TWO IDEALS CONNECTED WITH STRONG RIGHT UPPER POROSITY AT A POINT

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Abstract. Let $\boldsymbol{SP}$ be the set of upper strongly porous at 0 subsets of $\mathbb{R}^+$ and let $\hat{I}(\boldsymbol{SP})$ be the intersection of maximal ideals $\boldsymbol{I} \subseteq \boldsymbol{SP}$. Some characteristic properties of sets $E \in \hat{I}(\boldsymbol{SP})$ are obtained. It is shown that the ideal generated by the so-called completely strongly porous at 0 subsets of $\mathbb{R}^+$ is a proper subideal of $\hat{I}(\boldsymbol{SP})$.

Keywords: one-side porosity, local strong upper porosity, completely strongly porous set, ideal.

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1. Introduction

The basic ideas concerning the notion of set porosity for the first time appeared in some early works of Denjoy [2], [3] and Khintchine [1] and then arose independently in the study of cluster sets in 1967 (Dolženko [4]). A useful collection of facts related to the notion of porosity can be found, for example, in [7], [8], [15] and [16]. The porosity appears naturally in many problems and plays an implicit role in various arias of analysis (e. g., the cluster sets [18], the Julia sets [12], the quasisymmetric maps [17], the differential theory [9], the theory of generalized subharmonic functions [6] and so on). The reader can also consult [19] and [20] for more information.

The porosity found interesting applications in connection with ideals of sets. Well-known results for ideals of compact sets can be found, for example, in [10] and [11]. In many papers the authors investigate different characteristics (set-theoretic,

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descriptive, analytic) of the ideals of porous sets (see, e. g., [13], [21], [22]). Some questions related to the order isomorphism between the principal ideals of porous sets of $\mathbb{R}$ were studied in [14]. Our paper is also a contribution to this line of research, in particular, we investigate two ideals, whose elements are upper strongly porous at 0 subsets of $\mathbb{R}^+$.

2. Right upper porosity at a point

Let us recall the definition of the right upper porosity at a point. Let E be a subset of $\mathbb{R}^+ = [0, \infty)$.

Definition 2.1. *The right upper porosity of E at 0 is the number*

$$p^+(E, 0) := \limsup_{h \to 0^+} \frac{\lambda(E, 0, h)}{h} \tag{2.1}$$

where $\lambda(E, 0, h)$ is the length of the largest open subinterval of $(0, h)$, which could be the empty set $\varnothing$, that contains no point of E. The set E is porous on the right at 0 if $p^+(E, 0) > 0$ and E is strongly porous on the right at 0 if $p^+(E, 0) = 1$.

For the remaining of the paper, when the porosity is considered, this will always be assumed to be the right upper porosity at 0.

For $E \subseteq \mathbb{R}^+$ define the subsets $\tilde{E}$ and $\tilde{H}(E)$ of the set of sequences $\tilde{h} = \{h_n\}_{n \in \mathbb{N}}$ with $h_n \downarrow 0$ by the rules:

$$(\tilde{h} \in \tilde{E}) \Leftrightarrow (h_n \in E \setminus \{0\} \text{ for all } n \in \mathbb{N}), \tag{2.2}$$

and

$$(\tilde{h} \in \tilde{H}(E)) \Leftrightarrow \left(\frac{\lambda(E, 0, h_n)}{h_n} \to 1 \quad \text{with} \quad n \to \infty \right), \tag{2.3}$$

where the number $\lambda(E, 0, h_n)$ is the same as in Definition 2.1.

Define also an *equivalence relation* $\asymp$ on the set of sequences of positive numbers as follows. Let $\tilde{a} = \{a_n\}_{n \in \mathbb{N}}$ and $\tilde{\gamma} = \{\gamma_n\}_{n \in \mathbb{N}}$. Then $\tilde{a} \asymp \tilde{\gamma}$ if there are positive constants c_1 and c_2 such that

$$c_1 a_n \leq \gamma_n \leq c_2 a_n$$

for all $n \in \mathbb{N}$.

Definition 2.2. *Let $E \subseteq \mathbb{R}^+$. The set E is completely strongly porous on the right at 0 if for every $\tilde{\tau} = \{\tau_n\}_{n \in \mathbb{N}} \in \tilde{E}$ there is $\tilde{h} = \{h_n\}_{n \in \mathbb{N}} \in \tilde{H}(E)$ such that $\tilde{\tau} \asymp \tilde{h}$.*

In what follows we denote by $\boldsymbol{SP}$ and $\boldsymbol{CSP}$ the collection (i.e., the set) of sets $E \subseteq \mathbb{R}^+$ which are strongly porous on the right at 0 and completely strongly porous on the right at 0 respectively. The set $\boldsymbol{CSP}$ was introduced and studied in [5] with slightly different, but equivalent definition.

Definition 2.3. *Let $E \subseteq \mathbb{R}^+$ and $q > 1$. The q-blow up of E is the set*

$$E(q) := \bigcup_{x \in E} (q^{-1}x, qx).$$

The goal of the paper is to find some blow up characterizations for the intersection of maximal ideals $\boldsymbol{I} \subseteq \boldsymbol{SP}$ and for the ideal generated by $\boldsymbol{CSP}$.

3. Ideals and sets closed under subsets

Let $\boldsymbol{A}$ be a collection of sets. We shall say that $\boldsymbol{A}$ is *closed under subsets* if the implication

$$(B \in \boldsymbol{A} \ \wedge \ C \subseteq B) \Rightarrow (C \in \boldsymbol{A}) \tag{3.1}$$

holds for all sets C and B. If $\boldsymbol{\Gamma}$ is an arbitrary collection of sets, we write

$$V = V(\boldsymbol{\Gamma}) := \bigcup_{A \in \boldsymbol{\Gamma}} A.$$

Definition 3.1. *A collection $\boldsymbol{I}$ of subsets of a set X is an ideal on X if the following conditions hold:*
(i) *$\boldsymbol{I}$ is closed under subsets;*
(ii) *$B \cup C \in \boldsymbol{I}$ for all $B, C \in \boldsymbol{I}$;*
(iii) *$X \notin \boldsymbol{I}$ and $\varnothing \in \boldsymbol{I}$.*

We include the condition $\varnothing \in \boldsymbol{I}$ to guarantee that $\boldsymbol{I}$ is nonempty.

Let $\boldsymbol{\Gamma}$ be nonempty and closed under subsets. Define a set $I(\boldsymbol{\Gamma}) \subseteq 2^V$ by the rule

$$(B \in I(\boldsymbol{\Gamma})) \Leftrightarrow (\exists\, n \in \mathbb{N}\ \exists\, A_1, ..., A_n \in \boldsymbol{\Gamma} : B = \bigcup_{j=1}^{n} A_j). \tag{3.2}$$

If $V \notin I(\boldsymbol{\Gamma})$, then $I(\boldsymbol{\Gamma})$ is an ideal on V such that $\boldsymbol{\Gamma} \subseteq I(\boldsymbol{\Gamma})$ and the implication

$$(\boldsymbol{\Gamma} \subseteq \mathfrak{I}) \Rightarrow (I(\boldsymbol{\Gamma}) \subseteq \mathfrak{I})$$

holds for every ideal $\mathfrak{I}$ on V. In what follows we shall say that $I(\boldsymbol{\Gamma})$ is the *ideal generated by $\boldsymbol{\Gamma}$*.

Definition 3.2. *Let $\boldsymbol{\Gamma}$ be an arbitrary nonempty collection of sets. An ideal $\boldsymbol{I}$ on $V = V(\boldsymbol{\Gamma})$ is $\boldsymbol{\Gamma}$-maximal if $\boldsymbol{I} \subseteq \boldsymbol{\Gamma}$ and the implication*

$$(\boldsymbol{I} \subseteq \boldsymbol{\mathfrak{I}} \subseteq \boldsymbol{\Gamma}) \Rightarrow (\boldsymbol{I} = \boldsymbol{\mathfrak{I}}) \tag{3.3}$$

holds for every ideal $\boldsymbol{\mathfrak{I}}$ on V.

Write $M(\boldsymbol{\Gamma})$ for the set of $\boldsymbol{\Gamma}$-maximal ideals and define an ideal $\hat{I}(\boldsymbol{\Gamma})$ as

$$\hat{I}(\boldsymbol{\Gamma}) := \bigcap_{\boldsymbol{I} \in M(\boldsymbol{\Gamma})} \boldsymbol{I}, \tag{3.4}$$

i. e., $\hat{I}(\boldsymbol{\Gamma})$ is the intersection of $\boldsymbol{\Gamma}$-maximal ideals.

The paper contains the following main results.

- *A characteristic property of sets which belong to the intersection $\hat{I}(\boldsymbol{\Gamma})$ of $\boldsymbol{\Gamma}$-maximal ideals with closed under subsets $\boldsymbol{\Gamma}$.* (See Theorem 4.4).
- *The blow up characterizations of the ideals $\hat{I}(\boldsymbol{SP})$ and $I(\boldsymbol{CSP})$.* (See theorems 6.6 and 7.6).
- *The proper inclusion $I(\boldsymbol{CSP}) \subset \hat{I}(\boldsymbol{SP})$.* (See Corollary 7.7 and Example 7.8).

