Abstract

Decomposition and reconstruction algorithms for nested spaces of multivariate trigonometric polynomials are described. The scaling functions of these spaces are chosen as fundamental polynomials of Lagrange interpolation. Their properties are crucial for the approach presented here. In particular, FFT-algorithms can be used efficiently.

AMS classification: 42 A 15, 41 A 10, 41 A 63, 65 D 05
Key words and phrases: Trigonometric wavelets, Interpolation, Reconstruction, Decomposition, Circulant matrices

1 Introduction

Signal and image processing is nowadays mathematically examined in terms of wavelet analysis. We are interested here in decomposing multivariate periodic functions into trigonometric wavelet parts. In particular, the wavelets are trigonometric polynomials built from certain de la Vallée Poussin means of the Dirichlet kernel. So we did not start by periodizing a wavelet on the real line but, used directly, the properties of the corresponding trigonometric kernels. A first attempt in this direction has been done by C. K. Chui and H. N. Mhaskar in [1]. They used Fourier sums of the Haar scaling functions and wavelets to construct a univariate wavelet decomposition of $L_2(T)$. Then every wavelet space consists of trigonometric polynomials with frequencies from a certain dyadic band. The disadvantages of that approach are the lack of interpolatory properties of the scaling functions and wavelets and their large oscillations based on the unboundedness of the Fourier sums with respect to the uniform norm. To overcome these difficulties we follow the ideas of A. A. Privalov ([2]), where he used a special de la Vallée Poussin mean of the Fourier sum to construct a polynomial basis of $C(T)$ with a certain order of growth of their degrees. A wavelet analysis for the whole scale of possible de la Vallée Poussin means with the corresponding two-scale relations and decomposition formulas is outlined in [3] and [4]. Moreover, a unified introduction to periodic univariate wavelets based on calculations with Fourier coefficients is due to G. Plonka and M. Tasche ([5]).

Here, we consider bivariate trigonometric polynomials on the torus based on tensor products of the univariate scaling functions and wavelets. Then, the general multivariate case can be handled analogously. As usual, we obtain, in the bivariate case, three different wavelet spaces on every level. For the decomposition and reconstruction of a given function, signal or image, it is important to know the basis transformations between the different levels of sample and wavelet spaces. Therefore, the main part of this paper deals with the investigation of the corresponding matrices. The block circulant structure of these matrices allows to apply the Fast Fourier Transform. Starting with the function values on a certain
regular grid with \( c2^j \times c2^j \) points, the algorithms need \( \mathcal{O}(j2^j) \) operations to decompose a function into its wavelet parts. For a fixed number of interpolation points, one can choose with the different de la Vallée Poussin means between better localization and lower degree of the polynomials.

## 2 Definitions

Let us briefly recall the definitions of the univariate scaling functions as described in [2],[4].

**Definition 2.1** For \( \ell \in \mathbb{N} \), the Dirichlet kernel \( D_\ell \in T_\ell \) is defined as

\[
D_\ell = \frac{1}{2} + \sum_{k=1}^{\ell} \cos kx = \begin{cases} 
\frac{\sin(\frac{\ell+1}{2}x)}{2 \sin \frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\
\ell + \frac{1}{2} & \text{for } x \in 2\pi\mathbb{Z}.
\end{cases}
\]

Here, and in the following, \( T_\ell \) denotes the linear space of trigonometric polynomials of degree at most \( \ell \). Furthermore, for \( N,M \in \mathbb{N}, N > M \), the de la Vallée Poussin means \( \varphi_N^M \) are given as

\[
\varphi_N^M(x) = \frac{1}{2MN} \sum_{\ell = N-M}^{N+M-1} D_\ell(x) = \begin{cases} 
\frac{\sin N_x \sin M_x}{4NM \sin^2 \frac{x}{2}} & \text{for } x \notin 2\pi\mathbb{Z}, \\
1 & \text{for } x \in 2\pi\mathbb{Z}.
\end{cases}
\]

