Interpolatory Band-limited Wavelet Bases on the Sphere

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March 22, 2004

Abstract

In the present paper we construct space-localized bases for the space $W_n^n := \bigoplus_{k=n+1}^{n+s} \mathrm{Harm}_k(\mathbb{S}^2)$ of band-limited functions on the sphere. Each of the basis functions is a zonal polynomial centered at a point $\eta_i \in \mathbb{S}^2$. The goal of this work is to describe explicit fundamental systems $\{\eta_i\}$ for the space W_n^n.

AMS classification: 65D05, 41A063
Key words and phrases: fundamental systems, polynomial interpolation, band-limited functions on the sphere, spherical wavelets

1 Introduction

A classical way of addressing data fitting problems on the sphere is by *polynomial interpolation*: given a set of nodes $\{\xi_i\}_{i=1,\dots,N}$ on the two-dimensional sphere $\mathbb{S}^2$ and certain values $\{y_i\}_{i=1,\dots,N}$, the goal is to construct a polynomial in the space V_n of spherical polynomials of degree at most n that interpolates the known data.

An $L^2(\mathbb{S}^2)$-orthonormal basis of V_n is formed by the *spherical harmonics* $\{Y_k^j\}_{j=-k,\dots,k,\ k=0,\dots,n}$ (see e.g. Freeden, Gervens and Schreiner [3], Müller [8] or Reimer [9]). Thanks to their explicit representation in terms of the associated *Legendre functions* P_k^j, many of their properties appear in a natural way as an extension of well-known properties of elementary functions.

As the polynomial interpolation problems in V_n and in the wavelet spaces can be discussed in a similar way, we focus first on the interpolation problem in V_n, which reads: find a spherical polynomial $P := \sum_{k=0}^{n} \sum_{j=-k}^{k} \alpha_k^j Y_k^j$ in

V_n, such that for given data $\{y_i\}_{i=1,\ldots,N}$, the interpolation conditions

$$P(\xi_i) = \sum_{k=0}^{n} \sum_{j=-k}^{k} \alpha_k^j \, Y_k^j(\xi_i) = y_i, \quad i = 1, \ldots, N,$$

are satisfied, where $N = (n+1)^2$ denotes the dimension of V_n.

Unfortunately, not for any given set of pairwise distinct points $\{\xi_i\}_{i=1,\ldots,N}$, the polynomial interpolation problem has a unique solution and consequently it is of interest to identify those point sets, then called *fundamental systems (FS) for* V_n, for which the above system of equations is nonsingular.

A possible way of constructing FS for V_n is due to von Golitschek and Light [4] and Sündermann [10]. The key idea of their construction is to locate the N nodes on $n+1$ latitudes, such that the kth circle contains $2k+1$ points. For this distribution principle they can explicitly construct the unique interpolation polynomial for arbitrary given data. The disadvantage of this construction is its lack of symmetry. Since the number of nodes on each latitude is different, it is not clear how to design fast algorithms like e.g. Fourier or multiscale techniques.

On the other hand in [11, 12], Xu studies the construction of FS for V_n, such that the resulting points are not only symmetric to the equator but also exhibit a rotational symmetry with respect to the axis joining the north and south poles. In his construction, the fundamental systems for V_n arise from relating an interpolation problem on the unit disc to an interpolation problem on the unit sphere. In particular, for even polynomial degree n, he obtains a point constellation, in which the $(n+1)^2$ nodes are distributed on $n+1$ symmetric latitudes, where each of them carries $n+1$ equidistantly distributed points.

The corresponding interpolation problem for odd polynomial degree is studied in [6]. In a similar way to [11], the $(n+1)^2$ points are distributed equidistantly on $n+1$ symmetric latitudinal circles, each of them carrying $n+1$ nodes. The symmetric distribution of the resulting points not only generates a clear and regular geometry of the grid of nodes, but also simplifies theoretical and technical matters, as the involved Gram matrices attain a circulant structure. On the other hand, from the numerical point of view, a point distribution on such a structured grid allows the construction of spherical multiscale methods, leading us to the introduction of spherical wavelets.

More precisely, it turns out that the scaling space V_n is spanned by translates of the reproducing kernel K_n of the space of polynomials of degree n, i.e.

$$V_n = \text{span}\, \{K_n(\xi_i, \circ) : \; i = 1, \ldots, N\},$$

where $\{\xi_i\}_{i=1,\ldots,N}$ denotes a FS for V_n.

