} : E_2^2(\varphi) \to E^2 \cap L^p(\mathbb{R}^2) \) via

\[
\mathcal{G}_{h,N}[f](z) := \sum_{k \in \mathbb{Z}^2} f(kh) \prod_{j=1}^{2} \text{sinc}(\pi h^{-1} z_j - k_j \pi) \exp \left( -\frac{\beta(z_j - k_j h)^2}{Nh^2} \right),
\]

where \( N \) is a positive integer, \( k := (k_1, k_2), z \in \mathbb{C}^2 \) and \( \mathbb{Z}^2_2(z) := \left\{ k \in \mathbb{Z}^2 : |h^{-1} \Re z_j + 1/2| - k_j \leq N, \; j = 1, 2 \right\} \).

There, we studied estimates for \( |f(z) - \mathcal{G}_{h,N}[f](z)| \) where \( f \in E_2^2(\varphi) \) and bounds of exponential order were found. In other words, if \( f \in E_2^2(\varphi) \), then for all \( z \in \mathbb{C}^2, |\Im z_j| < N, j = 1, 2 \) we have [8, Theorem 3.3]

\[
|f(z) - \mathcal{G}_{h,N}[f](z)| = O \left( \frac{\varphi(Nh, Nh)}{\sqrt{N}} \right), \quad \text{as} \quad N \to \infty.
\]

Moreover, in [8] another class \( A_2^2(\varphi) \) of complex functions is introduced. This class includes all analytic functions \( f : S_2^d \to \mathbb{C}^2 \) which satisfy the condition (1.3) with \( \sigma = 0 \) where \( S_2^d, d > 0 \), is the bivariate strip

\[
S_2^d := \left\{ z \in \mathbb{C}^2 : |\Im z_j| < d, \; j = 1, 2 \right\}.
\]

If \( f \in A_2^2(\varphi) \), then for all \( z \in S_2^d \) we have, [8, Theorem 3.6],

\[
|f(z) - \mathcal{G}_{h,N}[f](z)| = O \left( \frac{\exp(-\pi/4 N)}{\sqrt{N}} \right), \quad \text{as} \quad N \to \infty.
\]
The sampling operator \( G_{h,N} \) is called a two-dimensional sinc-Gauss sampling, see [8]. In [3], the first-named author has established a generalized two-dimensional Hermite-Gauss sampling operator involving samples from the function and its mixed and non-mixed partial derivatives. The one-dimensional sinc-Gauss sampling was studied by many authors, see e.g. Qian and his co-authors [13–15], Schmeisser and Stenger [16], Tanaka et al. [17] and Asharabi [5]. Moreover, the one-dimensional Hermite-Gauss sampling is established in [7].

In this paper, we investigate the approximation of partial derivatives \( D^\alpha f \) of arbitrary order \( \alpha \) using the partial derivatives of the operator (1.4), i.e.,

\[
D^\alpha G_{h,N}[f](z) := \sum_{k \in \mathbb{Z}_h^2(z)} f(kh)D^\alpha \left[ \prod_{j=1}^{2} \frac{\sin(\pi h^{-1}z_j - k_j \pi) \exp \left( -\frac{\beta(z_j - k_j h)^2}{Nh^2} \right)}{(w_j - z_j) \sin(\pi h^{-1}w_j)} \right],
\]

where \( D^\alpha \) is defined above. We assume that \( f \) lies in one of the classes \( E_2^2(\varphi) \) and \( A_2^2(\varphi) \). Bounds of the error \( |D^\alpha f(z) - D^\alpha G_{h,N}[f](z)| \) will be given for all functions from these classes. As we see in the sequel of the paper, the convergence rate of this formula is of an exponential order.

The paper is organized as follows: In the next section, we introduce some preliminary results which will be used in the proof of the main estimates. The main results of this paper will be presented in Sections 3 and 4. Section 3 is devoted to present a bound for the absolute error \( |D^\alpha f(z) - D^\alpha G_{h,N}[f](z)| \) when \( f \) belongs to the class \( E_2^2(\varphi) \). Some further useful results are shown. In Section 4, we extend this approach to functions from the class \( A_2^2(\varphi) \). Illustrative examples are given in Section 5 to demonstrate the accuracy of the proposed method. Finally, Section 6 concludes the paper.

2. Auxiliary results

This section is devoted to introduce two auxiliary results which will be used in the proof of the main results of the paper. The following kernel function is considered in [8]

\[
K_z(w) := S_z(w) \prod_{j=1}^{2} \frac{\exp \left( -\frac{\beta(z_j - w_j)^2}{Nh^2} \right)}{(w_j - z_j) \sin(\pi h^{-1}w_j)},
\]

where \( \beta, h \) are defined above, \( w := (w_1, w_2), z := (z_1, z_2) \) and

\[
S_z(w) := \sin(\pi h^{-1}z_1) \sin(\pi h^{-1}z_2) \left\{ \frac{\sin(\pi h^{-1}w_1)}{\sin(\pi h^{-1}z_1)} + \frac{\sin(\pi h^{-1}w_2)}{\sin(\pi h^{-1}z_2)} - 1 \right\}.
\]

We consider \( z_1, z_2 \) to be arbitrary fixed complex parameters and regard the kernel \( K_z \) as a function of the variables \( w_1, w_2 \). This kernel has a singularity of order one at all the points of the sets

\[
\{(z_1, C), (C, z_2) : z_1, z_2 \in C \} \cup \{(k_1 h, C), (C, k_2 h) \ : \ k_1, k_2 \in \mathbb{Z} \}.
\]

These sets are subsets of \( \mathbb{C}^2 \) and understood as the Cartesian product of the \( w_j \)-planes for \( j = 1, 2 \). On \( w_j \)-plane, let \( R_j \) be the positively oriented rectangle with vertices at \( \pm h \left( N + \frac{1}{2} \right) + hN_{h^{-1}z_j} + i(3z_j \pm hN) \) where \( N_{z_j} := \lfloor R_{z_j} + \frac{1}{2} \rfloor \). When \( (k_1, k_2) \in \mathbb{Z}_h^2(z) \), the rectangle \( R_j \) contains the singular points \( w_j = z_j \) and \( w_j = k_j h \) of the kernel \( K \) in its interior. In the following result, we show that the error of approximation of partial derivatives of functions from \( E_2^2(\varphi) \) by partial derivatives of the operator \( G_{h,N} \) can be written as the integral of the partial derivatives of the kernel \( K \) over the rectangles \( R_1 \) and \( R_2 \).