For \( j \in \mathbb{N}_0 \), the univariate scaling function \( \phi_{j,0} \) is defined as

\[
\phi_{j,0} = \frac{M_j}{\varphi_{N_j}},
\]

with

\[
N_j = c \cdot 2^j, \quad M_j = \begin{cases} 
2^{j-\lambda} & \text{for } j \geq \lambda, \\
1 & \text{for } j < \lambda,
\end{cases}
\]

where \( \lambda \in \mathbb{N}_0 \) and \( c \in \mathbb{N} \) fulfil the condition

\[
c \geq \begin{cases} 
3 & \text{for } \lambda = 0, \\
2 & \text{for } \lambda = 1, \\
1 & \text{for } \lambda \geq 2.
\end{cases}
\]

The choice of \( \lambda \) and \( c \) corresponds to different de la Vallée Poussin means. The most interesting ones are the limit cases \( \lambda = 0, c = 3 \) and \( \lambda = \infty \) with arbitrary \( c \). While the first one describes an ordinary de la Vallée Poussin mean, the second one is the usual modified Dirichlet kernel. For all possible \( N_j \) and \( M_j \) the crucial interpolatory property of \( \phi_{j,0} \) is

\[
\phi_{j,0}\left(\frac{k\pi}{N_j}\right) = \delta_{k,0}, \quad k = 0, 1, \ldots, 2N_j - 1.
\]

Let us denote the translates of \( \phi_{j,0} \) by \( \phi_{j,k}(x) = \phi_{j,0}(x - \frac{k\pi}{N_j}), k = 0, 1, \ldots, 2N_j - 1. \)
Definition 2.2 For $j \in \mathbb{N}_0$, $K = (k_1, k_2)$, $k_1, k_2 = 0, 1, \ldots, 2N_j - 1$, the bivariate scaling functions $\Phi_{j,K} : \mathbb{T}^2 \to \mathbb{R}$ are defined as
\[
\Phi_{j,K}(x, y) = \phi_{j,k_1}(x) \cdot \phi_{j,k_2}(y).
\]

Analogously, these functions fulfill interpolatory conditions
\[
\Phi_{j,K} \left( \frac{m \pi}{N_j}, \frac{n \pi}{N_j} \right) = \delta_{k_1,m} \cdot \delta_{k_2,n}, \quad \text{for all} \quad k_1, k_2, m, n = 0, 1, \ldots, 2N_j - 1.
\]

For $j \in \mathbb{N}_0$, the sample spaces $V_j$ are defined by
\[
V_j = \text{span} \{ \Phi_{j,K} : K = (k_1, k_2), \quad k_1, k_2 = 0, 1, \ldots, 2N_j - 1 \}.
\]

Because of the interpolatory properties of the functions $\Phi_{j,K}$, one obtains immediately
\[
\dim V_j = 4N_j^2.
\]

For $j \in \mathbb{N}_0$, the interpolation operator $L_j$ mapping into the sample space $V_j$ is defined as
\[
L_j f(x, y) = \sum_{k_1=0}^{2N_j-1} \sum_{k_2=0}^{2N_j-1} f \left( \frac{k_1 \pi}{N_j}, \frac{k_2 \pi}{N_j} \right) \Phi_{j,k_1,k_2}(x, y).
\]

The following properties of the operator $L_j$ can be easily deduced from the univariate case (see references [2], [6], Chap. 10):

(i) $L_j f \in T_{N_j+M_j-1}^2$, where $T_i^2 := T_i \otimes T_i$,

(ii) $L_j f \left( \frac{m \pi}{N_j}, \frac{n \pi}{N_j} \right) = f \left( \frac{m \pi}{N_j}, \frac{n \pi}{N_j} \right)$ for all $m, n \in \mathbb{Z}$,

(iii) $L_j f = f$ for all $f \in T_{N_j-M_j}^2 \cup V_j$,

hence,

(iv) $T_{N_j-M_j}^2 \subset V_j \subset T_{N_j+M_j-1}^2$.