Then, for fixed $s \in \mathbb{N}$, we define the wavelet space W_n^s $(n \in \mathbb{N})$ of band-limited functions as the orthogonal complement of V_n in V_{n+s}. The aim of this paper is to construct FS for W_n^n, i.e. point systems for which the corresponding interpolation problem has a unique solution. While the problem of finding fundamental systems for V_n and W_n^2 (see [7]) has already been dealt with in several articles, the question of finding fundamental systems for W_n^n is treated here for the first time. Furthermore, we point out that W_n^s is also spanned by the translates of its reproducing kernel, that is

$$W_n^s = \text{span}\,\{K_{n+s}(\eta_j, \circ) - K_n(\eta_j, \circ) : \; j = 1, \ldots, M_s\},$$

where $\{\eta_j\}_{j=1,\ldots,M_s}$ is now a FS for W_n^s and M_s denotes the dimension of W_n^s.

The present paper is divided as follows: after introducing the necessary notation in Section 2, we study in Section 3 the interpolation problem in the space W_n^n. Subsections 4.1 and 4.2 offer then the necessary technical results for the completion of the proof of the main theorem of Section 3. Finally, in Section 5 we conclude with a remark concerning the presented fundamental systems for W_n^n.

2 The wavelet functions

Let $\mathbb{S}^2 := \{x \in \mathbb{R}^3 : \; \|x\|_2 = 1\}$ denote the unit sphere of the Euclidean space $\mathbb{R}^3$ and let

$$\begin{array}{rcccccl} \Psi : & [0,\pi] & \times & [0,2\pi) & \longrightarrow & \mathbb{R}^3 \\ & (\;\rho & , & \theta\;) & \longmapsto & (\sin\rho\,\cos\theta,\; \sin\rho\,\sin\theta,\; \cos\rho) \end{array}$$

be its parameterization in spherical coordinates (ρ, θ). The inner product and norm in $\mathbb{R}^3$ are defined as usual by $x \cdot y := x^T y = \sum_{i=1}^3 x_i\, y_i$ and $\|x\|_2^2 = x \cdot x$.

For given $n \in \mathbb{N}_0$, we denote with $V_n = \bigoplus_{k=0}^n \text{Harm}_k(\mathbb{S}^2)$ the space of spherical polynomials of degree at most n. Thereby, $\text{Harm}_k(\mathbb{S}^2)$ stands for the space of harmonic homogeneous polynomials on the sphere.

Throughout this paper, we will focus on the wavelet spaces $W_n^s := V_{n+s} \ominus V_n = \bigoplus_{k=n+1}^{n+s} \text{Harm}_k(\mathbb{S}^2)$ $(s \in \mathbb{N})$ of spherical polynomials of degrees in between $n+1$ and $n+s$. Considering now that $\dim \text{Harm}_k(\mathbb{S}^2) = 2k+1$, we obtain that $M_s := \dim W_n^s = \sum_{k=n+1}^{n+s} (2k+1) = s\,(s+2n+2)$.

An $L^2(\mathbb{S}^2)$-orthonormal basis of W_n^s, that is not localized on the sphere is given by the spherical harmonics

$$Y_k^j(\Psi(\rho,\theta)) := \sqrt{\frac{2k+1}{4\pi}}\, P_k^{|j|}(\cos\rho)\, e^{ij\theta}, \; j = -k, \ldots, k, \; k = n+1, \ldots, n+s$$

where

$$P_k^j(t) = \left(\frac{(k-j)!}{(k+j)!}\right)^{1/2} (1-t^2)^{j/2} \, \frac{d^j}{dt^j} P_k(t), \quad j = 0, \ldots, k, \quad t \in [-1,1],$$

denote the *associated Legendre functions* and P_k stands for the *Legendre polynomial* of degree k normalized according to the condition $P_k(1) = 1$.

In the following we identify band-limited functions, then called *polynomial wavelets*, which exhibit a space-localized behavior on the sphere. In fact, we define these wavelets in terms of the reproducing kernel of the underlying space W_n^s.

Definition 1. *Let $S := \{\eta_i\}_{i=1,\ldots,M_s}$ be a set of pairwise different points on $\mathbb{S}^2$. We call*

$$\psi_i^{n,s}(\xi) := \sum_{k=n+1}^{n+s} \frac{2k+1}{4\pi} P_k(\eta_i \cdot \xi), \quad i = 1, \ldots, M_s, \quad \xi \in \mathbb{S}^2,$$

the wavelet functions *corresponding to the set S.*

These band-limited functions possess the following properties, which do not depend on the choice of the point set S.