**Lemma 2.1.** Let \( f \in E_2^2(\varphi) \) and \( \alpha := (\alpha_1, \alpha_2) \in \mathbb{N}_0^2 \). Then we have for all \( z \in \mathbb{C}^2 \)

\[
D^\alpha f(z) - D^\alpha G_{h,N}[f](z) = \frac{1}{(2\pi i)^2} \oint_{R_2} \oint_{R_1} D^\alpha K_z(w)f(w) \, dw_1 \, dw_2,
\]
where \( D^2 \mathcal{K}_x(w) := \frac{\partial^2 \mathcal{K}_x(w)}{\partial z_1^2} \). The operator \( D^n \mathcal{G}_{h,N} \) and the kernel \( \mathcal{K}_x \) are given in (1.5) and (2.1), respectively.

**Proof.** Since \( f \in E_2^2(\varphi) \), we have [8, Lemma 3.2]

\[
 f(z) - \mathcal{G}_{h,N}[f](z) = \frac{1}{(2\pi i)^2} \int_{R_2} \int_{R_1} \mathcal{K}_x(w)f(w)\,dw_1 dw_2.
\]  

(2.3)

Taking partial derivatives to both sides of (2.3) and noting that the differentiation under the integral sign is allowed, we find (2.2).

For \( f \in E_2^2(\varphi) \), we define \( S_{s,m} \) to be

\[
 S_{s,m} := \int_{R_1} f(w_1, w_2) \partial_{z_1}^s \left\{ \exp \left( -\frac{\beta (z_1 - w_1)^2}{Nh^2} \right) \right\} (w_1 - z_1)^{m+1} \, dw_1,
\]

where \( z_1 \) lies inside the rectangle \( R_1 \), \( w_2 \) is a fixed point and \( s, m \in \mathbb{N}_0 \). This integral will appear in the proof of Theorem 3.1, which is the main result of the next section. The following lemma is devoted to estimate this integral.

**Lemma 2.2.** Let \( f \in E_2^2(\varphi) \) and let \( z_1 \) be inside the rectangle \( R_1 \). Then we have

\[
 |S_{s,m}| \leq \varphi \left( |\Re z_1| + h, |\Re w_2| \right) e^{\sigma(|z_1| + h)} e^{\sigma|z_2|} \sum_{k=0}^{m} \frac{1}{(m-k)!} h^k \left( \frac{\sqrt{\beta}}{\sqrt{Nh}} \right)^{s+m-k} |H_{s+m-k}|,
\]

(2.4)

where \( s, m \in \mathbb{N}_0 \), \( H_n \) is the Hermite number

\[
 H_n := \begin{cases} 
 0, & \text{if } n \text{ is odd,} \\
 (-1)^{n/2} (n-1)!!, & \text{if } n \text{ is even,} 
\end{cases}
\]

(2.5)

and \( n!! \) is the double factorial of \( n \).

**Proof.** Using Cauchy’s integral formula and Leibniz rule, we obtain

\[
 S_{s,m} = \frac{2\pi i}{m!} \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(z_1, w_2) \left\{ D^{(s,m-k)}_{z_1, w_1} \exp \left( -\frac{\beta (z_1 - w_1)^2}{Nh^2} \right) \right\}_{|w_1 = z_1}.
\]

The \( n \)-th degree Hermite polynomial is given by \( H_n(t) = (-1)^n e^{x^2} \frac{d^n}{dt^n} e^{-x^2} \), which yields

\[
 D^{(s,t)}_{z_1, w_1} e^{-\frac{\beta (z_1 - w_1)^2}{Nh^2}} = (-1)^s \left( \frac{\sqrt{\beta}}{\sqrt{Nh}} \right)^{s+t} H_{s+t} \left( \frac{\sqrt{\beta} (z_1 - w_1)}{\sqrt{Nh}} \right) e^{-\frac{\beta (z_1 - w_1)^2}{Nh^2}}.
\]

(2.6)

Using the monomial representation of the \( n \)-th degree Hermite polynomial

\[
 H_n(\zeta) = \frac{|n/2|}{\pi} \sum_{\ell=0}^{[n/2]} (-1)^{\ell} (2\zeta)^{n-2\ell} \ell!(n-2\ell)!
\]

we obtain

\[
 S_{s,m} = \frac{(-1)^s 2\pi i}{m!} \sum_{k=0}^{m} \binom{m}{k} f^{(k)}(z_1, w_2) \left( \frac{\sqrt{\beta}}{\sqrt{Nh}} \right)^{s+m-k} H_{s+m-k},
\]

(2.7)

where \( H_n \) is defined in (2.5). The derivatives \( f^{(k)}(z_1, w_2) \) can be represented, using Cauchy’s integral formula, as follows

\[
 f^{(k)}_{z_1}(z_1, w_2) = \frac{k!}{2\pi i} \int_{R_1} f(w_1, w_2) \frac{1}{(w_1 - z_1)^{k+1}} \, dw_1,
\]