Property (iv) and our special choice of $M_j$ and $N_j$ (see (1), (2)) guarantee the imbedding
\[
V_j \subset V_{j+1}.
\]

Hence, we obtain a chain of finite dimensional subspaces $V_j$ of the Hilbert space $L_2(\mathbb{T}^2)$ with
\[
\text{clos}_{L_2(\mathbb{T}^2)} \bigcup_{j \in \mathbb{N}_0} V_j = L_2(\mathbb{T}^2).
\]

Now let us consider for $j \in \mathbb{N}_0$ the univariate wavelet functions
\[
\psi_{j,m}(x) = \phi_{j+1,2m+1}(x) - \phi_{j,m} \left( x - \frac{\pi}{2N_j} \right),
\]

which fulfill the following interpolatory properties:
\[
\psi_{j,m} \left( \frac{(2\ell + 1) \pi}{N_{j+1}} \right) = \delta_{\ell,m} \quad \text{for all} \quad \ell \in \mathbb{Z},
\]
\[
\psi_{j,m} \left( \frac{\ell \pi}{N_j} \right) = -\phi_{j,m} \left( \frac{(2\ell - 1) \pi}{N_{j+1}} \right) \quad \text{for all} \quad \ell \in \mathbb{Z}.
\]

In 1991 A. A. Privalov proved the orthogonality between the univariate scaling functions and the wavelets from the same level $j$. 

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Theorem 2.1 [2] For all \( m, n = 0, \ldots, 2N_j - 1 \), it holds that
\[
\frac{1}{2\pi} \int_{0}^{2\pi} \phi_{j,m}(x) \psi_{j,n}(x) \, dx = 0.
\]

In \( L_2(\mathbb{T}^2) \) we use the standard inner product
\[
\langle f, g \rangle = \frac{1}{4\pi^2} \int_{0}^{2\pi} \int_{0}^{2\pi} f(x, y)g(x, y) \, dx \, dy.
\]

Hence, the tensor product structure of our scaling functions \( \Phi_{j,K} \) yields straightforward the corresponding wavelets and wavelet spaces.

Definition 2.3 For \( j \in \mathbb{N}_0 \), \( s = 1, 2, 3 \), the wavelet spaces \( W_j^s \) are defined by
\[
W_j^s = \text{span}\{\Psi_{j,K}^s : K = (k_1, k_2), \ k_1, k_2 = 0, 1, \ldots, 2N_j - 1\},
\]
where
\[
\Psi_{j,K}^1(x, y) = \phi_{j,k_1}(x) \cdot \psi_{j,k_2}(y),
\]
\[
\Psi_{j,K}^2(x, y) = \psi_{j,k_1}(x) \cdot \phi_{j,k_2}(y)
\]
and
\[
\Psi_{j,K}^3(x, y) = \psi_{j,k_1}(x) \cdot \psi_{j,k_2}(y).
\]

From the interpolatory properties of the functions \( \phi_{j,k} \) and \( \psi_{j,k} \), we obtain
\[
\dim W_j^s = 4N_j^2.
\]

Theorem 2.2 For the above defined sample spaces \( V_j \) and wavelet spaces \( W_j \), it holds that
\[
V_{j+1} = V_j \oplus 1 W_j^1 \oplus 1 W_j^2 \oplus 1 W_j^3,
\]
where \( \oplus 1 \) denotes orthogonal summation.

Proof: From (4) and from the the definition of \( V_j \) and \( W_j^s \), it is clear that every space on the right-hand side is a subspace of \( V_{j+1} \). From Theorem 2.1, it follows the orthogonality of the univariate functions \( \phi_{j,m} \) and \( \psi_{j,n} \) and so the pairwise orthogonality of the spaces \( V_j \) and \( W_j^s \), \( s = 1, 2, 3 \). Hence, the summation is really an orthogonal summation and the sum itself is a subspace of \( V_{j+1} \). In order to obtain the equality, we consider
\[
\dim V_{j+1} = 4N_j^2 + 1 = 4 \cdot 4N_j^2
\]
\[
= \dim V_j + \sum_{s=1}^{3} \dim W_j^s
\]
which completes the proof. ■