Lemma 1. *Given $S := \{\eta_i\}_{i=1,\ldots,M_s} \subset \mathbb{S}^2$, let $\{\psi_i^{n,s}\}_{i=1,\ldots,M_s} \subset W_n^s$ be the corresponding wavelet functions. Then it holds*

(i) (Orthogonality)

$$\langle \psi_i^{n,s}, P \rangle = 0, \qquad i = 1, \ldots, M_s, \quad \text{for all} \quad P \in V_n,$$

(ii) (Reproduction property)

$$\langle \psi_i^{n,s}, \psi_j^{n,s} \rangle = \psi_i^{n,s}(\eta_j), \qquad i,j = 1, \ldots, M_s,$$

(iii) (Localization property)

$$\left\| \frac{\psi_i^{n,s}}{\psi_i^{n,s}(\eta_i)} \right\| = \min \left\{ \|P\| : P \in W_n^s, \, P(\eta_i) = 1 \right\}, \qquad \eta_i \in \mathbb{S}^2 \text{ fixed},$$

where $\| \circ \|$ denotes the $L^2(\mathbb{S}^2)$-norm.

We can now ask ourselves for which point sets S, the corresponding polynomial wavelets constitute a basis of the space W_n^s. Making use of the *addition theorem* for $\mathrm{Harm}_k(\mathbb{S}^2)$ (see e.g. [8]), it is straightforward to check that the linear independence of the wavelet functions corresponding to a set of points $\{\eta_i\}_{i=1,\ldots,M_s} \subset \mathbb{S}^2$ is reflected in the regularity of the interpolation matrix

$$\mathbf{B}_n^s := \begin{pmatrix} Y_{n+1}^{-(n+1)}(\eta_1) & Y_{n+1}^{-(n+1)}(\eta_2) & \ldots & Y_{n+1}^{-(n+1)}(\eta_{M_s}) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n+1}^{n+1}(\eta_1) & Y_{n+1}^{n+1}(\eta_2) & \ldots & Y_{n+1}^{n+1}(\eta_{M_s}) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n+s}^{-(n+s)}(\eta_1) & Y_{n+s}^{-(n+s)}(\eta_2) & \ldots & Y_{n+s}^{-(n+s)}(\eta_{M_s}) \\ \vdots & \vdots & \ddots & \vdots \\ Y_{n+s}^{n+s}(\eta_1) & Y_{n+s}^{n+s}(\eta_2) & \ldots & Y_{n+s}^{n+s}(\eta_{M_s}) \end{pmatrix}. \tag{2.1}$$

Definition 2. *A set of points $\{\eta_i\}_{i=1,\ldots,M_s} \subset \mathbb{S}^2$, for which the interpolation matrix $\mathbf{B}_n^s$ is nonsingular or equivalently the wavelet functions $\{\psi_i^{n,s}\}_{i=1,\ldots,M_s}$ constitute a basis of W_n^s, is called a* FS *for W_n^s.*

Given a FS $\{\eta_i\}_{i=1,\ldots,M_s}$ for W_n^s, we can now construct unique spherical polynomials $L_j^{n,s} : \mathbb{S}^2 \longrightarrow \mathbb{C}$ $(j = 1, \ldots, M_s)$ in W_n^s, satisfying $L_j^{n,s}(\eta_i) = \delta_{ij}$. These functions will be called *Lagrangians*. On account of the reproduction property (ii) of the wavelet function $\psi_i^{n,s}$

$$\langle L_j^{n,s}, \psi_i^{n,s} \rangle = L_j^{n,s}(\eta_i) = \delta_{i,j}, \quad i, j = 1, \ldots, M_s,$$

it is straightforward to see, that the Lagrangians are the dual functions of our band-limited wavelets.

Moreover, introducing the vector valued functions

$$\begin{aligned} \boldsymbol{y}_{n,s} &= \left(Y_{n+1}^{-(n+1)}, \ldots, Y_{n+1}^0, \ldots, Y_{n+1}^{n+1}, \ldots, Y_{n+s}^{-(n+s)}, \ldots, Y_{n+s}^0, \ldots, Y_{n+s}^{n+s}\right)^T, \\ \mathbf{L}^{n,s} &= \left(L_1^{n,s}, L_2^{n,s}, \ldots, L_{M_s}^{n,s}\right)^T, \\ \boldsymbol{\psi}^{n,s} &= \left(\psi_1^{n,s}, \psi_2^{n,s}, \ldots, \psi_{M_s}^{n,s}\right)^T \end{aligned}$$