Remark 3.3. *The sets $\boldsymbol{SP}$ and $\boldsymbol{CSP}$ are closed under subsets and no one from these sets is an ideal on $\mathbb{R}^+$.*

Remark 3.4. *The $\boldsymbol{\Gamma}$-maximal ideals are a generalization of the prime ideals. Indeed, if $\boldsymbol{\Gamma} = 2^V$ and $\boldsymbol{I}$ is an ideal on V, then it may be proved that $\boldsymbol{I}$ is a prime ideal on V if and only if $\boldsymbol{I}$ is $\boldsymbol{\Gamma}$-maximal.*

4. A property of the intersection of $\boldsymbol{\Gamma}$-maximal ideals

We start with a useful property of an arbitrary $\boldsymbol{\Gamma}$-maximal ideal.

Lemma 4.1. *Let $\boldsymbol{\Gamma}$ be a nonempty collection of sets. The following two statements are equivalent:*

(i) *$\boldsymbol{\Gamma}$ is closed under subsets and $V(\boldsymbol{\Gamma}) \notin \boldsymbol{\Gamma}$;*
(ii) *For every $A \in \boldsymbol{\Gamma}$ there exists a $\boldsymbol{\Gamma}$-maximal ideal $\boldsymbol{I}$ such that $A \in \boldsymbol{I}$.*

Proof. (ii) $\Rightarrow$ (i). Assume that (ii) holds. Let $A \in \boldsymbol{\Gamma}$. Using (ii), we can find a $\boldsymbol{\Gamma}$-maximal ideal $\boldsymbol{I} \ni A$. Then $2^A \subseteq \boldsymbol{I} \subseteq \boldsymbol{\Gamma}$ holds. Hence $\boldsymbol{\Gamma}$ is closed under subsets. Suppose now that $V \in \boldsymbol{\Gamma}$. By (ii), there is a $\boldsymbol{\Gamma}$-maximal ideal $\boldsymbol{I}$ such that

$$V \in \boldsymbol{I}. \tag{4.1}$$

The ideal $\boldsymbol{I}$ is an ideal on V. Hence $V \notin \boldsymbol{I}$, contrary to (4.1).

(i) $\Rightarrow$ (ii). Suppose that (i) holds. Let $A \in \boldsymbol{\Gamma}$. Then $2^A \subseteq \boldsymbol{\Gamma}$ and 2^A is an ideal on V. Using Zorn's Lemma, we can find a $\boldsymbol{\Gamma}$-maximal ideal $\boldsymbol{I}$ such that $\boldsymbol{I} \supseteq 2^A$. It is clear that $A \in \boldsymbol{I}$ holds. The implication (i) $\Rightarrow$ (ii) follows. □

Let $\boldsymbol{\Gamma}$ be a collection of sets. We shall denote by $I^*(\boldsymbol{\Gamma})$ the collection of sets S satisfying the condition

$$S \cup B \in \boldsymbol{\Gamma} \tag{4.2}$$

for every $B \in \boldsymbol{\Gamma}$.

Remark 4.2. *It is clear that $I^*(\boldsymbol{\Gamma})$ is closed under subsets, if $\boldsymbol{\Gamma}$ is closed under subsets.*

Lemma 4.3. *If $\boldsymbol{\Gamma}$ is a nonempty collection of sets, then*

$$(V(\boldsymbol{\Gamma}) \in \boldsymbol{\Gamma}) \Leftrightarrow (V(\boldsymbol{\Gamma}) \in I^*(\boldsymbol{\Gamma}))$$

holds.

Proof. Let $V \in \boldsymbol{\Gamma}$. Then we have $B \cup V = V \in \boldsymbol{\Gamma}$ for every $B \in \boldsymbol{\Gamma}$. Hence $V \in I^*(\boldsymbol{\Gamma})$. Let now $V \in I^*(\boldsymbol{\Gamma})$ and $B \in \boldsymbol{\Gamma}$. The inclusion $B \subseteq V$ holds. Thus,

$$V = B \cup V \in \boldsymbol{\Gamma}.$$

□

Theorem 4.4. *Let $\boldsymbol{\Gamma}$ be nonempty closed under subsets and let*

$$V(\boldsymbol{\Gamma}) \notin \boldsymbol{\Gamma}. \tag{4.3}$$

Then the equality

$$I^*(\boldsymbol{\Gamma}) = \hat{I}(\boldsymbol{\Gamma}) \tag{4.4}$$

holds where $\hat{I}(\boldsymbol{\Gamma})$ is defined by (3.4).

Proof. Let us prove the inclusion

$$I^*(\boldsymbol{\Gamma}) \subseteq \hat{I}(\boldsymbol{\Gamma}). \tag{4.5}$$

Using (3.4), we can see that (4.5) holds if and only if

$$A \in \boldsymbol{I} \quad \text{for every } \boldsymbol{\Gamma}\text{-maximal ideal } \boldsymbol{I} \text{ and every } A \in I^*(\boldsymbol{\Gamma}). \tag{4.6}$$

Let A be an arbitrary element of $I^*(\boldsymbol{\Gamma})$ and let $\boldsymbol{I}$ be a $\boldsymbol{\Gamma}$-maximal ideal. Define a set $\boldsymbol{I}(A)$ as

$$\boldsymbol{I}(A) := \{B \cup K : B \subseteq A \text{ and } K \in \boldsymbol{I}\}. \tag{4.7}$$

The trivial inclusion $\varnothing \subseteq A$ implies that $\boldsymbol{I} \subseteq \boldsymbol{I}(A)$. It follows from Definition 3.2 that $\boldsymbol{I} \subseteq \boldsymbol{\Gamma}$. Since $I^*(\boldsymbol{\Gamma})$ is closed under subsets (see Remark 4.2), the relations

$$B \subseteq A \in I^*(\boldsymbol{\Gamma}) \quad \text{and} \quad K \in \boldsymbol{I} \subseteq \boldsymbol{\Gamma}$$

yield

$$B \cup K \in \boldsymbol{\Gamma}. \tag{4.8}$$

Hence

$$\boldsymbol{I}(A) \subseteq \boldsymbol{\Gamma}. \tag{4.9}$$

Moreover, (4.8), (4.7) and (4.3) imply that $V \notin \boldsymbol{I}(A)$. Since $\boldsymbol{I}$ and $\boldsymbol{\Gamma}$ are closed under subsets, the definition of $I^*(\boldsymbol{\Gamma})$ and (4.7) imply that $\boldsymbol{I}(A)$ is closed under subsets. If, for $i = 1, 2$, $B_i \cup K_i \in \boldsymbol{I}(A)$ with $B_i \subseteq A$ and $K_i \in \boldsymbol{I}$, then, by the definition of ideals, $K_1 \cup K_2 \in \boldsymbol{I}$ and, moreover, $B_1 \cup B_2 \subseteq A$. Consequently, from the equality

$$(B_1 \cup K_1) \cup (B_2 \cup K_2) = (B_1 \cup B_2) \cup (K_1 \cup K_2)$$

we obtain

$$(B_1 \cup K_1) \cup (B_2 \cup K_2) \in \boldsymbol{I}(A).$$

Hence $\boldsymbol{I}(A)$ is an ideal on V. Since $\boldsymbol{I} \subseteq \boldsymbol{I}(A)$ and $\boldsymbol{I}$ is $\boldsymbol{\Gamma}$-maximal, from (4.9) and (3.3) we obtain the equality

$$\boldsymbol{I}(A) = \boldsymbol{I}. \tag{4.10}$$

The membership $A \in \boldsymbol{I}(A)$ and (4.10) yield (4.6).

Consider now the inclusion

$$\hat{I}(\boldsymbol{\Gamma}) \subseteq I^*(\boldsymbol{\Gamma}). \tag{4.11}$$

If (4.11) does not hold, then we can find $A \in \hat{I}(\boldsymbol{\Gamma})$ and $B \in \boldsymbol{\Gamma}$ so that

$$A \cup B \notin \boldsymbol{\Gamma}. \tag{4.12}$$

By Lemma 4.1, there is a $\boldsymbol{\Gamma}$-maximal ideal $\boldsymbol{I}$ such that $B \in \boldsymbol{I}$. The membership $A \in \hat{I}(\boldsymbol{\Gamma})$ yields that $A \in \boldsymbol{I}$. Since $\boldsymbol{I}$ is an ideal, from $A \in \boldsymbol{I}$ and $B \in \boldsymbol{I}$ it follows that $A \cup B \in \boldsymbol{I} \subseteq \boldsymbol{\Gamma}$, contrary to (4.12). □

Corollary 4.5. *Let $\boldsymbol{\Gamma}$ be nonempty and closed under subsets. Then the collection $I^*(\boldsymbol{\Gamma})$ is an ideal on V if and only if $V \notin \boldsymbol{\Gamma}$.*

Proof. The intersection of an arbitrary nonempty set of ideals is an ideal. The set of $\boldsymbol{\Gamma}$-maximal ideals is nonempty, because $\boldsymbol{\Gamma} \neq \varnothing$. Consequently, $\hat{I}(\boldsymbol{\Gamma})$ is an ideal on $V = V(\boldsymbol{\Gamma})$. Hence, by Theorem 4.4, $I^*(\boldsymbol{\Gamma})$ is an ideal on V.