(2.8)
where \( z_1 \) lies inside the rectangle \( R_1 \) and \( w_2 \) is a fixed point. We use the deformation theorem, cf. e.g. [1], to replace the rectangle \( R_1 \) in (2.8) by the positively oriented circle \( \gamma(z_1, h) := \{ z_1 \in \mathbb{C} : |w_1 - z_1| = h \} \). Since \( f \in E^2_\alpha(\varphi) \), inequality (1.3) holds. Using Cauchy estimates for (2.8) and applying inequality (1.3), we get

\[
|f^{(k)}(z_1, w_2)| \leq k! \varphi(\|z_1\| + h, |Re w_2|) \ e^{\sigma(|Re z_1| + h)} e^{\sigma(|Im w_2|)/(2\pi h^k)}. \tag{2.9}
\]

Combining (2.9) and (2.7) implies (2.4).

\[ \square \]

3. Sinc-Gauss approximation for \( D^\alpha f, f \in E^2_\alpha(\varphi) \)

This section is devoted to provide an error bound for partial derivatives of any order of \( f, f \in E^2_\alpha(\varphi) \) using the sampling operator \( D^\alpha \mathcal{G}_{h,N} \). In the following theorem, we study how well the operator \( D^\alpha \mathcal{G}_{h,N} \) approximates the corresponding derivative of the function \( f \) which belongs to the class \( E^2_\alpha(\varphi) \) and \( \alpha \in \mathbb{N}_0^2 \).

**Theorem 3.1.** Let \( f \in E^2_\alpha(\varphi) \) and \( \alpha = (\alpha_1, \alpha_2) \in \mathbb{N}_0^2 \). Then we have

\[
|D^\alpha f(z) - D^\alpha \mathcal{G}_{h,N} [f](z)| \leq 2 e^{\pi h^{-1}(|Re z_1| + |Im z_2|)} \varphi(a(z_1), a(z_2)) B_{h,N,\alpha}(\Im z) \frac{e^{-\beta N}}{\sqrt{\pi \beta N}}, \tag{3.1}
\]

where \( a(z_j) = |Re z_j| + h(N + 1) \) and \( z = (z_1, z_2) \in \mathbb{C}^2 \) such that \( |\Im z_j| < N, j = 1, 2 \). The function \( B_{h,N,\alpha} \) is given by

\[
B_{h,N,\alpha}(\Im z) := \sum_{\substack{k_1 + k_2 + s_1 = \alpha_1 \alpha_2! \left\{ e^{\sigma h} \sum_{l=1}^2 M_{h,N}(s_\ell) L_{h,N}(s_3 - \ell l_3 - \ell) \theta_N (h^{-1} \Im z_{\ell}) / N l_{\ell} \right\}} \]

\[
+ \frac{e^{-\beta N}}{\sqrt{\pi \beta N}} \prod_{\tau=1}^2 M_{h,N}(s_{\tau}) \theta_N (h^{-1} \Im z_{\tau}) / N l_{\tau}, \tag{3.2}
\]

where

\[
L_{h,N}(s, m) := \sum_{k=0}^m \frac{|H_{s+m-k}|}{(m-k)!} \left( \frac{\sqrt{\beta}}{\sqrt{ Nh} h \right)^{s+m-k} = O(N^{-s/2}), \text{ as } N \to \infty, \tag{3.3}
\]

\[
\theta_N(t) := \cosh(2\beta t) + \frac{2 e^{2\beta t}}{\sqrt{\pi \beta N} (1 - (t/N)^2)} + \frac{1}{2} \left[ \frac{e^{2\beta t}}{e^{2\pi (N-t)} - 1} + \frac{e^{-2\beta t}}{e^{2\pi (N+t)} - 1} \right] = \cosh(2\beta t) + O \left( N^{-1/2} \right), \text{ as } N \to \infty,
\]

and

\[
M_{h,N}(s) := \frac{(s/2)!^{s/2}}{h^s} \sum_{m=0}^{\left\lfloor s/2 \right\rfloor} \frac{s!}{m!(s-2m)!} N^m = (6\sqrt{\beta} h^{-1})^s + O(N^{-1}), \text{ as } N \to \infty. \tag{3.4}
\]

**Proof.** Since \( f \in E^2_\alpha(\varphi) \), the representation (2.2) holds. With the use of the generalized Leibniz rule,
the integral in (2.2) may be expanded to obtain the representation

\[ D^a f(z) - D^a G_{h,N}[f](z) \]

\[ = \frac{1}{(2\pi i)^2} \sum_{l=1,2} \left( \begin{array}{c} \sin (\pi h^{-1} z_1 + \pi k_1/2) / k_1! s_1! \end{array} \right) \int_{R_2} \int_{R_1} f(w) \prod_{\ell=1}^{2} \partial_{z_{\ell}} \left\{ \exp \left( -\frac{\beta (z_{\ell} - w_{\ell})^2}{N\pi^2} \right) \right\} \frac{dw_1 dw_2}{\sin (\pi h^{-1} w_1) \prod_{\ell=1}^{2} (w_{\ell} - z_{\ell})^{l+1}} \]

\[ + \frac{1}{(2\pi i)^2} \sum_{l=1,2} \left( \begin{array}{c} \sin (\pi h^{-1} z_2 + \pi k_2/2) / k_2! s_2! \end{array} \right) \int_{R_2} \int_{R_1} f(w) \prod_{\ell=1}^{2} \partial_{z_{\ell}} \left\{ \exp \left( -\frac{\beta (z_{\ell} - w_{\ell})^2}{N\pi^2} \right) \right\} \frac{dw_1 dw_2}{\sin (\pi h^{-1} w_2) \prod_{\ell=1}^{2} (w_{\ell} - z_{\ell})^{l+1}} \]

\[ + \frac{1}{(2\pi i)^2} \sum_{l=1,2} \left( \begin{array}{c} 2 \sin (\pi h^{-1} z_1 + \pi k_2/2) / k_1! s_1! \end{array} \right) \int_{R_2} \int_{R_1} f(w) \prod_{\ell=1}^{2} \partial_{z_{\ell}} \left\{ \exp \left( -\frac{\beta (z_{\ell} - w_{\ell})^2}{N\pi^2} \right) \right\} \frac{dw_1 dw_2}{\prod_{\ell=1}^{2} (w_{\ell} - z_{\ell})^{l+1} \sin (\pi h^{-1} w_{\ell})} \]