and the matrix notation

$$\mathbf{\Phi}_n^s = \left(\langle \psi_i^{n,s}, \psi_j^{n,s} \rangle\right)_{i,j=1,\ldots,M_s} \in \mathbb{R}^{M_s \times M_s},$$

it is not difficult to obtain the following change of basis relation between the bases $\{\psi_i^{n,s}\}_{i=1,\ldots,M_s}$ and $\{L_i^{n,s}\}_{i=1,\ldots,M_s}$

$$\mathbf{L}^{n,s} = (\mathbf{\Phi}_n^s)^{-1} \boldsymbol{\psi}^{n,s} = ((\mathbf{B}_n^s)^* (\mathbf{B}_n^s))^{-1} \boldsymbol{\psi}^{n,s} = (\mathbf{B}_n^s)^{-1} \boldsymbol{y}_{n,s}.$$

As it is shown in [7], it is possible to analyze explicitly the regularity of $\mathbf{B}_n^s$ for the case $s=2$. In the next section, we discuss a possible construction of fundamental systems for the space W_n^n.

3 A construction for $s=n$ when n is even

As pointed out in Section 2, the dimension of W_n^n is given by

$$M_n = \dim W_n^n = \dim V_{2n} - \dim V_n = n\,(3n+2).$$

A possible way of distributing M_n points regularly on the sphere is by choosing n symmetric latitudes and considering on each of them $3n+2$ equidistantly distributed points. In fact, if we choose the heights of the n latitudinal circles as the zeros of the Legendre polynomial P_n, the resulting point system constitutes a FS for W_n^n. The next theorem gives us the promised description of fundamental systems for W_n^n when $n \in \mathbb{N}$ is even.

Theorem 1. *Let n be an even integer and let*

$$-1 < \cos\rho_n < \cos\rho_{n-1} < \cdots < \cos\rho_1 < 1$$

denote the zeros of the Legendre polynomial P_n. Then the set of M_n points

$$\{\eta_{j,k} := \Psi(\rho_j, \theta_k^j) \;:\; j=1,\dots,n,\;\; k=1,\dots,3n+2\},$$

with

$$\theta_k^j = \begin{cases} \frac{2\pi k}{3n+2} & \text{if } j \text{ is odd,} \\ \frac{(2(k-1)+\alpha)\pi}{3n+2} & \text{if } j \text{ is even,} \end{cases}$$

and $\alpha \in (0,2)$, constitutes a FS for W_n^n.

Proof. Our goal is to establish the regularity of the interpolation matrix $\mathbf{B}_n^n$. For this purpose, we reduce our $M_n{\times}M_n$-dimensional problem into $3n{+}2$ problems of dimension $n \times n$. Since a reordering of the rows of $\mathbf{B}_n^n$ does not change the rank of the matrix, we can assume without loss of generality that $\mathbf{B}_n^n$ is given in the following form:

$$\left(\begin{array}{c} \mathbf{Z}_0 \\ \hline \mathbf{Z}_1 \\ \hline \vdots \\ \hline \mathbf{Z}_{3n+1} \end{array}\right) := \left(\begin{array}{c} \mathbf{Y}_l^m \;:\; m \equiv 0 \bmod (3n+2),\; l \geq |m| \\ \hline \mathbf{Y}_l^m \;:\; m \equiv 1 \bmod (3n+2),\; l \geq |m| \\ \hline \vdots \\ \hline \mathbf{Y}_l^m \;:\; m \equiv 3n+1 \bmod (3n+2),\; l \geq |m| \end{array}\right) \begin{array}{l} \in \mathbb{C}^{n\times M_n} \\ \in \mathbb{C}^{n\times M_n} \\ \\ \in \mathbb{C}^{n\times M_n}. \end{array}$$

Here, $\mathbf{Y}_l^m$ again denotes the row vector containing the evaluation of the spherical harmonic Y_l^m at the nodes $\{\eta_{j,k}\}_{j=1,\dots,n\;\,k=1,\dots,3n+2}$. We assume that