Conversely, if $I^*(\boldsymbol{\Gamma})$ is an ideal on V, then condition (iii) from the definition of ideals implies that $V \notin I^*(\boldsymbol{\Gamma})$. Using Lemma 4.3, we obtain that $V \notin \boldsymbol{\Gamma}$. □

Remark 4.6. *If $\boldsymbol{\Gamma}$ is closed under subsets and $V(\boldsymbol{\Gamma}) \in \boldsymbol{\Gamma}$, then, as is easily seen, the equality $\hat{I}(\boldsymbol{\Gamma}) = \{\varnothing\}$ holds, so that, in this case, the question about the structure of $\hat{I}(\boldsymbol{\Gamma})$ is trivial.*

5. Blow up of sets

Recall that for $q > 1$ and $E \subseteq \mathbb{R}^+$ we define the q-blow up of E as

$$E(q) := \bigcup_{x \in E} (q^{-1}x, qx). \tag{5.1}$$

Remark 5.1. *For all $E \subseteq \mathbb{R}^+$ and $q > 1$, we have*

$$(0 \notin E) \Leftrightarrow (E(q) \supseteq E). \tag{5.2}$$

Indeed, the implication $(0 \notin E) \Rightarrow (E(q) \supseteq E)$ is evident. Conversely, suppose that $0 \in E$. Since $0 \notin (q^{-1}x, qx)$ for every nonzero x and $(q^{-1}0, q0) = (0,0) = \varnothing$, we obtain $0 \notin E(q)$. Thus (5.2) follows.

Lemma 5.2. *Let $0 < a < b < \infty$. The following statements hold.*

(i) *If $q \geq \frac{b}{a}$ and $\varnothing \neq E \subseteq (a,b)$, then the set $E(q)$ is an open interval such that $E(q) \supseteq (a,b)$.*

(ii) *If $E = (a,b)$, then $E(q) = (q^{-1}a, qb)$ for every $q > 1$.*

The proof is simple and omitted here.

Lemma 5.3. *Let A and B be subsets of $\mathbb{R}^+$, let $t > 0$ and let*

$$(0,t) \cap B \subseteq (0,t) \cap A \tag{5.3}$$

hold. Then the inclusion

$$(0, tq^{-1}) \cap B(q) \subseteq (0, tq^{-1}) \cap A(q) \tag{5.4}$$

holds for every $q > 1$.

Proof. Let $q > 1$ and let $x \in (0, tq^{-1}) \cap B(q)$. Then we have

$$0 < x < tq^{-1} \tag{5.5}$$

and there is $y \in B$ such that

$$q^{-1}y < x < qy. \tag{5.6}$$

It follows from (5.5) and (5.6), that $q^{-1}y < x < tq^{-1}$. Consequently, $y < t$ holds. The last inequality, $y \in B$ and (5.3) imply

$$y \in (0, t) \cap B \subseteq (0, t) \cap A,$$

so that $y \in (0, t)$ and $y \in A$. These relations yield

$$(q^{-1}y, qy) \subseteq (0, tq) \quad \text{and} \quad (q^{-1}y, qy) \subseteq A(q).$$

Consequently we have

$$(0, tq^{-1}) \cap B(q) \subseteq (0, tq) \cap A(q). \tag{5.7}$$

The inclusion $(0, tq^{-1}) \subseteq (0, tq)$ and (5.7) imply that

$$(0, tq^{-1}) \cap B(q) \subseteq (0, tq^{-1}) \cap (0, tq) \cap A(q) \subseteq (0, tq^{-1}) \cap A(q).$$

Inclusion (5.4) follows. □

Lemma 5.4. *Let $E \subseteq \mathbb{R}^+$ and $E \notin$ **SP**. Then there are $q > 1$ and $t > 0$ such that the equality*

$$E(q) \cap (0, t) = (0, t) \tag{5.8}$$

holds.

Proof. Equality (5.8) evidently holds for every $q > 1$ if $(0, t) \subseteq E$. Hence we can assume that $(0, t) \setminus E \neq \varnothing$ for every $t > 0$. Since E is not strongly porous on the right at 0, there is $s \in (0, 1)$ such that

$$\limsup_{h \to 0+} \frac{\lambda(E, 0, h)}{h} < s,$$

where $\lambda(E, 0, h)$ is the length of the largest open subinterval of $(0, h)$ that contains no point of E (see Definition 2.1). Consequently, there exists $t > 0$ such that, for every $y \in (0, t) \setminus E$, there exists $x \in E$ satisfying the inequalities

$$x < y \quad \text{and} \quad \frac{y - x}{y} < s.$$

These inequalities imply that

$$x < y < \frac{x}{1 - s}.$$

Hence, $y \in (q^{-1}x, qx)$ holds with $q = \frac{1}{1-s}$. Thus, the inclusion $(0, t) \setminus E \subseteq E(q)$ holds for such q. Since $E \cap (0, t) \subseteq E(q)$ holds for all $t > 0$ and $q > 1$, we obtain

$$(0, t) = (E \cap (0, t)) \cup ((0, t) \setminus E) \subseteq E(q) \cup E(q) = E(q).$$

Thus, $(0, t) \subseteq (0, t) \cap E(q) \subseteq (0, t)$, which implies (5.8). □

6. Blow up of strongly porous at 0 sets

Let us prove that the q-blow up preserves $\boldsymbol{SP}$.

Lemma 6.1. *Let $E \subseteq \mathbb{R}^+$ and $q > 1$. Then E belongs to $\boldsymbol{SP}$ if and only if $E(q)$ belongs to $\boldsymbol{SP}$.*

Proof. Since $E(q) = (E \setminus \{0\})(q)$ and

$$(E \in \boldsymbol{SP}) \Leftrightarrow (E \setminus \{0\} \in \boldsymbol{SP}),$$

we may assume that $0 \notin E$. In accordance with (5.2), this assumption implies the inclusion

$$E \subseteq E(q). \tag{6.1}$$

Since $\boldsymbol{SP}$ is a down set, the implication $(E(q) \in \boldsymbol{SP}) \Rightarrow (E \in \boldsymbol{SP})$ follows.

Let $E \in \boldsymbol{SP}$. Then there is a sequence $\{(a_n, b_n)\}_{n \in \mathbb{N}}$ such that $0 < a_n < b_n$, $b_n \downarrow 0$, $(a_n, b_n) \cap E = \varnothing$ and $\lim\limits_{n \to \infty} \frac{a_n}{b_n} = 0$. It is easy to prove that $q a_n < q^{-1} b_n$ and $(q a_n, q^{-1} b_n) \cap E(q) = \varnothing$ for all sufficiently large n. Since

$$\lim_{n \to \infty} \frac{q a_n}{q^{-1} b_n} = \lim_{n \to \infty} q^2 \frac{a_n}{b_n} = 0,$$

the set $E(q)$ is strongly porous on the right at 0. The implication $(E \in \boldsymbol{SP}) \Rightarrow (E(q) \in \boldsymbol{SP})$ follows. Thus,

$$(E \in \boldsymbol{SP}) \Leftrightarrow (E(q) \in \boldsymbol{SP})$$

holds. □

Corollary 6.2. *Let $E \subseteq \mathbb{R}^+$ and $q > 1$. Then $E \in I^*(\boldsymbol{SP})$ holds if and only if $E(q) \in I^*(\boldsymbol{SP})$.*

Proof. As in the proof of Lemma 6.1, we may suppose that $E(q) \supseteq E$. This yields $(E(q) \in I^*(\boldsymbol{SP})) \Rightarrow (E \in I^*(\boldsymbol{SP}))$. Let $E \in I^*(\boldsymbol{SP})$. The relation $E(q) \in I^*(\boldsymbol{SP})$ holds if and only if

$$E(q) \cup B \in \boldsymbol{SP} \quad \text{for every } B \in \boldsymbol{SP}. \tag{6.2}$$

Using the relation

$$(B \in \boldsymbol{SP}) \Leftrightarrow (B \setminus \{0\} \in \boldsymbol{SP})$$

we may consider only the case where $0 \notin B$. The membership $E \in I^*(\boldsymbol{SP})$ implies $E \cup B \in \boldsymbol{SP}$. Consequently, by Lemma 6.1, we obtain

$$E(q) \cup B(q) \in \boldsymbol{SP}. \tag{6.3}$$

Since $0 \notin B$, the inclusion $B \subseteq B(q)$ holds. The last inclusion and (6.3) yield (6.2). □

Let A and B be nonempty subsets of $\mathbb{R}^+$. We define $A \prec B$ if $b < a$ holds for every $b \in B$ and $a \in A$. Furthermore, we set

$$A \preceq B \quad \text{if} \quad A = B \quad \text{or} \quad A \prec B.$$

The relation $\preceq$ is a partial order on the set of nonempty subsets of $\mathbb{R}^+$. A chain (i.e., a linearly ordered set) $(P, \leq_P)$ is said to be well-ordered if every nonempty subset X of P contains a smallest element, i.e., an element $x \in X$ such that $x \leq_P y$ for every $y \in X$.