Since \( f \in E_\alpha^2(\varphi) \), we have from (1.3) for all points \( w \) in the bivariate rectangle \( R_1 \times R_2 \)

\[ |f(w)| \leq \varphi(b(z_1), b(z_2)) \prod_{j=1}^{2} e^{\sigma|3w_j|}, \]

where \( a(z_j) = |\Re z_j| + h(N + 1) \). Aside from that, for all \( w_j \in R_j \) we have

\[ \max (\varphi(|3z_1| + h, |\Re z_2|), \varphi(|\Re z_1|, |3z_2| + h)) \leq \varphi(a(z_1), a(z_2)), \]

because \( \varphi \) is non-decreasing for both of the variables. In view of (2.6) with \( l = 0 \), we have

\[ \left| \partial_{z_{\ell}} \left( e^{-\frac{\beta (z_{\ell} - w_{\ell})^2}{N\pi^2}} \right) \right| \leq \left( \frac{\sqrt{3}}{\sqrt{N h}} \right)^{s_{\ell}} \max_{w_{\ell} \in R_{\ell}} \left| H_{s_{\ell}} \left( \sqrt{3} (z_{\ell} - w_{\ell}) / \sqrt{N h} \right) \right| \frac{e^{-\frac{\beta (z_{\ell} - w_{\ell})^2}{N\pi^2}}}{s_{\ell}! s_{\ell}!} \]

The maximum of the Hermite polynomial on the rectangle \( R_\ell \) is estimated in [4] as follows

\[ \max_{w_{\ell} \in R_{\ell}} \left| H_{s_{\ell}} \left( \sqrt{3} (z_{\ell} - w_{\ell}) / \sqrt{N h} \right) \right| \leq \sum_{m=0}^{s_{\ell}/2} \frac{s_{\ell}! (6\sqrt{3})^{s_{\ell}/2}}{m! (s_{\ell} - 2m)!} \left( \frac{N^{s_{\ell}/2}}{s_{\ell}!} \right). \]

Applying the triangle inequality and using (2.4) and (3.5)–(3.9), we obtain

\[ |D^a f(z) - D^a G_{h,N}[f](z)| \leq \frac{1}{2\pi} \varphi(a(z_1), a(z_2)) e^{\sigma(|3z_1| + h)} \sin (\pi h^{-1} z_1 + \pi k_1/2) \]

\[ \times \sum_{l=1,2} \left( \frac{L_{h,N}(s_1, s_2, l) M_{h,N}(s_1)}{\prod_{\ell=1}^{2} k_{\ell}! s_{\ell}!} \right) \int_{R_2} \int_{R_1} \left| \exp \left( \sigma |w_{1}\left| - \frac{\beta (z_{\ell} - w_{\ell})^2}{N h} \right| \right) \right| \frac{dw_1}{(w_{1} - z_{1})^{l+1} \sin (\pi h^{-1} w_{1})} \]

\[ + \frac{1}{2\pi} \varphi(a(z_1), a(z_2)) e^{\sigma(|3z_1| + h)} \sin (\pi h^{-1} z_2 + \pi k_2/2) \]

\[ \times \sum_{l=1,2} \left( \frac{L_{h,N}(s_1, s_1, l) M_{h,N}(s_1)}{\prod_{\ell=1}^{2} k_{\ell}! s_{\ell}!} \right) \int_{R_2} \int_{R_1} \left| \exp \left( \sigma |w_{2}| - \frac{\beta (z_{\ell} - w_{\ell})^2}{N h} \right) \right| \frac{dw_2}{(w_{2} - z_{2})^{l+1} \sin (\pi h^{-1} w_{2})} \]

\[ + \frac{1}{4\pi^2} \varphi(a(z_1), a(z_2)) \sum_{l=1,2} \prod_{\ell=1,2} \left( \frac{\sin (\pi h^{-1} z_{\ell} + \pi k_{\ell}/2)}{k_{\ell}! s_{\ell}!} \right) \int_{R_2} \int_{R_1} \left| \exp \left( \sigma |w_{\ell}| - \frac{\beta (z_{\ell} - w_{\ell})^2}{N h} \right) \right| \frac{dw_{\ell}}{(w_{\ell} - z_{\ell})^{l+1} \sin (\pi h^{-1} w_{\ell})} \]
where \( L_{h,N}(s,m,k) \) and \( M_{h,N} \) are defined in (3.3) and (3.4), respectively. In [16, pp. 203-205] the integral in the right-hand side of (3.10) is estimated for \( l\ell = 0 \). Using the same technique, all the integrals in (3.10) can be estimated as

\[
\int_{R_{l\ell}} \left| \exp \left( \sigma|z_{l\ell}| - \frac{\beta (z_{l\ell} - z_{l\ell})^2}{N \ell} \right) \right| \, dw_{l\ell} \leq 4\pi \theta_N \left( \frac{e^{-\beta N}}{N^{l\ell} \sqrt{\pi \beta N}} \right), \tag{3.11}
\]

where the function \( \theta_N \) is defined in (3.4). Using the estimate \( |\sin (\pi h^{-1} z)| \leq e^{\pi h^{-1} |z|} \) and substituting from (3.11) into (3.10), we finally get (3.1).