To illustrate the different de la Vallée Poussin cases, we present the following pictures.
Figure 1 includes scaling functions $\Phi_{1,(6,6)}$, whereas Figure 2 shows $\Psi_{1,(6,6)}^1$ and Figure 3 $\Psi_{1,(6,6)}^3$. The upper pictures a) are for the parameters $c = 3, \lambda = 0$ and the lower pictures b) are with $c = 3, \lambda = \infty$ (compare (1),(2)). I.e., a) corresponds to the ordinary de la Vallée Poussin case and b) reflects the pure Fourier case.
3 Two-scale Relations

This section is devoted to the computation of the two-scale sequences for reconstruction.

As $V_j \subset V_{j+1}$, there exist coefficients $q_{j,K,R}$ such that

$$\Phi_{j,K} = \sum_{R} q_{j,K,R} \Phi_{j+1,R}, \quad \text{where } R = (r_1, r_2).$$

Property (iii) of the interpolation operator introduced in Section 2 implies that the functions $\Phi_{j,K}$ are reproduced by $I_{j+1}$, i.e.,

$$\Phi_{j,K} = \sum_{r_1, r_2=0}^{2N_{j+1}-1} \Phi_{j,K} \left( \frac{r_1 \pi}{N_{j+1}}, \frac{r_2 \pi}{N_{j+1}} \right) \Phi_{j+1,R}. \quad (7)$$

Thus, the coefficients

$$q_{j,K,R} = \Phi_{j,K} \left( \frac{r_1 \pi}{2N_j}, \frac{r_2 \pi}{2N_j} \right)$$

can be obtained from knot evaluation of the scaling function. Further, since $W^s_j \subset V_{j+1}$, $s = 1, 2, 3$, there have to exist coefficients $\sigma_{j,K,R}^s$ such that

$$\Psi_{j,K}^s = \sum_{R} \sigma_{j,K,R}^s \Phi_{j+1,R}. \quad (8)$$

Analogously to (7), we conclude

$$\sigma_{j,K,R} = \Psi_{j,K}^s \left( \frac{r_1 \pi}{2N_j}, \frac{r_2 \pi}{2N_j} \right).$$

From the interpolatory properties (3), (5) and (6) of $\phi_{j,k}$ and $\psi_{j,k}$, it is natural to distinguish between even and odd translates.

With the notation $|K| = k_1 \cdot 2N_j + k_2$ the vectors

$$\Phi_j = (\Phi_{j,K})_{|K|=0}^{4N_j^2-1} \quad \text{and} \quad \Psi_j^s = (\Psi_{j,K}^s)_{|K|=0}^{4N_j^2-1}, \quad s = 1, 2, 3,$$

are uniquely determined. To reorder the components with odd and even indices we introduce the permutation matrix

$$P_j = \left( p_{j,[K] \mid \ell} \right)_{|K|=0}^{16N_j^2-1},$$

where

$$p_{j,[K] \mid \ell} = (\delta_{2k_1, \ell_1} + \delta_{2k_1, \ell_1+4N_j-1})(\delta_{2k_2, \ell_2} + \delta_{2k_2, \ell_2+4N_j-1}),$$

for $k_1, k_2, \ell_1, \ell_2 = 0, 1, \ldots, 4N_j - 1$.

Furthermore, let $F_j$ be the Fourier matrix of order $2N_j$

$$F_j = \frac{1}{\sqrt{2N_j}} \left( e^{\frac{2\pi i k \ell}{2N_j}} \right)_{k,\ell=0}^{2N_j-1},$$

and $K_j$ be the knot evaluation matrix of the univariate scaling functions

$$K_j = \left( \phi_{j,k} \left( \frac{(2\ell + 1)\pi}{2N_j} \right) \right)_{k,\ell=0}^{2N_j-1}.$$