It is easy to prove, that for every nonempty $A \subseteq \mathbb{R}^+$, the set $\mathrm{Cc}A$ of connected components of A is a chain w. r. t. the partial order $\preceq$. Define a set $\mathrm{Cc}^1 A$ by the rule:

$$B \in \mathrm{Cc}^1 A \quad \text{if} \quad B \in \mathrm{Cc}A \quad \text{and} \quad B \subset (0, 1].$$

Lemma 6.3. *Let $\varnothing \neq E \subseteq \mathbb{R}^+$ and let $q > 1$. Then the chain $(\mathrm{Cc}^1 E(q), \preceq)$ is well-ordered.*

Proof. If there is $X \subseteq \mathrm{Cc}^1 E(q)$, which does not have a smallest element, then there is a sequence $\{(a_i, b_i)\}_{i \in \mathbb{N}}$ such that

$$(a_1, b_1) \succ (a_2, b_2) \succ ...(a_i, b_i) \succ (a_{i+1}, b_{i+1}) \succ ...$$

with $(a_i, b_i) \in X$ for every $i \in \mathbb{N}$. The equalities

$$\ln a_1^{-1} = (\ln a_1^{-1} - \ln b_1^{-1}) + \ln b_1^{-1}$$

$$= (\ln a_1^{-1} - \ln b_1^{-1}) + (\ln b_1^{-1} - \ln a_2^{-1}) + (\ln a_2^{-1} - \ln b_2^{-1}) + \ln b_2^{-1}$$

$$= ... = \sum_{k=1}^{i+1} (\ln a_k^{-1} - \ln b_k^{-1}) + \sum_{k=1}^{i} (\ln b_k^{-1} - \ln a_{k+1}^{-1}) + \ln b_{i+1}^{-1}$$

and the inequalities

$$\ln a_k^{-1} > \ln b_k^{-1} \geq \ln a_{k+1}^{-1} > \ln b_{k+1}^{-1} \geq 0,$$

$k = 1, ..., i+1$ imply that

$$\ln a_1^{-1} \geq \sum_{k=1}^{i+1} (\ln a_k^{-1} - \ln b_k^{-1}). \tag{6.4}$$

Since $X \subseteq \mathrm{Cc}^1 E(q)$, the intersection $(a_k, b_k) \cap E$ is nonempty for every $k = 1, ..., i$. It follows directly from the definition of q-blow up, that the inclusion

$$(q^{-1}x, qx) \subseteq (a_k, b_k) \tag{6.5}$$

holds for every $x \in E \cap (a_k, b_k)$. Conditions (6.4) and (6.5) yield the inequalities

$$\ln a_1^{-1} \geq \sum_{k=1}^{i+1} \ln \frac{b_k}{a_k} \geq \sum_{k=1}^{i+1} \ln q^2 = 2(i+1) \ln q.$$

Letting $i \to \infty$, we obtain the equality $\ln a_1^{-1} = \infty$, contrary to $(a_1, b_1) \in \mathrm{Cc}^1 E(q)$. □

The proof of Lemma 6.3 shows, in particular, that for given $q > 1$ and $(a, b) \in \mathrm{Cc}^1 E(q)$, the set $\{(c, d) \in \mathrm{Cc}^1 E(q) : (c, d) \preceq (a, b)\}$ is finite. This finiteness together with Lemma 6.3 implies the following

Corollary 6.4. *Let $\varnothing \neq E \subseteq \mathbb{R}^+$ and let $q > 1$. If $\mathrm{Cc}^1 E(q) \neq \varnothing$, then the chain $(\mathrm{Cc}^1 E(q), \preceq)$ is isomorphic either the first infinite ordinal number ω or an initial segment of ω.*

For a set $E \subseteq \mathbb{R}^+$, we use the symbol $\mathrm{ac}E$ to denote the set of its accumulation points.

Remark 6.5. *Let $E \subseteq \mathbb{R}^+$ and $q > 1$. Then $(\mathrm{Cc}^1 E(q), \preceq)$ is isomorphic to ω if and only if $0 \in \mathrm{ac}E(q)$ and $0 \in \mathrm{ac}(\mathbb{R}^+ \setminus E(q))$. In particular, if $E \in \boldsymbol{SP}$, then $\mathrm{Cc}^1 E(q)$ is isomorphic to ω if and only if $0 \in \mathrm{ac}E$.*

Corollary 6.4 means, in particular, that for every infinite $\mathrm{Cc}^1 E(q)$ there is a unique sequence $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ such that the logical equivalence

$$((a,b) \in \mathrm{Cc}^1 E(q)) \Leftrightarrow (\exists\, i \in \mathbb{N} : (a,b) = (a_i, b_i)) \tag{6.6}$$

holds for every interval $(a,b) \subseteq \mathbb{R}^+$ and logical equivalence

$$((a_i, b_i) \prec (a_j, b_j)) \Leftrightarrow (i < j) \tag{6.7}$$

holds for all $i, j \in \mathbb{N}$. If a sequence $\{(a_i, b_i)\}_{i\in\mathbb{N}}$ satisfies (6.6) $-$ (6.7) we shall write

$$\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i\in\mathbb{N}}.$$

The following theorem is a blow up characterization of the ideal $\hat{I}(\mathbf{SP})$.

Theorem 6.6. *Let $E \subseteq \mathbb{R}^+$ and $0 \in \mathrm{ac}E$. Then the following conditions are equivalent.*

(i) $E \in \hat{I}(\boldsymbol{SP})$.

(ii) *For every $q > 1$, the chain $\mathrm{Cc}^1 E(q)$ is infinite, $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i\in\mathbb{N}}$, and the inequality*

$$\limsup_{i\to\infty} \frac{b_i}{a_i} < \infty \tag{6.8}$$

holds.

Proof. (i) $\Rightarrow$ (ii). In accordance with Theorem 4.4, the equality $\hat{I}(\mathbf{SP}) = I^*(\mathbf{SP})$ holds, so that $(E \in \hat{I}(\boldsymbol{SP})) \Leftrightarrow (E \in I^*(\boldsymbol{SP}))$. Suppose that $E \in I^*(\boldsymbol{SP})$ and $q > 1$. Then, by Corollary 6.2, $E(q) \in I^*(\boldsymbol{SP})$ holds. Since $\boldsymbol{SP}$ is closed under subsets, it follows directly from the definition of $I^*(\boldsymbol{SP})$ that $I^*(\boldsymbol{SP}) \subseteq \boldsymbol{SP}$. Consequently, the equality $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i\in\mathbb{N}}$ holds. (See Remark 6.5). Suppose that

$$\limsup_{i\to\infty} \frac{b_i}{a_i} = \infty. \tag{6.9}$$

Let us consider the set

$$B := \mathbb{R}^+ \setminus \left(\bigcup_{i\in\mathbb{N}} (a_i, b_i) \right).$$

Definition 2.1 and (6.9) imply that $B \in \boldsymbol{SP}$. Consequently, by the definition of $I^*(\boldsymbol{SP})$ we must have $B \cup E(q) \in \boldsymbol{SP}$. It is clear from the definition of B that

$$(0, b_1) \subseteq B \cup E(q).$$

Hence the interval $(0, b_1)$ must be strongly porous on the right at 0, contrary to Definition 2.1. Hence (i) implies (ii).

(ii) ⇒ (i). Suppose now, that condition (ii) holds, but $E \notin I^*(\boldsymbol{SP})$. Then, there is $B \in \boldsymbol{SP}$ such that $B \cup E \notin \boldsymbol{SP}$. By Lemma 5.4, we can find $q > 1$ and $t > 0$ such that the q-blow-up of $B \cup E$ is a superset of the interval $(0, t)$, i. e.

$$B(q) \cup E(q) \supseteq (0, t). \tag{6.10}$$

Lemma 6.1 shows that $B(q) \in \boldsymbol{SP}$. Consequently, there is a sequence $\{(a_j^*, b_j^*)\}_{j\in\mathbb{N}}$ of open intervals (a_j^*, b_j^*) such that

$$0 < a_j^* < b_j^* < \infty,\ a_j^* \downarrow 0,\ (a_j^*, b_j^*) \cap B(q) = \varnothing \quad \text{and} \quad \lim_{j\to\infty} \frac{b_j^*}{a_j^*} = \infty \tag{6.11}$$

hold for every $j \in \mathbb{N}$. Inclusion (6.10) and relations (6.11) imply that $(a_j^*, b_j^*) \subseteq E(q)$ hold for all sufficiently large $j \in \mathbb{N}$. Using condition (ii) of the present lemma, we can find a subsequence $\{(a_{i_k}, b_{i_k})\}_{k\in\mathbb{N}}$ of the sequence $\{(a_i, b_i)\}_{i\in\mathbb{N}}$, where $\{(a_i, b_i)\}_{i\in\mathbb{N}} = \mathrm{Cc}^1 E(q)$ and a subsequence $\{(a_{j_k}^*, b_{j_k}^*)\}_{k\in\mathbb{N}}$ of the sequence $\{(a_j^*, b_j^*)\}_{j\in\mathbb{N}}$, so that $(a_{j_k}^*, b_{j_k}^*) \subseteq (a_{i_k}, b_{i_k})$ for every $k \in \mathbb{N}$. Consequently, we obtain

$$\limsup_{i\to\infty} \frac{b_i}{a_i} \geq \limsup_{k\to\infty} \frac{b_{i_k}}{a_{i_k}} \geq \limsup_{k\to\infty} \frac{b_{j_k}^*}{a_{j_k}^*} = \lim_{j\to\infty} \frac{b_j^*}{a_j^*} = \infty,$$

contrary to (6.8). □

7. Ideal generated by $\boldsymbol{CSP}$

The goal of the present section is to obtain the blow up characterization of the ideal $I(\boldsymbol{CSP})$.