The estimate (3.1) depends on the growth of the function \( \varphi \). In the following, we introduce two advantageous special cases of Theorem 3.1. The first result is assigned to \( \varphi := \|f\|_\infty \). In this case the class \( E_2^F (\|f\|_\infty) \) will be the Bernstein space \( B_{h}^\infty (\mathbb{R}^2) \). The Bernstein space \( B_{h}^\infty (\mathbb{R}^2) \) is the class of all entire functions of exponential type \( \sigma \) satisfying the growth condition

\[
|f(z)| \leq \|f\|_\infty \prod_{j=1}^{2} e^{\sigma |z_j|}, \quad z := (z_1, z_2) \in \mathbb{C}^2. \tag{3.12}
\]

**Corollary 3.2.** If \( f \in B_{h}^\infty (\mathbb{R}^2) \), we have for \( z \in \mathbb{C}^2, |\Im z_j| < N, j = 1, 2 \)

\[
|D^\alpha f(z) - D^\alpha \mathcal{G}_{h,N}[f](z)| \leq 2\|f\|_\infty e^{\pi h^{-1} (|\Im z_1| + |\Im z_2|)} B_{h,N,\alpha} (\Im z) \frac{e^{-\beta N}}{\sqrt{\pi \beta N}}. \tag{3.13}
\]

Moreover, this bound is uniform on \( \mathbb{R}^2 \), i.e.,

\[
|D^\alpha f(x) - D^\alpha \mathcal{G}_{h,N}[f](x)| \leq 2\|f\|_\infty B_{h,N,\alpha} (0) \frac{e^{-\beta N}}{\sqrt{\pi \beta N}}, \quad x \in \mathbb{R}^2, \tag{3.14}
\]

where \( B_{h,N,\alpha} \) is defined in (3.2).

**Proof.** Since \( f \in B_{h}^\infty (\mathbb{R}^2) \), inequality (3.12) is valid. If we choose \( \varphi \) as \( \|f\|_\infty \), we get (3.13). □

In the second result, \( \varphi \) has an exponential growth on the two axes of \( \mathbb{R}^2 \).

**Corollary 3.3.** Let \( f \) be an entire function satisfying

\[
|f(z)| \leq M \prod_{j=1}^{2} e^{2 \Xi_{z_j} + \sigma |\Im z_j|}, \quad z \in \mathbb{C}^2, \quad M > 0,
\]

where \( \sigma, \kappa \) are non-negative numbers and \( \sigma + \kappa \neq 0 \). Then we have for \( |\Im z_j| < N \)

\[
|D^\alpha f(z) - D^\alpha \mathcal{G}_{h,N}[f](z)| \leq 2 M e^{\pi h^{-1} (|\Im z_1| + |\Im z_2|)} e^{\kappa (h + |\Re z_1| + |\Re z_2|)} B_{h,N,\alpha} (\Im z) \frac{e^{-(\beta + \kappa) N}}{\sqrt{\pi \beta N} N}, \tag{3.15}
\]

where \( h \in (0, \pi/(\sigma + 2\kappa)) \) and \( B_{h,N,\alpha} \) is given by

\[
B_{h,N,\alpha} (\Im z) := \sum_{k_{j}+l_{j}+s_{j}=\alpha_{j}} f_{k_{j}l_{j}s_{j}} \left\{ e^{\pi h} \prod_{\ell=1}^{2} M_{h,N}(s_{\ell}^2) L_{h,N}(s_{3-\ell}, l_{3-\ell}) \theta_N (h^{-1} \Im z_{\ell}) \right\} N^{l_{\ell}}
\]

\[
+ \frac{e^{-(\beta + \kappa) N}}{\sqrt{\pi \beta N} N} e^{\pi h} \prod_{\tau=1}^{2} M_{h,N}(s_{\tau}) \theta_N (h^{-1} \Im z_{\tau}) \right\} N^{l_{\tau}}. \tag{3.16}
\]
Proof. In this case, we have

$$|f(w)| \leq e^c(|\Im z_1|+a(z_2))e^c(|\Im z_1|+|\Im w_2|), \quad w_2 \in R_2,$$

$$|f(w)| \leq e^c(a(z_1+|\Im z_2|))e^c(|\Im w_1|+|\Im z_2|), \quad w_1 \in R_1.$$  

Using these estimates in the proof of Theorem 3.1, we get (3.15) after we restrict \( h \) to be in the interval \((0, \pi/(\sigma+2\kappa))\). \( \square \)

Denote the expansion (1.2) by \( L_\alpha^0[f](z) \) such that \( L_\alpha^0 : B^2_\sigma(\mathbb{R}^2) \to B^2_\sigma(\mathbb{R}^2) \). In the following result, we show that the operator \( L_\alpha^0 \) is a limit case for the sinc-Gauss operator \( D^\alpha G_{h,N} \).

**Lemma 3.4.** Let \( \varphi \) be a constant function and \( h := \pi/\sigma \). Then we have

$$\lim_{N \to \infty} D^\alpha G_{h,N} f = L_\alpha^0 f = D^\alpha f, \quad \text{for all } f \in B^2_\sigma(\mathbb{R}^2) \subset E^2_\sigma(\varphi).$$

Proof. Since \( h := \pi/\sigma \), we have \( \beta = 0 \). Letting \( \beta = 0 \) in the operator \( D^\alpha G_{h,N} \) and taking \( N \to \infty \) implies the expansion (1.2). \( \square \)

4. **Sinc-Gauss approximation for** \( D^\alpha f, \ f \in A^2_d(\varphi) \)

This section is devoted to extend the results of the last section to the class \( A^2_d(\varphi) \) of analytic functions defined in Section 1. For functions of this class \( A^2_d(\varphi) \), we estimate the error \( |D^\alpha f(z) - D^\alpha G_{h,N} f(z)| \) in the special case \( h := \frac{d}{\pi} \) and \( \beta = \pi/2 \).