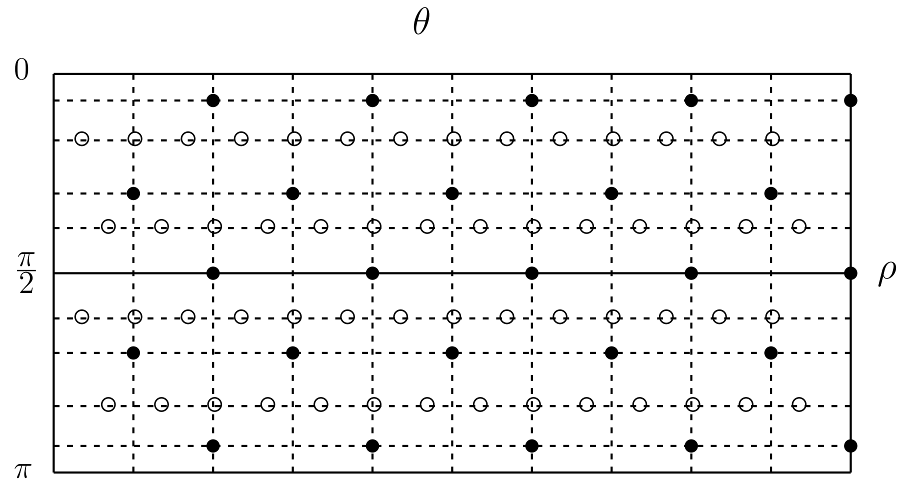

Figure 1: Fundamental systems for V_4 and W_4^4. The full circles correspond to the FS proposed in [11] for even polynomial degree n. The empty circles represent the FS for W_4^4 introduced in Theorem 1. Note that we distribute 56 points on four symmetric latitudes, where each of them carries 14 equidistantly distributed points. Moreover, the points lying on two consecutive parallel circles are rotated by an angle of $\alpha\pi/(3n+2)$. In the figure, we display the case when $\alpha = 1$.

within the matrices $\mathbf{Z}_m \in \mathbb{C}^{n\times M_n}$ $(m = 0, \ldots, 3n+1)$ the functions are ordered in the following way:
(i) if $m \in \{0, 1, \ldots, n+1\}$, then

$$\mathbf{Z}_m = \left[\left(\mathbf{Y}_{n+1}^{m}\right)^T, \left(\mathbf{Y}_{n+2}^{m}\right)^T, \ldots, \left(\mathbf{Y}_{2n}^{m}\right)^T\right]^T,$$

(ii) if $m \in \{2n+1, 2n+2, \ldots, 3n+1\}$, then

$$\mathbf{Z}_m = \left[\left(\mathbf{Y}_{n+1}^{m-(3n+2)}\right)^T, \left(\mathbf{Y}_{n+2}^{m-(3n+2)}\right)^T, \ldots, \left(\mathbf{Y}_{2n}^{m-(3n+2)}\right)^T\right]^T,$$

(iii) if $m \in \{n+2, n+3, \ldots, 2n\}$, then

$$\mathbf{Z}_m = \left[\left(\mathbf{Y}_{m}^{m}\right)^T, \left(\mathbf{Y}_{m+1}^{m}\right)^T, \ldots, \left(\mathbf{Y}_{2n}^{m}\right)^T, \left(\mathbf{Y}_{3n+2-m}^{m-(3n+2)}\right)^T, \ldots, \left(\mathbf{Y}_{2n}^{m-(3n+2)}\right)^T\right]^T.$$

Bearing in mind the distribution of the points on the n latitudinal circles $z = \cos\rho_j$ $(j = 1, \ldots, n)$, we can split the $n \times M_n$-dimensional matrices $\mathbf{Z}_m$ $(m = 0, \ldots, 3n+1)$ into n matrices of dimension $n \times (3n+2)$

$$\mathbf{Z}_m = (\mathbf{Z}_m^1, \mathbf{Z}_m^2, \ldots, \mathbf{Z}_m^n), \qquad m = 0, \ldots, 3n+1,$$

where $\mathbf{Z}_m^j \in \mathbb{C}^{n\times(3n+2)}$ contains the information relative to the points lying on the jth latitudinal circle. Moreover, let $\omega := \exp(2\pi i/(3n+2))$ and consider the $(3n+2)\times(3n+2)$-dimensional Fourier matrix $\mathbf{F}_{3n+2}$ with entries

$$\mathbf{F}_{3n+2}(j,k) := \frac{1}{\sqrt{3n+2}}\left(\omega^{-(j-1)(k-1)}\right) = \frac{1}{\sqrt{3n+2}}\left(e^{-\frac{2\pi i}{3n+2}(j-1)(k-1)}\right).$$