The following lemma is a direct consequence of Theorem 36 and Theorem 42 from [5].

Lemma 7.1. *Let $E \subseteq \mathbb{R}$. Then $E \in \boldsymbol{CSP}$ if and only if there are $q > 1$, $t > 0$ and a decreasing sequence $\{x_n\}_{n\in\mathbb{N}}$, such that $x_n > 0$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = 0$ and*

$$E \cap (0, t) \subseteq \left(\bigcup_{n\in\mathbb{N}} (q^{-1}x_n, qx_n) \right) \cap (0, t).$$

In this section, for every $n \in \mathbb{N}$, we denote by $\boldsymbol{n}$ the set $\{1, 2, ..., n\}$.

Lemma 7.2. *Let $E \subseteq \mathbb{R}^+$ and $q > 1$. Then the following logical equivalence*

$$(E \in I(\boldsymbol{CSP})) \Leftrightarrow (E(q) \in I(\boldsymbol{CSP}))$$

holds.

Proof. As in the proof of Lemma 6.1, we may assume that $0 \notin E$. In accordance with Remark 5.1, this assumption implies the inclusion

$$E \subseteq E(q). \tag{7.1}$$

Now the implication

$$(E(q) \in I(\boldsymbol{CSP})) \Rightarrow (E \in I(\boldsymbol{CSP}))$$

follows from (7.1), because $I(\boldsymbol{CSP})$ is a down set. To prove the converse implication suppose that $E \in I(\boldsymbol{CSP})$. Then there are $B_1, ..., B_n \in \boldsymbol{CSP}$, such that $E = B_1 \cup ... \cup B_n$. The last equality implies that $E(q) = B_1(q) \cup ... \cup B_n(q)$. Consequently $E(q) \in I(\boldsymbol{CSP})$ holds if $B_j(q) \in \boldsymbol{CSP}$ for every $j \in \boldsymbol{n}$. By Lemma 7.1, for every $j \in \boldsymbol{n}$, we can find $q_j > 1$, $t_j > 0$, and a decreasing sequence $\{x_{k,j}\}_{k\in\mathbb{N}}$ of positive numbers such that $\lim\limits_{k\to\infty} \frac{x_{k+1,j}}{x_{k,j}} = 0$ and

$$(0, t_j) \cap B_j \subseteq (0, t_j) \cap \bigcup_{k\in\mathbb{N}} (q_j^{-1} x_{k,j}, q_j x_{k,j}). \tag{7.2}$$

Statement (ii) of Lemma 5.2, Lemma 5.3 and (7.2) imply

$$(0, t_j q^{-1}) \cap B_j(q) \subseteq (0, t_j q^{-1}) \cap \bigcup_{k\in\mathbb{N}} (q^{-1} q_j^{-1} x_{k,j}, q q_j x_{k,j}).$$

Hence, by Lemma 7.1, the statement $B_j(q) \in \boldsymbol{CSP}$ holds for every $j \in \boldsymbol{n}$. $\square$

Lemma 7.3. *Let $E \subseteq \mathbb{R}^+$, $q > 1$ and let $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i\in\mathbb{N}}$. Suppose that*

$$\limsup_{i\to\infty} \frac{b_i}{a_i} < \infty \tag{7.3}$$

and there is $N \in \mathbb{N}$ such that

$$\lim_{n\to\infty} \bigvee_{j=0}^{N} \frac{a_{n+j}}{b_{n+j+1}} = \infty \tag{7.4}$$

where

$$\bigvee_{j=0}^{N}\frac{a_{n+j}}{b_{n+j+1}}=\max\left\{\frac{a_n}{b_{n+1}},\frac{a_{n+1}}{b_{n+2}},...,\frac{a_{n+N}}{b_{n+N+1}}\right\}.$$

Then there are $B_1,...,B_{2N+2}\in$ ***CSP*** *such that*

$$E\subseteq B_1\cup...\cup B_{2N+2}. \tag{7.5}$$

Proof. Suppose $N\in\mathbb{N}$ is a number such that (7.4) holds. Let us define a sequence $\{F_k\}_{k\in\mathbb{N}}$ of sets $F_k\subseteq\mathbb{N}$ as $F_1:=\{1,...,N+1\}$, $F_2:=\{(N+1)+1,...,2(N+1)\}$, $F_3:=\{2(N+1)+1,...,3(N+1)\}$ and so on. It is clear that $\bigcup\limits_{k=1}^{\infty}F_k=\mathbb{N}$ and $F_{k_1}\cap F_{k_2}=\varnothing$ if $k_1\neq k_2$, and

$$|F_k|=N+1\quad\text{for every } k\in\mathbb{N}. \tag{7.6}$$

Let $m_k\in F_k$ be a number satisfying the condition

$$\frac{a_{m_k}}{b_{m_k+1}}=\bigvee_{n\in F_k}\frac{a_n}{b_{n+1}}. \tag{7.7}$$

It follows from (7.4), (7.6) and (7.7) that

$$\lim_{k\to\infty}\frac{a_{m_k}}{b_{m_k+1}}=\infty. \tag{7.8}$$

The definition of F_k and (7.6) imply the double inequality

$$1\leq m_{k+1}-m_k\leq 2N+1. \tag{7.9}$$

For every $k\in\mathbb{N}$ denote by $\mathfrak{F}_k$ the set of all connected components of $E(q)$, which lie between $[b_{m_k+2},a_{m_k+1}]$ and $[b_{m_k+1},a_{m_k}]$,

$$\mathfrak{F}_k:=\{(a_n,b_n):[b_{m_k+2},a_{m_k+1}]\succ(a_n,b_n)\succ[b_{m_k+1},a_{m_k}]\}. \tag{7.10}$$

It easy to show that

$$\bigcup_{k=m_1}^{\infty}(a_{k+1},b_{k+1})=\bigcup_{k=1}^{\infty}\bigcup\mathfrak{F}_k \tag{7.11}$$

and $\mathfrak{F}_i \cap \mathfrak{F}_j = \varnothing$ if $i \neq j$. From (7.9) it also follows that $1 \leq |\mathfrak{F}_k| \leq 2N+1$ for every $k \in \mathbb{N}$. Consequently, for every $k \in \mathbb{N}$, the elements of $\mathfrak{F}_k$ can be numbered (with some repetitions if it is necessary) in a finite sequence $(a_{k,1}, b_{k,1})$, $(a_{k,2}, b_{k,2})$, ..., $(a_{k,2N+1}, b_{k,2N+1})$. Using the inclusion

$$E(q) \subseteq \bigcup_{n=1}^{\infty} (a_{n+1}, b_{n+1}) \cup (a_1, \infty)$$

and (7.11) we obtain

(7.12)

$$E(q) \subseteq \bigcup_{k \in \mathbb{N}} \left(\bigcup_{j=1}^{2N+1} (a_{k,j}, b_{k,j}) \right) \cup (a_{m_1}, \infty) = \bigcup_{j=1}^{2N+1} \left(\bigcup_{k \in \mathbb{N}} (a_{k,j}, b_{k,j}) \right) \cup (a_{m_1}, \infty).$$

Write

$$B_j := \bigcup_{k \in \mathbb{N}} (a_{k,j}, b_{k,j})$$

for every $j \in \boldsymbol{2N+1}$, where $\boldsymbol{2N+1} = \{1, ..., 2N+1\}$, and put $B_{2N+2} := \{0\} \cup (a_{m_1}, \infty)$. Now we have $E \subseteq E(q) \cup \{0\} \subseteq B_1 \cup ... \cup B_{2N+2}$. It still remains to prove that $B_j \in$ ***CSP*** for $j = 1, ..., 2N+2$. The statement $B_{2N+2} \in$ ***CSP*** is clear. Let $j \in \boldsymbol{2N+1}$. In accordance with the Definition 2.2, the statement $B_j \in$ ***CSP*** holds if for every $\tilde{h} = \{h^l\}_{l \in \mathbb{N}} \in \tilde{B}_j$ there is $\tilde{a} = \{a^l\}_{l \in \mathbb{N}} \in \tilde{H}(B_j)$ such that $\tilde{h} \asymp \tilde{a}$. Inequality (7.3) and the definition of B_j imply that there is a positive constant $c > 1$ such that

$$a_{k,j} \leq x \leq c a_{k,j}$$

for every $x \in (a_{k,j}, b_{k,j})$ and every $k \in \mathbb{N}$. Consequently, if $\{h^l\}_{l \in \mathbb{N}} \in \tilde{B}_j$, then we have $\{h^l\}_{l \in \mathbb{N}} \asymp \{a^l\}_{l \in \mathbb{N}}$, where, for every $l \in \mathbb{N}$, a^l is the left endpoint of the interval $(a_{k,j}, b_{k,j})$ which contains h^l. Hence, $B_j \in$ ***CSP*** holds if $\{a_{k,j}\}_{k \in \mathbb{N}} \in \tilde{H}(B_j)$, which is equivalent to