**Theorem 4.1.** Let \( f \in A^2_d(\varphi) \). Then we have for \( z \in S_{\pi/4}^d \)

$$\left| D^\alpha f(z) - D^\alpha G_{\varphi,N} f(z) \right| \leq 2\sqrt{2} \varphi \left( b(z_1), b(z_2) \right) \sum_{k_1+l_1+\ldots+l_2 = \alpha, \ j = 1, 2} \frac{\alpha_1!\alpha_2!}{k_1!k_2!s_1!s_2!} \left( \sum_{\ell=1}^2 M_{\varphi,N}(s_{\ell}) L_{\varphi,N}(s_{2-\ell}, l_{3-\ell}) \right) \left| \sin \left( \frac{\pi N z_\ell}{d} + \frac{\pi k_\ell}{2} \right) \right| \left( \frac{2 \sqrt{2}}{2 \pi N} \left( \frac{\pi N}{d} + \frac{\pi k_\ell}{2} \right) \right) \left( \frac{2 \sqrt{2} \sqrt{2}}{2 \pi N} \right) \left( \frac{2 \sqrt{2}}{2 \pi N} \right) \left( \frac{2 \sqrt{2}}{2 \pi N} \right),$$

where \( b(z_j) = |\Re z_j| + 2d, L_{\varphi,N} \) and \( M_{\varphi,N} \) are defined in (3.3), (3.4) and \( \vartheta_N \) is given by

$$\vartheta_N(t) = \frac{1}{1-t} \left( \frac{1}{1-e^{-2\pi N}} + \frac{2\sqrt{2}}{2\pi \sqrt{N}(1+t)} \right) = \frac{1}{1-t} \left( 1 + O(N^{-1/2}) \right), \quad \text{as } N \to \infty.$$  

Proof. Let \( R_j \) be the rectangle which has vertices at \( \pm h \left( N + \frac{1}{2} \right) + hNz_j/h + id \) and \( \pm h \left( N + \frac{1}{2} \right) + hNz_j/h - i(d - 3z_j) \). The Lemma 2.1 is also valid for the special operator \( D^\alpha G_{\varphi,N} \), which is defined on the class \( A^2_d(\varphi) \), with the rectangle \( R_j \). Applying the generalized Leibniz rule to (2.2) gives us (3.5) with \( h := \frac{d}{\pi} \) and \( R_j := R_j \). Since \( f \in A^2_d(\varphi) \), we have for all points \( w \) on the bivariate rectangle \( R_1 \times R_2 \)

$$|f(w)| \leq \varphi(b(z_1), b(z_2)).$$  

(4.2)
Applying the triangle inequality to (3.5) and using (2.4), (4.2), (3.8) and (3.9), we obtain

\[
\left| D^\alpha f(z) - D^\alpha G_{\mathbb{Z},N}[f](z) \right| \leq \frac{\alpha_1! \alpha_2!}{2\pi} \varphi(b(z_1), b(z_2)) \left| \sin \left( \frac{\pi N z_1}{d} + \frac{\pi k_1}{2} \right) \right| \\
\times \sum_{k_j + l_j + s_j = \alpha_j} \frac{L_{\mathbb{Z},N}(s_j, l_j) M_{\mathbb{Z},N}(s_{j-1})}{\prod_{l=1}^2 k_\ell ! s_\ell !} \int_{\mathcal{R}_\ell} \left| \exp \left( -\frac{\pi N (z_\ell - w_\ell)^2}{2d^2} \right) \right| \left| (w_\ell - z_\ell)^{l_\ell + 1} \sin \left( \frac{\pi N w_\ell}{d} \right) \right| \, |dw_\ell|,
\]

where \( L_{\mathbb{Z},N} \) and \( M_{\mathbb{Z},N} \) are defined in (3.3) and (3.4), respectively. The integral in the right-hand side of (4.3) is estimated in [16, pp. 209-211] for \( \ell_\ell = 0 \). Using the same technique, the integral in (4.3) can be estimated as follows:

\[
\int_{\mathcal{R}_\ell} \left| \exp \left( -\frac{\pi N (z_\ell - w_\ell)^2}{2d^2} \right) \right| \, |dw_\ell| \leq 4\sqrt{2} \vartheta_N \left( \frac{3z_\ell}{d} \right) \exp \left( -\frac{\pi N}{2} \left( 1 - \frac{2|z_\ell|}{d} \right) \right),
\]

where \( \vartheta_N \) is defined in (4.2). Combining (4.3) and (4.4) yields (4.1).

\[\square\]

**Remark 4.2.** The function \( \varphi \) does not affect the convergence rate of the error bound (4.1) because it is independent on \( N \) while the factor \( \left| \sin \left( \frac{\pi N z_\ell}{d} + \frac{\pi k_\ell}{2} \right) \right| \) affects the convergence because

\[
\left| \sin \left( \frac{\pi N z_j}{d} \right) \right| \leq \exp \left( \frac{\pi N |z_j|}{d} \right).
\]

Then the decisive factor in the error bound (4.1) becomes \( \exp \left( -\frac{\pi N}{2} \left( 1 - \frac{2|z_j|}{d} \right) \right) \), which guarantees convergence to zero as long as \( |z_j| \leq d/4 \). Moreover, the functions \( L_{\mathbb{Z},N} \) and \( M_{\mathbb{Z},N} \) have the following asymptotic behaviour

\[
L_{\mathbb{Z},N}(s, l) = O(N^{s/2 + 1}) \quad \text{as} \quad N \to \infty,
\]

\[
M_{\mathbb{Z},N}(s) = O(N^s) \quad \text{as} \quad N \to \infty.
\]

Those functions also affect the convergence rate of the bound (4.1). Therefore, when \( \alpha \) becomes large we need to select \( N \) to be large, too.