Then the two-scale relations in matrix form are given in the following way.
Theorem 3.1  For \( j \in \mathbb{N}_0 \), it holds that

\[
\begin{pmatrix}
\Phi_j \\
\Psi_j^1 \\
\Psi_j^2 \\
\Psi_j^3
\end{pmatrix} = C_j P_j \Phi_{j+1}
\]

with the reconstruction matrix

\[
C_j = \begin{pmatrix}
I_j \otimes I_j & I_j \otimes K_j & K_j \otimes I_j & K_j \otimes K_j \\
-I_j \otimes K_j^T & I_j \otimes I_j & -K_j \otimes K_j^T & K_j \otimes I_j \\
-K_j^T \otimes I_j & -K_j^T \otimes K_j & I_j \otimes I_j & I_j \otimes K_j \\
K_j^T \otimes K_j^T & -K_j^T \otimes I_j & -I_j \otimes K_j^T & I_j \otimes I_j
\end{pmatrix}.
\]

Here, \( I_j \in \mathbb{R}^{2N \times 2N} \) denotes the identity matrix and the Kronecker product of two \( n \times n \) matrices \( A = (a_{k,\ell}) \) and \( B \) is given by the \( n^2 \times n^2 \) matrix

\[
A \otimes B = (a_{k,\ell} B)_{k,\ell}.
\]

While \( K_j \) is circulant, the blocks of \( C_j \) are block circulants with circulant blocks. **Proof:** The desired result is only the reformulation of (7) and (8) in matrix notation, where we used the interpolatory conditions (3) and (5) to simplify

\[
\Phi_{j,K} \left( \frac{r_1 \pi}{2N_j}, \frac{r_2 \pi}{2N_j} \right) = 0 \quad \text{if} \quad (r_1, r_2) \neq K \quad \text{and} \quad (r_1 \text{ or } r_2 \text{ even }),
\]

\[
\Psi_{j,K}^1 \left( \frac{r_1 \pi}{2N_j}, \frac{r_2 \pi}{2N_j} \right) = 0 \quad \text{if} \quad (r_1, r_2) \neq K \quad \text{and} \quad (r_1 \text{ even or } r_2 \text{ odd }),
\]

and analogously for \( \Psi_{j,K}^2 \) and \( \Psi_{j,K}^3 \). □

4 Riesz Stability

For further usage, we have to consider inner products of scaling functions on the same level. Let us define the Gramian

\[
G_{j} = \{(\phi_{j,k}, \phi_{j,\ell})\}_{k,\ell=0}^{2N_j-1}.
\]

From the definition of the scaling functions, we know

\[
G_{j} = \text{circ}(\{(\phi_{j,0}, \phi_{j,0}), (\phi_{j,0}, \phi_{j,1}), \ldots, (\phi_{j,0}, \phi_{j,2N_j-1})\}).
\]

and one can see easily [4] that \( G_{j} \) is symmetric and positive definite with

\[
G_{j} = F_j \Gamma_j F_j^T,
\]

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where \( \Gamma_j = \text{diag}(\gamma_{j,\ell})_{\ell=0}^{2N_j-1} \).

In [4], it is proved that the eigenvalues \( \gamma_{j,\ell} \) are given by

\[
\gamma_{j,\ell} = \sum_{k=0}^{2N_j-1} \langle \phi_{j,0,k}, \phi_{j,\ell,k} \rangle \cdot e^{rac{j\ell k}{N_j}}
\]

\[
= \frac{1}{4N_j} \left\{ \begin{array}{ll}
2 & \text{if } 0 \leq \ell \leq N_j - M_j, \quad N_j + M_j \leq \ell \leq 2N_j - 1, \\
1 + \left( \frac{N_j - \ell}{N_j} \right)^2 & \text{if } N_j - M_j \leq \ell \leq N_j + M_j.
\end{array} \right.
\]

\( (9) \)

Now we are able to prove the Riesz stability of the scaling function basis in \( V_j \).