(7.13)
$$\lim_{k \to \infty} \frac{a_{k,j}}{b_{k+1,j}} = \infty.$$

Let us prove (7.13). It follows from (7.10) that

$$[b_{m_k+2}, a_{m_k+1}] \succ (a_{k,j}, b_{k,j}) \succ [b_{m_k+1}, a_{m_k}]$$

and

$$[b_{m_k+3}, a_{m_k+2}] \succ (a_{k+1,j}, b_{k+1,j}) \succ [b_{m_k+2}, a_{m_k+1}].$$

Hence we have

$$(a_{k+1,j}, b_{k+1,j}) \succ [b_{m_k+2}, a_{m_k+1}] \succ (a_{k,j}, b_{k,j}).$$

Consequently the inequality

$$\frac{a_{k,j}}{b_{k+1,j}} \leq \frac{a_{m_k+1}}{b_{m_k+2}}$$

holds. The last inequality and (7.8) imply (7.13). □

Corollary 7.4. *Let $E \subseteq \mathbb{R}^+$. If there are $N \in \mathbb{N}$ and $q > 1$ so that $\mathrm{Cc}^1 E(q)$ is infinite and conditions* (7.3) *and* (7.4) *hold, then $E \in I(\boldsymbol{CSP})$.*

In the following lemma, as in Lemma 7.3, the equality $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i \in \mathbb{N}}$ means that conditions (6.6) and (6.7) are satisfied.

Lemma 7.5. *Let $E \in I(\boldsymbol{CSP})$ and let $0 \in \mathrm{ac}E$. Then $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i \in \mathbb{N}}$ for every $q > 1$, and there are $q_0 > 1$ and $M \in \mathbb{N}$ such that the conditions*

$$\limsup_{i \to \infty} \frac{b_i}{a_i} < \infty \tag{7.14}$$

and

$$\lim_{n \to \infty} \bigvee_{j=0}^{M} \frac{a_{n+j}}{b_{n+j+1}} = \infty \tag{7.15}$$

hold for every $q > q_0$.

Proof. It follows from the definition of $I(\boldsymbol{CSP})$ that there is $N \in \mathbb{N}$ such that

$$E = B_1 \cup ... \cup B_N \quad \text{with some } B_1, ..., B_N \in \boldsymbol{CSP}. \tag{7.16}$$

Let $\boldsymbol{N} = \{1, ..., N\}$. We may assume $0 \in \mathrm{ac}B_j$ for every $j \in \boldsymbol{N}$. Indeed, if $0 \notin \mathrm{ac}B_j$ for all $j \in \boldsymbol{N}$, then

$$0 \notin \mathrm{ac}(B_1 \cup ... \cup B_N) = \mathrm{ac}E,$$

contrary to the condition $0 \in \mathrm{ac}E$. Hence, there is $j_1 \in \boldsymbol{N}$ such that $0 \in \mathrm{ac}B_{j_1}$. Write

$$\boldsymbol{J}_0 := \{j \in \boldsymbol{N} : \mathrm{ac}B_j \not\ni 0\}, \quad \boldsymbol{J}_1 := \{j \in \boldsymbol{N} : \mathrm{ac}B_j \ni 0\} \quad \text{and} \quad B'_j := B_j \cup \left(\bigcup_{i \in \boldsymbol{J}_0} B_i \right)$$

for every $j \in \boldsymbol{J}_1$. Renumbering the elements of $\boldsymbol{N}$, we may also assume that $\boldsymbol{J}_1 = \{1, ..., N_1\}$ with $N_1 \leq N$. Then the representation

$$E = B'_1 \cup ... \cup B'_{N_1}$$

holds with $B'_j \in \boldsymbol{CSP}$ and $\mathrm{ac}B'_j \ni 0$ for every $j \in \boldsymbol{N}_1$. Without loss of generality, we put $\boldsymbol{N}_1 = \boldsymbol{N}$ and $B_j = B'_j$ for every $j \in \boldsymbol{N}_1$.

Using Lemma 7.1, for every $j \in \boldsymbol{N}$, we can find $q_j \in (1, \infty)$ and a strictly decreasing sequence $\{x_{j,n}\}_{n \in \mathbb{N}}$ with

$$\lim_{n\to\infty} \frac{x_{j,n+1}}{x_{j,n}} = 0 \quad \text{and} \quad \lim_{n\to\infty} x_{j,n} = 0, \tag{7.17}$$

so that the inclusion

$$B_j \cap (0, x_{j,1}) \subseteq \bigcup_{n\in\mathbb{N}} (q_j^{-1} x_{j,n}, q_j x_{j,n}) \tag{7.18}$$

holds. Write

$$B_{j,n} := B_j \cap (q_j^{-1} x_{j,n}, q_j x_{j,n}) \tag{7.19}$$

for all $n \in \mathbb{N}$ and $j \in \boldsymbol{N}$, and define

$$B_{j,0} := B_j \cap [q_j x_{j,1}, \infty) \tag{7.20}$$

for every $j \in \boldsymbol{N}$. Inclusion (7.18) implies that

$$B_j \setminus \{0\} = \bigcup_{n=0}^{\infty} B_{j,n} \tag{7.21}$$

and from (7.16) it follows that

$$E \setminus \{0\} = \bigcup_{j=1}^{N} \left(\bigcup_{n=0}^{\infty} B_{j,n} \right). \tag{7.22}$$

Replacing the sequences $\{x_{j,n}\}_{n\in\mathbb{N}}$ by suitable subsequences, we may assume that

$$B_{j,n} \neq \varnothing \quad \text{for every } j \in \boldsymbol{N} \text{ and } n \in \mathbb{N}. \tag{7.23}$$

Recall that $0 \in \mathrm{ac}B_j$ holds for every $j \in \boldsymbol{N}$. Let $q \geq \bigvee_{j=1}^{N} q_j^2$. Lemma 5.2, the implication $E \subseteq (a, b) \Rightarrow E(q) \subseteq (q^{-1}a, qb)$ and (7.23) imply that $B_{j,n}(q)$ are open intervals. Write

$$B_{j,n}(q) := (r_{j,n}, s_{j,n}), \quad n \in \mathbb{N},\ j \in \boldsymbol{N}. \tag{7.24}$$

Consequently, from statement (ii) of Lemma 5.2 and

$$B_{j,n} \subseteq (q_j^{-1} x_{j,n}, q_j x_{j,n}) \quad \text{and} \quad q \geq \bigvee_{j=1}^{N} q_j^2$$

it follows that

$$(r_{j,n}, s_{j,n}) = B_{j,n}(q) \subseteq (q^{-1}q_j^{-1}x_{j,n}, qq_jx_{j,n}) \subseteq (q^{-\frac{3}{2}}x_{j,n}, q^{\frac{3}{2}}x_{j,n}).$$

Hence the inequality

$$\frac{s_{j,n}}{r_{j,n}} \leq q^3 \tag{7.25}$$

holds for all $n \in \mathbb{N}$ and $j \in \boldsymbol{N}$. Since

$$x_{j,n} \in (s_{j,n}, r_{j,n}) \quad \text{and} \quad x_{j,n+1} \in (s_{j,n+1}, r_{j,n+1}),$$

inequality (7.25) and limit relation (7.17) imply that

$$\lim_{n\to\infty} \frac{r_{j,n}}{s_{j,n+1}} = \infty. \tag{7.26}$$

Hence there is $m_1 \in \mathbb{N}$ such that

$$\frac{r_{j,n}}{s_{j,n+1}} \geq q^{3(N+1)} \tag{7.27}$$

holds for all $n \in \mathbb{N} \setminus \boldsymbol{m_1}$ and $j \in \boldsymbol{N}$. Using (7.25) and (7.27), we see, in particular, that

$$(r_{j,n_1}, s_{j,n_1}) \cap (r_{j,n_2}, s_{j,n_2}) = \varnothing \tag{7.28}$$

if $n_1, n_2 \in \mathbb{N} \setminus \boldsymbol{m_1}$, $n_1 \neq n_2$ and $j \in \boldsymbol{N}$. This disjointness together with (7.21) and (7.24) yield

$$B_j(q) = \bigcup_{n=0}^{\infty} B_{j,n}(q) = \bigcup_{n=m_1+1}^{\infty} (r_{j,n}, s_{j,n}) \cup O_{j,q,m_1} \tag{7.29}$$

for every $j \in \boldsymbol{N}$ with $O_{j,q,m_1} := B_j(q) \cap [r_{j,m_1}, \infty)$. Note that, as was shown in Remark 5.1, $0 \notin E(q)$ for every $q > 1$ and $E \subseteq \mathbb{R}^+$.