Multiplication from the right hand side by the $M_n \times M_n$-dimensional block diagonal matrix $\mathbf{F} := \mathrm{diag}\,(\mathbf{F}_{3n+2}, \mathbf{F}_{3n+2}, \dots, \mathbf{F}_{3n+2})$ yields

$$\mathbf{B}_n^n\,\mathbf{F} = \begin{pmatrix} \mathbf{Z}_0^1\,\mathbf{F}_{3n+2} & \mathbf{Z}_0^2\,\mathbf{F}_{3n+2} & \dots & \mathbf{Z}_0^n\,\mathbf{F}_{3n+2} \\ \mathbf{Z}_1^1\,\mathbf{F}_{3n+2} & \mathbf{Z}_1^2\,\mathbf{F}_{3n+2} & \dots & \mathbf{Z}_1^n\,\mathbf{F}_{3n+2} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{Z}_{3n+1}^1\,\mathbf{F}_{3n+2} & \mathbf{Z}_{3n+1}^2\,\mathbf{F}_{3n+2} & \dots & \mathbf{Z}_{3n+1}^n\,\mathbf{F}_{3n+2} \end{pmatrix}. \tag{3.1}$$

Let us now focus our attention on one of the $n\times(3n+2)$-dimensional matrices $\mathbf{Z}_m^j\,\mathbf{F}_{3n+2}$ and compute its entries. Since the longitudinal angles of the points on hand vary from latitude to latitude, it is necessary to examine the cases of odd and even j separately. Also the structure of the functions involved in each of the matrices $\mathbf{Z}_m^j$ recommends the distinction of three cases:

(I) First, we study the entries of the first $n\,(n+2)$ rows of $\mathbf{B}_n^n\mathbf{F}$, i.e. the rows corresponding to the functions $Y_{n+\kappa}^m$ $(m = 0, \dots, n+1,\ \kappa = 1, \dots, n)$.

(II) Second, we compute the entries of the $n\,(n+1)$ last rows of $\mathbf{B}_n^n\mathbf{F}$, i.e. the rows corresponding to the functions $Y_{n+\kappa}^{m-(3n+2)}$ for $m = 2n+1, \dots, 3n+1$ and $\kappa = 1, \dots, n$.

(III) Finally, we analyze the remaining $n\,(n-1)$ rows corresponding to the functions $Y_{m+\kappa}^m$ $(\kappa = 0, \dots, 2n-m)$ and $Y_{3n+2-m+\kappa}^{m-(3n+2)}$ $(\kappa = 0, \dots, m-n-2)$ for $m = n+2, \dots, 2n$.

Case (I): Let $m \in \{0, 1, \dots, n+1\}$.

- For **odd** j, it holds

$$\begin{aligned} \left(\mathbf{Z}_m^j\,\mathbf{F}_{3n+2}\right)(r,s) &= \frac{1}{\sqrt{3n+2}}\sum_{k=1}^{3n+2} Y_{n+r}^m\left(\Psi\left(\rho_j, \frac{2\pi k}{3n+2}\right)\right)\,\omega^{(1-k)(s-1)} \\ &= c_{n+r}\,P_{n+r}^m(\cos\rho_j)\sum_{k=1}^{3n+2}\omega^{mk+s-1-sk+k} \\ &= c_{n+r}\,P_{n+r}^m(\cos\rho_j)\,\omega^{s-1}\sum_{k=1}^{3n+2}\omega^{(m-s+1)\,k}, \end{aligned} \tag{3.2}$$

where $r = 1, \dots, n,\ s = 1, \dots 3n+2$ and c_κ denotes the constant

$$c_\kappa := \sqrt{\frac{(2\kappa+1)}{4\pi\ (3n+2)}}, \qquad \kappa = n+1, \dots, 2n. \tag{3.3}$$

Note that the sum in (3.2) is nonzero if and only if $(m{-}s{+}1) \equiv 0 \bmod(3n{+}2)$. Since $m \in \{0, \dots, n+1\}$ and $s \in \{1, \dots, 3n{+}2\}$, a nonzero sum occurs only when $s{=}m{+}1$. Accordingly, we obtain for $r = 1, \dots, n$

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\, c_{n+r}\, P_{n+r}^m(\cos\rho_j)\, \omega^m & \text{if } s = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

- On the other hand for **even** j we have

$$\begin{aligned} \left(\mathbf{Z}_m^j\, \mathbf{F}_{3n+2}\right)(r,s) &= \frac{1}{\sqrt{3n+2}} \sum_{k=1}^{3n+2} Y_{n+r}^m \left(\Psi\left(\rho_j, \frac{(2(k-1)+\alpha)\pi}{3n+2}\right)\right) \omega^{(1-k)(s-1)} \\ &= c_{n+r}\ P_{n+r}^m(\cos\rho_j) \sum_{k=1}^{3n+2} \omega^{\frac{\alpha m}{2}-m+mk+s-1-sk+k} \\ &= c_{n+r}\ P_{n+r}^m(\cos\rho_j)\, \omega^{s-1+\frac{\alpha m}{2}-m} \sum_{k=1}^{3n+2} \omega^{(m-s+1)\,k}, \end{aligned} \tag{3.4}$$

where $r = 1, \dots, n,\ s = 1, \dots, 3n+2$ and c_{n+r} denotes the constant introduced in (3.3). Since the sum in (3.4) is nonzero if and only if $s{=}m{+}1$, we obtain for $r = 1, \dots, n$