In the following result, we introduce a special case of the last theorem for the real domain.
Corollary 4.3. Let $f \in A^2_d (\|f\|_{\infty})$. Then we have the following uniform bound

$$|D^\alpha f (x) - D^\alpha G_{h,N}^\dagger [f](x)| \leq 4\sqrt{2} c_{\alpha,N,d} \|f\|_{\infty} \frac{e^{-\frac{\pi N}{2}}}{\pi N^{\frac{1}{2}}}, \quad x \in \mathbb{R}^2,$$

where $c_{\alpha,N,d}$ is given by

$$c_{\alpha,N,d} := \sum_{k_j + l_j + s_j = \alpha_j} \frac{\alpha_1! \alpha_2!}{k_1! k_2! s_1! s_2!} \left\{ \sum_{\ell = 1}^2 \frac{M_{\pi,N}(s_\ell) L_{\pi,N}(s_3-\ell, l_3-\ell)}{N^{2\ell}} + 2\sqrt{2} \frac{e^{-\frac{\pi N}{2}}}{\pi N} \prod_{\tau = 1}^2 \frac{M_{\pi,N}(s_\tau)}{N^{1\tau}} \right\}.$$

Proof. The result follows directly from Theorem 4.1. 

Let us mention, that there exist absolute constants $c_1(\alpha), c_2(\alpha)$ depending only on $\alpha$ such that

$$c_{\alpha,N,d} \leq c_1(\alpha) + c_2(\alpha) \left( \frac{N}{d} \right)^{\alpha_1 + \alpha_2}.$$

E.g., $c_{(0,0),N,d} < 4.4$, $c_{(1,0),N,d} = c_{(0,1),N,d} < 11.2 + 24.4 N/d$ and

$$c_{(1,1),N,d} < 81.7 + \frac{125.3 N^2}{d^2}.$$

Remark 4.4. The univariate analogies of the results of Theorem 3.1 and Theorem 4.1 are investigated in [4]. If we restrict the function $f(z_1, z_2) = g(z_1)$ in Theorem 3.1 to a univariate function, we obtain a bound of $|g^{(r)}(z_1) - G_{h,N}^{(r)}[g](z_1)|$ analogous to [4, Theorem 2.3] with the same order of convergence but somewhat different constants. The same applies to Theorem 4.1 and [4, Theorem 3.1].

5. Illustrative examples

This section is devoted to discuss three illustrative examples. The functions in Examples 1 and 2 are chosen from the space $E^2_2(\varphi)$ while in the last example $f \in A^2_d(\varphi)$. We summarize the numerical results in tables and illustrate the absolute and relative errors by figures. As predicted by the error estimates, the accuracy increases when $N$ is fixed but $h$ decreases without any additional cost except that the partial derivatives $D^\alpha f$ are approximated on a smaller domain.

Example 5.1. Consider the function

$$f(z) = \cos(z_1 + z_2), \quad z = (z_1, z_2) \in \mathbb{C}^2,$$

which belongs to the Bernstein space $B^\infty_2 (\mathbb{R}^2)$. In this example, we apply Corollary 3.2 with $h = 0.3$ and $N = 12$. The function $\varphi$ is chosen to be constant and the bound is uniform. Let $\mathcal{B}^{\alpha}_{h,N}$ be the uniform bound in (3.14), i.e.,

$$\mathcal{B}^{\alpha}_{h,N} := 2\|f\|_{\infty} B_{h,N,\alpha}(0) \frac{e^{-\beta N}}{\sqrt{\pi \beta N}},$$

where $B_{h,N,\alpha}$ is defined in (3.2). In Table 1, we display the maximum of the absolute errors on the region $[0, 4] \times [0, 4]$ and the uniform bounds computed from the error estimates. Figures 1, 2 show the graphs of the absolute error $|D^\alpha f(z) - D^\alpha G_{h,N}^\dagger [f](z)|$ on the region $[0, 4] \times [0, 4]$ for $h = 0.3$ and $N = 12$. 

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Absolute error for $D^\alpha G_{h,N}$ with $h = 0.3$ and $N = 12$, $\mathcal{J} = [0,4] \times [0,4]$

| $\alpha$ | $\max_{x \in \mathcal{J}} |D^\alpha f(x) - D^\alpha G_{h,N}[f](x)|$ | $\mathcal{B}_h,N^\alpha$ | $\alpha$ | $\max_{x \in \mathcal{J}} |D^\alpha f(x) - D^\alpha G_{h,N}[f](x)|$ | $\mathcal{B}_h,N^\alpha$ |
|---------|--------------------------------------------------|-----------------|---------|--------------------------------------------------|-----------------|
| $(1,0)$ | $5.7689\times 10^{-8}$                           | $6.261\times 10^{-7}$ | $(0,2)$ | $1.01454\times 10^{-6}$                          | $1.680\times 10^{-5}$ |
| $(1,1)$ | $1.06403\times 10^{-7}$                          | $4.731\times 10^{-6}$ | $(1,2)$ | $1.07863\times 10^{-6}$                          | $8.918\times 10^{-5}$ |

Table 1: Approximation of partial derivatives of the cosine function.

| Figure 1: $D^{(0,1)} f - D^{(0,1)} G_{h,N}[f]$. | Figure 2: $D^{(1,2)} f - D^{(1,2)} G_{h,N}[f]$. |