Obviously, for every $x \in E(q)$ there is a unique connected component (a_x, b_x), $a_x = a_x(q)$ and $b_x = b_x(q)$, of the set $E(q)$ such that $x \in (a_x, b_x)$. As is easily seen the following statements are valid:

- The chain $(\mathrm{Cc}^1 E(q), \preceq)$ is infinite if there is $t \in (0, \infty)$ such that $a_x > 0$ for every $x \in (0, t) \cap E(q)$;

• Inequality (7.14) holds if there are $t \in (0, \infty), k \in (1, \infty)$ and $p \in \mathbb{N}$ such that

$$k^{-p}x < a_x \tag{7.30}$$

for every $x \in (0, t) \cap E(q)$.

Note also that the inequalities $q_1 \geq q_2 > 1$ imply the inclusion $E(q_1) \supseteq E(q_2)$. Thus, the inclusion $(a_x(q_1), b_x(q_1)) \supseteq (a_x(q_2), b_x(q_2))$ holds if $q_1 \geq q_2 > 1$. Consequently, to prove the first part of the lemma it is sufficient to show that (7.30) holds if

$$q \geq \bigvee_{j=1}^{N} q_j^2 \quad \text{and} \quad x \in (0, r^1) \cap E(q) \tag{7.31}$$

where

$$r^1 := \bigwedge_{j=1}^{N} r_{j,m_1}. \tag{7.32}$$

Let $x \in \mathbb{R}^+$. To find $k \in (1, \infty)$ and $p \in \mathbb{N}$, which satisfy (7.30), we define a subset $\boldsymbol{J}_x$ of $\boldsymbol{N}$ by the rule:

$$(j \in \boldsymbol{J}_x) \Leftrightarrow (j \in \boldsymbol{N} \quad \text{and} \quad x \in (0, r^1) \cap B_j(q)), \tag{7.33}$$

where r^1 is defined in (7.32). From (7.33) it is clear that

$$(\boldsymbol{J}_x = \varnothing) \Leftrightarrow (x \in [r^1, \infty) \quad \text{or} \quad x \in \mathbb{R}^+ \setminus E(q)). \tag{7.34}$$

Let (7.32) hold and let

$$\theta \in (q^3, q^{3(N+1)}). \tag{7.35}$$

We claim that if $\boldsymbol{J}_x \neq \varnothing \neq \boldsymbol{J}_{\theta^{-1}x}$, then the equality

$$\boldsymbol{J}_x \cap \boldsymbol{J}_{\theta^{-1}x} = \varnothing \tag{7.36}$$

holds. Suppose contrary, that $\boldsymbol{J}_x \neq \varnothing \neq \boldsymbol{J}_{\theta^{-1}x}$ holds, but there is $j_0 \in \boldsymbol{N}$ such that $j_0 \in \boldsymbol{J}_x \cap \boldsymbol{J}_{\theta^{-1}x}$. Then, using (7.33), we see that there are $n_1, n_2 \in \mathbb{N} \setminus \boldsymbol{m_1}$, so that

$$x \in (r_{j_0,n_2}, s_{j_0,n_2}) \quad \text{and} \quad \theta^{-1}x \in (r_{j_0,n_1}, s_{j_0,n_1}). \tag{7.37}$$

If $n_1 = n_2$, then the inequalities $r_{j_0,n_1} < \theta^{-1}x < x < s_{j_0,n_1}$ hold. Hence, we have

$$\theta = \frac{x}{\theta^{-1}x} \leq \frac{s_{j_0,n_1}}{r_{j_0,n_1}}.$$

Now, using (7.35), we obtain

$$q^3 < \theta \leq \frac{s_{j_0,n_1}}{r_{j_0,n_1}},$$

contrary to (7.25). Hence, $n_1 \neq n_2$. The relations $\theta^{-1}x < x$ and $n_1 \neq n_2$ imply the inequality $n_1 > n_2$. Consequently, $n_2 < n_2 + 1 \leq n_1$. These inequalities and (6.7), imply

$$(r_{j_0,n_2}, s_{j_0,n_2}) \prec (r_{j_0,n_2+1}, s_{j_0,n_2+1}) \preceq (r_{j_0,n_1}, s_{j_0,n_1}).$$

Hence,

$$\theta = \frac{x}{\theta^{-1}x} \geq \frac{r_{j_0,n_1+1}}{s_{j_0,n_1}}. \tag{7.38}$$

From (7.35) and (7.38) it follows that

$$q^{3(N+1)} > \frac{r_{j_0,n_1+1}}{s_{j_0,n_1}},$$

contrary to (7.27). Thus, (7.36) holds, if $\boldsymbol{J}_x \neq \varnothing$ and $\boldsymbol{J}_{\theta^{-1}x} \neq \varnothing$.

Now, let $k \in (q^3, q^{\frac{3(N+1)}{N}})$. It is simple to prove that

$$q^3 < k < ... < k^N < q^{3(N+1)}.$$

Hence (7.35) holds, if $\theta = k^m$ and $m \in \boldsymbol{N}$. Consequently, if we have

$$\boldsymbol{J}_{k^{-m}x} \neq \varnothing \tag{7.39}$$

for every $m \in \boldsymbol{N} \cup \{0\}$, then

$$\boldsymbol{J}_{k^{-m_1}x} \cap \boldsymbol{J}_{k^{-m_2}x} = \varnothing \tag{7.40}$$

for all distinct $m_1, m_2 \in \boldsymbol{N} \cup \{0\}$. (To see it suppose $m_1 < m_2$ and replace in (7.35) x and $\theta^{-1}x$ by $k^{-m_1}x$ and $k^{-(m_2-m_1)}k^{-m_1}x$ respectively.) By (7.40), $\boldsymbol{J}_x, \boldsymbol{J}_{k^{-1}x}, ...\boldsymbol{J}_{k^{-N}x}$ are disjoint subsets of $\boldsymbol{N}$. Hence, if (7.39) holds, then

$$N = |\boldsymbol{N}| \geq \sum_{l=0}^{N} |\boldsymbol{J}_{k^{-l}x}| \geq \sum_{l=0}^{N} 1 = N + 1. \tag{7.41}$$

This contraction shows, that there is $l \in \boldsymbol{N} \cup \{0\}$ such that

$$\boldsymbol{J}_{k^{-l}x} = \varnothing. \tag{7.42}$$

Assume that $x \in (0, r^1) \cap E(q)$. By (7.33), equality (7.42) holds if and only if

$$k^{-l}x \in [r^1, \infty) \quad \text{or} \quad k^{-l}x \in \mathbb{R}^+ \setminus E(q).$$

Since $0 < k^{-l}x < x < r^1$, (7.42) yields that $k^{-l}x \notin E(q)$. Since (a_x, b_x) is a connected component of the set $E(q)$, it is proved, that the inequality

$$k^{-N}x < a_x \tag{7.43}$$

holds whenever $x \in (a_x, b_x) \in \mathrm{Cc}^1 E(q)$, $x < r^1$ and $q \geq \bigvee_{j=1}^{N} q_j^2$. Since $(\mathrm{Cc}^1 E(q), \preceq)$ is infinite for every $q > 1$, the assertion (7.14) holds for

$$q > q_0 := \bigvee_{j=1}^{N} q_j^2. \tag{7.44}$$

To complete the proof it suffices to show that (7.15) holds with $M = N$.

Let (7.44) hold and let

$$(a_i, b_i) \in \{(a_n, b_n)\}_{n \in \mathbb{N}} = \mathrm{Cc}^1 E(q).$$

For $i \in \mathbb{N}$ define a set $\boldsymbol{J}_i \subseteq \boldsymbol{N}$ as

$$\boldsymbol{J}_i := \bigcup_{x \in (a_i, b_i)} \boldsymbol{J}_x \tag{7.45}$$

where $\boldsymbol{J}_x$ was defined by (7.33). It follows from (7.45) and (7.34) that there is $i_0 \in \mathbb{N}$ such that $\boldsymbol{J}_i \neq \varnothing$ for $i \geq i_0$, i. e.

$$(a_i, b_i) \cap (0, r^1) \cap E(q) \neq \varnothing,$$

for $i \geq i_0$. Hence, without loss of generality, we may suppose that if $x \in (a_i, b_i)$ and $i \geq i_0$, then $x < r^1$. Consequently, for every $i \geq i_0$ there is $l \in \boldsymbol{N}$ such that

$$\boldsymbol{J}_i \cap \boldsymbol{J}_{i+l} \neq \varnothing. \tag{7.46}$$

Otherwise, the sets $\boldsymbol{J}_i, \boldsymbol{J}_{i+1}, ..., \boldsymbol{J}_{i+N}$ are disjoint nonempty subsets of $\boldsymbol{N}$, which contradicts the equality $|\boldsymbol{N}| = N$. If (7.44) holds, then there are $y_i \in (a_i, b_i)$ and $y_{i+l} \in (a_{i+l}, b_{i+l})$ such that $\boldsymbol{J}_{y_i} \cap \boldsymbol{J}_{y_{i+l}} \neq \varnothing$. Let $j_1 \in \boldsymbol{J}_{y_i} \cap \boldsymbol{J}_{y_{i+l}}$. Then we have

$$y_i, y_{i+l} \in B_{j_1}(q).$$

Using (7.29), we can find $(r_{j_1,n_1}, s_{j_1,n_1})$ and $(r_{j_1,n_2}, s_{j_1,n_2})$ such that $n_1 > n_2$,

$$y_{i+l} \in (r_{j_1,n_1}, s_{j_1,n_1}) \quad \text{and} \quad y_i \in (r_{j_1,n_2}, s_{j_1,n_2}).$$