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\, c_{n+r}\, P_{n+r}^m(\cos\rho_j)\, \omega^{\frac{\alpha m}{2}} & \text{if } s = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Case (II): Now, let $m = 2n+1, \dots, 3n+1$. An analysis along the same lines as in **(I)** yields:

- If j is **odd**, then we have

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\, c_{n+r}\, P_{n+r}^m(\cos\rho_j)\, \omega^m & \text{if } s = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

- Similarly, for **even** j it follows

$$\left(\mathbf{Z}_m^j\, \mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\, c_{n+r}\, P_{n+r}^{3n+2-m}(\cos\rho_j)\, \omega^{\frac{\alpha m}{2}} & \text{if } s = m+1, \\ 0 & \text{otherwise.} \end{cases}$$

Case (III): Finally, it remains to study the case, when $m = n+2, \dots, 2n$.

• For **odd** j, it is straightforward to check that

$$\left(\mathbf{Z}_m^j\,\mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\,c_{m-1+r}\,P_{m-1+r}^{m}(\cos\rho_j)\,\omega^m & \text{if } s = m+1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r = 1, \dots, 2n{-}m{+}1$ and

$$\left(\mathbf{Z}_m^j\,\mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\,c_{n+r}\,P_{n+r}^{3n+2-m}(\cos\rho_j)\,\omega^m & \text{if } s = m+1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r = 2n-m+2, \dots, n$.

• On the other hand, for **even** j we obtain

$$\left(\mathbf{Z}_m^j\,\mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\,c_{m-1+r}\,P_{m-1+r}^{m}(\cos\rho_j)\,\omega^{\frac{\alpha m}{2}} & \text{if } s = m+1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r = 1, \dots, 2n{-}m{+}1$ and

$$\left(\mathbf{Z}_m^j\,\mathbf{F}_{3n+2}\right)(r,s) = \begin{cases} (3n+2)\,c_{n+r}\,P_{n+r}^{3n+2-m}(\cos\rho_j)\,\omega^{\frac{\alpha m}{2}}\,e^{-\alpha\pi i} & \text{if } s{=}m{+}1, \\ 0 & \text{otherwise,} \end{cases}$$

for $r = 2n-m+2, \dots, n$.

Now that we have computed the entries of the matrix (3.1) explicitly, let us multiply from the right hand side by the permutation matrix

$$\begin{aligned} \mathbf{P}_2 \;=\; & \big[\mathbf{e}_1, \mathbf{e}_{(3n+2)+1}, \dots, \mathbf{e}_{(n-1)(3n+2)+1}, \mathbf{e}_2, \mathbf{e}_{(3n+2)+2}, \dots, \mathbf{e}_{(n-1)(3n+2)+2}, \\ & \dots, \mathbf{e}_{3n+2}, \mathbf{e}_{(3n+2)+(3n+2)}, \mathbf{e}_{2\,(3n+2)+(3n+2)}, \dots, \mathbf{e}_{(n-1)(3n+2)+(3n+2)}\big]\,. \end{aligned}$$

Rescaling the rows and columns of $\mathbf{B}_n^n\,\mathbf{F}$ by a multiplication from the left and right hand sides with appropriate diagonal matrices, we transform $\mathbf{B}_n^n\,\mathbf{F}$ into the block diagonal matrix

$$\text{diag}\;\left(\mathbf{H}_0, \mathbf{H}_1, \dots, \mathbf{H}_{n+2}, \dots, \mathbf{H}_{2n}, , \mathbf{H}_{2n+1}, \dots, \mathbf{H}_{3n+1}\right),$$

where with $x_j := \cos\rho_j$ $(j = 1, \dots, n)$

$$\mathbf{H}_m = \begin{pmatrix} P_{n+1}^m(x_1) & P_{n+1}^m(x_2) & \cdots & P_{n+1}^m(x_n) \\ P_{n+2}^m(x_1) & P_{n+2}^m(x_2) & \cdots & P_{n+2}^m(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n}^m(x_1) & P_{2n}^m(x_2) & \cdots & P_{2n}^m(x_n) \end{pmatrix} \tag{3.5}$$