Example 5.2. In this example, we approximate the function $f(z) = \sinh(z_1 + z_2)$, $z = (z_1, z_2) \in \mathbb{C}^2$, which satisfies the inequality $|f(x)| \leq e^{\|x_1\|+\|x_2\|}$ on $\mathbb{R}^2$. Here we apply Corollary 3.3 with $\sigma = 0$, $\kappa = 1$, $M = 1$ and $h = 0.5$. Note that $f$ has exponential growth on $\mathbb{R}^2$ and the samples are exponentially increasing with respect to the real axes $x_1, x_2$. Consequently, the absolute errors increase when the samples are increasing but the relative errors are nearly constant for fixed $N$. Let $R_{h,N,\alpha}$ be the relative bound associated with the real-valued bound in (3.15), i.e.,

$$R_{h,N,\alpha}^\alpha(x) := 2M \exp^{-h^{-1}(\|x_1\|+\|x_2\|)e^{\alpha(\|x_1\|+\|x_2\|)}}B_{h,N,\alpha}(0)\frac{e^{-(\beta-h \kappa)N}}{\sqrt{\pi \beta N}}/D^\alpha f(x),$$

where $B_{h,N,\alpha}$ is given in (3.16). In Table 2, we show the numerical results with the relative errors and the graphs of the relative errors, which we denote by

$$T_{h,N,\alpha}^\alpha[f](x) := (D^\alpha f(x) - D^\alpha G_{h,N}[f](x)) / D^\alpha f(x),$$

are given in Figures 3 and 4 for some values of $\alpha$. Note that $R_{h,N,\alpha}^\alpha[f](x)$ has not any poles in the region $(0,\infty) \times (0,\infty)$ because, $D^\alpha f(x)$ is either $\cosh(x_1 + x_2)$ when $\alpha_1 + \alpha_2$ is odd or $\sinh(x_1 + x_2)$ when $\alpha_1 + \alpha_2$ is even.

Example 5.3. The function $f(z) = \frac{1}{(z_1^2 + 9)(z_2^2 + 9)}$, $z = (z_1, z_2) \in \mathbb{C}^2$,
Relative error for $D^\alpha \mathcal{G}_{h,N}$ with $h=0.5$ and $N=15$, $J=[0,5] \times [0,5]$

| $\alpha$ | $\max_{x \in J} |T_{h,N}^0[f](x)|$ | $\max_{x \in J} |R_{h,N}^0[f](x)|$ | $\max_{x \in J} |T_{h,N}^1[f](x)|$ | $\max_{x \in J} |R_{h,N}^1[f](x)|$ |
|----------|-----------------|-----------------|-----------------|-----------------|
| $(0,1)$  | $7.6886 \times 10^{-10}$ | $5.5073 \times 10^{-8}$ | $1.200 \times 10^{-8}$ | $1.05042 \times 10^{-4}$ |
| $(0,2)$  | $2.28996 \times 10^{-6}$ | $9.5073 \times 10^{-8}$ | $1.200 \times 10^{-8}$ | $1.4837 \times 10^{-4}$ |
| $(1,1)$  | $3.04538 \times 10^{-4}$ | $1.200 \times 10^{-8}$ | $1.200 \times 10^{-8}$ | $1.4837 \times 10^{-4}$ |

Table 2: Approximation of partial derivatives of the hyperbolic sine function.

![Figure 3](image1.png) Figure 3: $T_{h,N}^{(0,1)}[f](x)$.

![Figure 4](image2.png) Figure 4: $T_{h,N}^{(1,2)}[f](x)$.

is analytic in the bivariate strip $S_2^2$. Here we apply Theorems 4.1 with $\varphi = \|f\|_{\infty}$, $d = 1$ and $N = 15$. Note that $f$ decreases on $\mathbb{R}^2$ and the sample values are decreasing in the two axes $x_1, x_2$. We denote by $\mathcal{R}_h^\alpha_N(z)$ to be the bound in (4.1). As we see in Corollary 4.3, the bound $\mathcal{R}_h^\alpha_N(x)$, $x \in \mathbb{R}^2$, will be uniform when $\varphi$ is a constant function. First, we approximate the partial derivatives of $f$ at real domains using the operator $\mathcal{G}_{\frac{\alpha}{2},N}$. The results are shown in Table 3 and the graphs of the absolute error $D^\alpha f - D^\alpha \mathcal{G}_{\frac{\alpha}{2},N}[f]$, for some $\alpha$, are shown in Figures 5 and 6. In the second test, we illustrate the quality of the method in the complex domain $(iy_1, iy_2)$, $y_1, y_2 \in [0, 0.3]$; see Table 4. As predicted by the theory, the bound converges when $|y_j| < 3/4$, $j = 1, 2$.

| $\alpha$ | $\max_{x \in J} \left|D^\alpha f(x) - D^\alpha \mathcal{G}_{\frac{\alpha}{2},N}[f](x)\right|$ | $\mathcal{R}_\infty \frac{\alpha}{2},N$ | $\max_{x \in J} \left|D^\alpha f(x) - D^\alpha \mathcal{G}_{\frac{\alpha}{2},N}[f](x)\right|$ | $\mathcal{R}_\infty \frac{\alpha}{2},N$ |
|----------|-----------------|-----------------|-----------------|-----------------|
| $(1,0)$  | $4.1019 \times 10^{-12}$ | $1.820 \times 10^{-11}$ | $3.5068 \times 10^{-10}$ | $8.059 \times 10^{-10}$ |
| $(2,0)$  | $1.59599 \times 10^{-12}$ | $1.942 \times 10^{-10}$ | $7.43024 \times 10^{-11}$ | $5.846 \times 10^{-9}$ |

Table 3: Approximation of partial derivatives of $f$.  

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Table 4: Approximation of mixed partial derivatives of \( f \) at imaginary points \( (iy_k, iy_l) \).

6. Conclusion

In this paper, we use the bivariate sinc-Gauss sampling formula to approximate the partial derivatives of any order for entire and holomorphic functions on an infinite horizontal strip domain, using only finitely many samples of the function itself. The theoretical error analysis is established via a complex analytic approach and the convergence rate is of an exponential order. This formula has a high accuracy in comparison with the accuracy of the bivariate classical sampling formula. The numerical examples show a quite good agreement with our theoretical analysis. With the error bound, this formula can be used for approximating the solutions of partial differential equations, and we expect that it will provide a higher accuracy than the classical bivariate sampling formula. This will be studied in a future work.

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References