Indeed, if $n_1 = n_2$, then the points y_i and y_{i+l} belong to one and the same connected component of $E(q)$. Using (7.26), we can show that

$$\lim_{i \to \infty} \frac{y_i}{y_{i+l}} = \infty. \tag{7.47}$$

Note also, if $b_i < r^1$ and $q \geq \bigvee_{j=1}^{N} q_j^2$, then, using (7.43), we can prove that

$$k^{-N} \leq \frac{a_i}{b_i} \quad \text{for } k \in (q^3, q^{\frac{3(N+1)}{N}}). \tag{7.48}$$

Now (7.47), (7.48) and the condition $l \in \boldsymbol{N}$ imply (7.15) with $M = N$. □

Using Lemma 7.3 and Lemma 7.5, we obtain the following blow up description of the ideal $I(\boldsymbol{CSP})$.

Theorem 7.6. *Let $E \subseteq \mathbb{R}^+$ and $0 \in \mathrm{ac}E$. Then the following conditions are equivalent:*

(i) $E \in I(\boldsymbol{CSP})$;
(ii) *the chain $\mathrm{Cc}^1 E(q)$ is infinite for every $q > 1$, $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i \in \mathbb{N}}$, and there are $q_0 > 1$ and $M \in \mathbb{N}$ such that*

$$\limsup_{i \to \infty} \frac{b_i}{a_i} < \infty \quad \textit{and} \quad \lim_{n \to \infty} \bigvee_{j=0}^{M} \frac{a_{n+j}}{b_{n+j+1}} = \infty \quad \textit{for all } q > q_0.$$

Theorem 7.6 and Theorem 6.6 imply the following corollary.

Corollary 7.7. *We have the inclusion $I(\boldsymbol{CSP}) \subseteq \hat{I}(\boldsymbol{SP})$.*

The following example shows that there exists a set $E \subseteq \mathbb{R}^+$ such that $E \in \hat{I}(\boldsymbol{SP})$ but $E \notin I(\boldsymbol{CSP})$.

Example 7.8. *Let $\alpha \in (0,1)$. For every $j \in \mathbb{N}$ define positive numbers $y_{0,j}, y_{1,j}, ..., y_{j,j}$ so that*

$$y_{1,j} = \alpha^1 y_{0,j},\ y_{2,j} = \alpha^2 y_{1,j},\ ...,\ y_{j,j} = \alpha^j y_{j-1,j} \quad \text{and} \quad y_{0,j+1} < y_{j,j},$$

and

$$\lim_{j\to\infty} \frac{y_{j,j}}{y_{0,j+1}} = \infty.$$

Write

$$E = \bigcup_{j\in\mathbb{N}} \left(\bigcup_{k=0}^{j} \{y_{k,j}\} \right).$$

Let $q > 1$. Simple estimations show that $\mathrm{Cc}^1 E(q)$ is infinite, $\mathrm{Cc}^1 E(q) = \{(a_i, b_i)\}_{i\in\mathbb{N}}$ and

$$\limsup_{i\to\infty} \frac{b_i}{a_i} \le \left(\frac{1}{\alpha}\right)^m + \left(\frac{1}{\alpha}\right)^{m-1} + ... + \frac{1}{\alpha} + 1,$$

where m is the smallest positive whole number such that

$$q < \left(\frac{1}{\alpha}\right)^m. \tag{7.49}$$

Consequently, by Theorem 6.6, we have

$$E \in \hat{I}(\boldsymbol{SP}).$$

In accordance with Theorem 7.6, the statement $E \in I(\boldsymbol{CSP})$ does not hold if and only if the inequality

$$\liminf_{n\to\infty} \bigvee_{j=0}^{M} \frac{a_{n+j}}{b_{n+j+1}} < \infty$$

holds for every $q > 1$ and $M \in \mathbb{N}$. Let $m \in \mathbb{N}$ satisfy (7.49). Then we can show that

$$\liminf_{n\to\infty} \bigvee_{j=0}^{M} \frac{a_{n+j}}{b_{n+j+1}} \le \left(\frac{1}{\alpha}\right)^{m+M+1}.$$

Thus, E does not belong to $I(\boldsymbol{CSP})$.

References

[1] *A. Chinčin*: Recherches sur la structure des fonctions mesurables. (Russian; French) Moscou. Rec. Math. **31** (1923), 265–285, 377–433. JFM 50.0189.04

[2] *A. Denjoy*: Sur une propriété des séries trigonométriques. Amst. Ak. Versl. **29** (1920), 628–639. JFM 47.0261.04

[3] *A. Denjoy*: Leçons sur le calcul des cofficients d'une série trigonométrique. Part II, Métrique et topologie d'ensembles parfaits et de fonctions. Gauthier-Villars, Paris, 1941. Zbl 0063.01081, MR0018238

[4] *E. P. Dolženko*: Boundary properties of arbitrary functions. Mathematics of the USSR – Izvestiya. **1** (1967), 1–12. Zbl 0181.08102, MR0217297

[5] *O. Dovgoshey, V. Bilet*: Investigations of strong right upper porosity at a point. Real Anal. Exchange, **39** (1) (2014), 197–228.

[6] *O. Dovgoshey, J. Riihentaus*: Mean value type inequalities for quasinearly subharmonic functions. Glasgow Math. J., **55** (2) (2013), 349–368. Zbl 1271.31008, MR3040867

[7] *J. Foran, P. Humke*: Some set-theoretic properties of σ-porous sets. Real Anal. Exchange, **6** (1) (1980-81), 114–119. Zbl 0467.28001, MR0606545

[8] *P. Humke, T. Vessey*: Another note on σ-porous sets. Real Anal. Exchange, **8** (1982-83), 262–271. Zbl 0543.28001, MR0694514

[9] *L. Karp, T. Kilpeläinen, A. Petrosyan, H. Shahgholian*: On the porosity of free boundaries in degenerate variational inequalities. J. Differential Equations, **164** (2000), 110–117. Zbl 0956.35054, MR1761419

[10] *A. Kechris*: Hereditary properties of the class of closed sets of ubiqueness for trigonometric series. Israel J. Math., **73** (2) (1991), 189–198. Zbl 0769.42006, MR1135211

[11] *A. Kechris, A. Louveau, W. Woodin*: The structure of σ-ideals of compact sets. Trans. Amer. Math. Soc., **301** (1) (1987), 263–288. Zbl 0633.03043, MR0879573

[12] *F. Przytycki, S. Rohde*: Porosity of Collet-Eckmann Julia sets. Fund. Math., **155** (1998), 189–190. Zbl 0908.58054, MR1606527

[13] *M. Repický*: Porous sets and additivity of Lebesgue measure. Real Anal. Exchange, **15** (1) (1981-90), 282–2982. Zbl 0716.28004, MR1042544

[14] *O. Semenova, A. Florinskii*: Ideals of porous sets in the real line and in metrizable topological spaces. J. Math. Sci., New York, **102** (5) (2000), 4508–4522. Zbl 1002.28001, MR1807069

[15] *B. S. Thomson*: Real Functions. Lecture Notes in Mathematics, **1170**. Springer-Verlag, Berlin, Heidelberg, New York, Tokyo, 1985. Zbl 0581.26001, MR0818744

[16] *J. Tkadlec*: Constructions of some non-σ-porous sets on the real line. Real Anal. Exchange, **9** (2) (1983-84), 473–482. Zbl 0582.28001, MR0766073

[17] *J. Väisälä*: Porous sets and quasisymmetric maps. Trans. Amer. Math. Soc., **299** (1987), 525–533. Zbl 0617.30025, MR0869219

[18] *L. Zajiček*: On cluster sets of arbitrary functions. Fund. Math., **83** (1974), 197–217. Zbl 0276.26005, MR0338294

[19] *L. Zajiček*: Porosity and σ-porosity. Real Anal. Exchange, **13** (1987-88), 314–350. Zbl 0666.26003, MR0943561

[20] *L. Zajiček*: On σ-porous sets in abstract spaces. Abstr. Appl. Anal., **2005** (5) (2005), 509–534. Zbl 1098.28003, MR2201041

[21] *L. Zajiček, M. Zelený*: On the complexity of some σ-ideals of σ-P-porous sets. Comment Math. Univ. Carolin, **44** (3) (2003), 531–554. Zbl 1099.54029, MR2025819

[22] *M. Zelený, J. Pelant*: The structure of the σ-ideals of σ-porous sets. Comment Math. Univ. Carolin, **45** (1) (2004), 37–72. Zbl 1101.28001, MR2076859

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