**Theorem 4.1** For arbitrary sequences \( \alpha \in \mathbb{R}^{4N_j^2} \), it holds that

\[
\frac{1}{2} \left( \sum_{k_1,k_2=0}^{2N_j-1} \alpha_K^2 \right)^{\frac{\tau}{2}} \leq \left\| \sum_{k_1,k_2=0}^{2N_j-1} 2N_j \alpha_K \Phi_{j,K} \right\|_{L_2(T^2)} \leq \left( \sum_{k_1,k_2=0}^{2N_j-1} \alpha_K^2 \right)^{\frac{\tau}{2}}.
\]

**Proof:** It holds

\[ G_j \otimes G_j = (\langle \Phi_{j,K}, \Phi_{j,L} \rangle)_{|K|,|L|=0}^{4N_j^2-1}. \]

The Kronecker product \( G_j \otimes G_j \) is circulant, symmetric and positive definite with the eigenvalues \( \Gamma_{j,L} = \gamma_{j,\ell} \gamma_{j,\ell} \) and the corresponding orthonormal eigenvectors \( U_{j,L} \in \mathbb{R}^{4N_j^2} \)

Then, we can write every \( \alpha \in \mathbb{R}^{4N_j^2} \) in the form

\[ \alpha = \sum_L \beta_L U_{j,L}. \]

From

\[
\sum_K \alpha_K^2 = \sum_K \beta_K^2
\]

and

\[
\left\| \sum_{k_1,k_2=0}^{2N_j-1} 2N_j \alpha_K \Phi_{j,K} \right\|_{L_2(T^2)}^2 = 4N_j^2 \left( \sum_{k_1,k_2=0}^{2N_j-1} \alpha_K \Phi_{j,K} \cdot \sum_{\ell_1,\ell_2=0}^{2N_j-1} \alpha_L \Phi_{j,L} \right)
\]

\[
= 4N_j^2 \sum_{k_1,k_2=0}^{2N_j-1} \sum_{\ell_1,\ell_2=0}^{2N_j-1} \alpha_K \alpha_L \langle \Phi_{j,K}, \Phi_{j,L} \rangle
\]

\[
= 4N_j^2 \left( (G_j \otimes G_j) \alpha, \alpha \right)
\]

\[
= 4N_j^2 \left( \sum_{k_1,k_2=0}^{2N_j-1} \sum_{\ell_1,\ell_2=0}^{2N_j-1} \beta_K \beta_L \left( (G_j \otimes G_j) \Phi_{j,K} \otimes \Phi_{j,L} \right) \right)
\]

\[
= 4N_j^2 \left( \sum_{k_1,k_2=0}^{2N_j-1} \sum_{\ell_1,\ell_2=0}^{2N_j-1} \beta_K \beta_L \left( \Gamma_{j,K} U_{j,K} \otimes U_{j,L} \right) \right)
\]

\[
= 4N_j^2 \left( \sum_{k_1,k_2=0}^{2N_j-1} \beta_K^2 \gamma_{j,k_1} \gamma_{j,k_2} \right)
\]

we obtain with (9) the desired Riesz constants in the theorem as

\[
2N_j \min_{\ell} \gamma_{j,\ell} = \frac{1}{2},
\]

\[
2N_j \max_{\ell} \gamma_{j,\ell} = 1.
\]
5 Decomposition Relations

To state the counterpart of Theorem 3.1, i.e., to write the scaling functions $\Phi_{j+1,K} \in V_{j+1}$ in the corresponding scaling function and wavelet basis of $V_j \oplus \hat{+} W^1 \oplus \hat{+} W^2 \oplus \hat{+} W^3$, we need some preliminaries.

Since $G_j$ is circulant, its inverse is given by

$$G_j^{-1} = F_j G_j^{-1} F_j = F_j \text{diag} \left( \frac{1}{\gamma_i^j} \right)_{i=0}^{2N_j-1} F_j.$$ 

For the knot evaluation matrix $K_j$ and the matrix of inner products $G_j$ there exist the following helpful relation.

**Lemma 5.1** For $j \in \mathbb{N}_0$, it holds that

$$I_j + K_j K_j^T = 4N_j G_j. \quad (10)$$

**Proof:** Because all matrices involved are circulant, it suffices to show that the equality holds for the first row. Then we use the well-known quadrature formula for trigonometric polynomials with equidistant nodes (see e.g. [6], Chap. 10) to conclude

$$(4N_j G_j)_{0,r} = 4N_j \langle \phi_{j,0}, \phi_{j,r} \rangle = 4N_j \frac{2\pi}{2\pi} \int_0^{2\pi} \phi_{j,0}(x) \phi_{j,r}(x) \, dx$$

$$= \sum_{k=0}^{4N_j-1} \phi_{j,0} \left( \frac{k\pi}{2N_j} \right) \phi_{j,r} \left( \frac{k\pi}{2N_j} \right)$$

$$= \delta_{0,r} + \sum_{s=0}^{2N_j-1} \phi_{j,0} \left( \frac{(2s+1)\pi}{2N_j} \right) \phi_{j,r} \left( \frac{(2s+1)\pi}{2N_j} \right)$$

$$= (I_j + K_j K_j^T)_{0,r},$$

which proves the assertion. \(\blacksquare\)

Now we can summarize our results in the following theorem.

**Theorem 5.2** For $j \in \mathbb{N}_0$, it holds that

$$P_j \Phi_{j+1} = D_j$$

with the decomposition matrix
\[ 16N_j^2 D_j = 16N_j^2 (I_4 \otimes G_j^{-1} \otimes I_j)C_j^T (I_4 \otimes I_j \otimes G_j^{-1}) = \]

\[
\begin{pmatrix}
G_j^{-1} \otimes G_j^{-1} & -G_j^{-1} \otimes G_j^{-1} K_j & -G_j^{-1} K_j \otimes G_j^{-1} & G_j^{-1} K_j \otimes G_j^{-1} K_j \\
G_j^{-1} K_j^T G_j^{-1} & G_j^{-1} \otimes G_j^{-1} & -G_j^{-1} K_j \otimes K_j^T G_j^{-1} & -G_j^{-1} K_j \otimes G_j^{-1} \\
K_j^T G_j^{-1} \otimes G_j^{-1} & -K_j^T G_j^{-1} \otimes G_j^{-1} K_j & G_j^{-1} \otimes G_j^{-1} & -G_j^{-1} \otimes G_j^{-1} K_j \\
K_j^T G_j^{-1} \otimes K_j^T G_j^{-1} & K_j^T G_j^{-1} \otimes G_j^{-1} & G_j^{-1} \otimes K_j^T G_j^{-1} & G_j^{-1} \otimes G_j^{-1} \\
\end{pmatrix}
\]

where \( I_4 \in \mathbb{R}^{4 \times 4} \) denotes the identity matrix.

**Proof:** To prove the theorem, it suffices to show that \( C_j D_j = I \in \mathbb{R}^{4N_j^2 \times 4N_j^2} \). Only by applying the rules of computation for the Kronecker product (see e.g. [7], Chap. 2) we get

\[ 16N_j^2 C_j D_j = I_4 \otimes H_j \]

with

\[ H_j = \left( G_j^{-1} + K_j K_j^T G_j^{-1} \right) \otimes \left( G_j^{-1} + K_j K_j^T G_j^{-1} \right) \).

Multiplying (10) in Lemma 5.1 by \( G_j^{-1} \) and using the result one obtains

\[ H_j = 16N_j^2 I_j \otimes I_j \]

such that

\[ C_j D_j = I_4 \otimes I_j \otimes I_j = I \]

what proves the Theorem. \( \blacksquare \)

### 6 Algorithms

Now we want to describe how to use the two-scale and decomposition relations, respectively, for a given function. As \( V_{j+1} = V_j \oplus W_{j+1} \), a function \( f_{j+1} \in V_{j+1} \) can be written uniquely as

\[ f_{j+1} = f_j + g_j^1 + g_j^2 + g_j^3 \]  with  \( f_j \in V_j \)  and  \( g_j^s \in W_j^s \), \( s = 1, 2, 3 \).

Using the basis functions of these spaces, one obtains

\[ f_{j+1} = \sum_{k_1, k_2 = 0}^{2N_{j+1}^2-1} c_{j+1, k_1, k_2} \phi_{j+1, k_1, k_2}, \]

\[ f_j = \sum_{k_1, k_2 = 0}^{2N_j^2-1} c_{k_1, k_2} \phi_{j, k_1, k_2} \]  and  \( g^s_j = \sum_{k_1, k_2 = 0}^{2N_j^2-1} d_{j, k_1, k_2} \psi_{j, k_1, k_2} \)  for  \( s = 1, 2, 3 \).

Denoting \( c_j = (c_{j, k_1, k_2})_{k_1, k_2 = 0}^{2N_j^2-1} \)  and  \( d_j^s = (d_{j, k_1, k_2})_{k_1, k_2 = 0}^{2N_j^2-1} \)  for  \( s = 1, 2, 3 \), respectively, one gets

\[ f_{j+1} = \Phi_{j+1}^T c_{j+1}, \quad f_j = \Phi_j^T c_j \]  and  \( g^s_j = (\psi_j^s)^T d_j^s \)  for  \( s = 1, 2, 3 \).
As $P_j^T = P_j^{-1}$, we can write

$$f_{j+1} = (P_j \Phi_{j+1})^T P_j c_{j+1}.$$ 

On the other hand, it holds that

$$f_j + g_j^1 + g_j^2 + g_j^3 = \left( (\Phi_j)^T, (\Psi_j^1)^T, (\Psi_j^2)^T, (\Psi_j^3)^T \right)^T \left( (c_j)^T, (d_j^1)^T, (d_j^2)^T, (d_j^3)^T \right)^T = (P_j \Phi_{j+1})^T C_j^T \left( (c_j)^T, (d_j^1)^T, (d_j^2)^T, (d_j^3)^T \right)^T.$$ 

Comparing coefficients leads to one step of the reconstruction algorithm in matrix form

$$P_j c_{j+1} = C_j^T \left( (c_j)^T, (d_j^1)^T, (d_j^2)^T, (d_j^3)^T \right)^T.$$ 

Multiplying by the inverse $(C_j^T)^{-1} = D_j^T$ gives the matrix representation of one step of the decomposition algorithm

$$\left( (c_j)^T, (d_j^1)^T, (d_j^2)^T, (d_j^3)^T \right)^T = D_j^T P_j c_{j+1}.$$ 

Computing reconstruction or decomposition sequences, respectively, we have to take into consideration the fact that the blocks of the reconstruction matrix $C_j$ as well as the ones of the decomposition matrix $D_j$ are block circulants with circulant blocks. For every matrix $A_j \in \mathbb{R}^{4N_j \times 4N_j}$ with such a special structure, it holds that (\cite{7}, Chap. 5)

$$A_j = (F_j \otimes F_j) B_j (F_j \otimes F_j)$$

with a diagonal matrix $B_j$. That’s why the multiplication of such $A_j$ by a vector can be realized by Tensor product discrete Fourier transforms of length $2N_j \times 2N_j$. Because of the special choice of $N_j$ (see (1)) we can compute reconstruction or decomposition sequences for the level $j$ with $O(j^{2j})$ essential operations by using methods of Fast Fourier Transform (\cite{8}).

Let us finish with the decomposition of the quadratic Box-spline

$$f(x, y) = B^{1,1,2}(x - \frac{3}{2}, y - \frac{3}{2})$$

into its wavelet parts. To do that, we first interpolate $f$ at 384 × 384 points to obtain $L_0 f$ with $c = 3$ and $\lambda = 0$. Then we use our algorithms to compute the orthogonal projections $g_s^j(L_0 f)$, $s = 1, 2, 3$ into the wavelet spaces $W_j^s$, which detect the discontinuities of the first derivatives of $f$.

The result can be seen in the following pictures.
Fig. 4 \( g_5^1(I_{L_0} f) \).

Fig. 5 \( g_5^3(I_{L_0} f) \).

Fig. 6 \( g_5^3(I_{L_0} f) \).

References


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