for $m = 0, \ldots, n+1$,

$$\mathbf{H}_m = \begin{pmatrix} P_{n+1}^{3n+2-m}(x_1) & P_{n+1}^{3n+2-m}(x_2) & \cdots & P_{n+1}^{3n+2-m}(x_n) \\ P_{n+2}^{3n+2-m}(x_1) & P_{n+2}^{3n+2-m}(x_2) & \cdots & P_{n+2}^{3n+2-m}(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n}^{3n+2-m}(x_1) & P_{2n}^{3n+2-m}(x_2) & \cdots & P_{2n}^{3n+2-m}(x_n) \end{pmatrix}$$

for $m = 2n+1, \ldots, 3n+1$ and finally

$$\mathbf{H}_m = \begin{pmatrix} P_m^m(x_1) & P_m^m(x_2) & \cdots & P_m^m(x_n) \\ P_{m+1}^m(x_1) & P_{m+1}^m(x_2) & \cdots & P_{m+1}^m(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n}^m(x_1) & P_{2n}^m(x_2) & \cdots & P_{2n}^m(x_n) \\ P_{3n+2-m}^{3n+2-m}(x_1) & e^{-\alpha\pi i}\, P_{3n+2-m}^{3n+2-m}(x_2) & \cdots & e^{-\alpha\pi i}\, P_{3n+2-m}^{3n+2-m}(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n}^{3n+2-m}(x_1) & e^{-\alpha\pi i}\, P_{2n}^{3n+2-m}(x_2) & \cdots & e^{-\alpha\pi i}\, P_{2n}^{3n+2-m}(x_n) \end{pmatrix} \tag{3.6}$$

for $m = n+2, \ldots, 2n$. Consequently, we have reduced the problem of examining the regularity of $\mathbf{B}_n^n$ to the analysis of the regularity of each of the matrices $\mathbf{H}_m$.

On the one hand, the regularity of the matrices $\mathbf{H}_m$ $(m{=}n{+}2, \ldots, 2n{+}1)$ will be proved in Theorem 2 of Subsection 4.1.

On the other hand, the matrices $\mathbf{H}_m$ and $\mathbf{H}_{3n+2-m}$ $(m = 1, \ldots, n+1)$ are permuted versions of each other, so that it simply remains to ensure the regularity of $\mathbf{H}_m$ for $m{=}0, \ldots, n+1$ in order to conclude the proof. In fact, as we will see in Theorem 3 of Subsection 4.2 all these matrices are regular and we are finally in a position to confirm that the underlying point system $\{\eta_{j,k}\}$ constitutes a FS for the space W_n^n. ■

4 Regularity of the matrices $\mathbf{H}_m$

4.1 The case of $m = n+2, \ldots, 2n$

The subject of this subsection is to establish the regularity of the matrices $\mathbf{H}_m$ in equation (3.6). In fact, after an index shift of m by $n+1$, we obtain

$$\begin{aligned} P_m^m \quad &\longrightarrow \quad P_{m-(n+1)}^{m-(n+1)} = P_{m'}^{m'}, \quad P_{2n}^m \longrightarrow P_{2n-(n+1)}^{m-(n+1)} = P_{n'}^{m'} \quad \text{and} \\ P_{3n+2-m}^{3n+2-m} \quad &\longrightarrow \quad P_{3n+2-m-(n+1)}^{3n+2-m-(n+1)} = P_{2n-m+1}^{2n-m+1} = P_{n'-m'+1}^{n'-m'+1}, \end{aligned}$$

for $m = n+2, \ldots, 2n$ or equivalently for $m' = 1, \ldots, n'$. Then it is straightforward to see, that the matrices $\mathbf{H}_m$ can be brought by regular transformations into the form $\mathbf{C}_{m'} :=$

$$\begin{pmatrix} 1 & 1 & \ldots & 1 \\ x_1 & x_2 & \ldots & x_{n'+1} \\ x_1^2 & x_2^2 & \ldots & x_{n'+1}^2 \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{n'-m'} & x_2^{n'-m'} & \ldots & x_{n'+1}^{n'-m'} \\ e^{\frac{\alpha\pi i}{2}}(1-x_1^2)^\beta & e^{-\frac{\alpha\pi i}{2}}(1-x_2^2)^\beta & \ldots & e^{\frac{-\alpha\pi i}{2}}(1-x_{n'+1}^2)^\beta \\ e^{\frac{\alpha\pi i}{2}}x_1(1-x_1^2)^\beta & e^{-\frac{\alpha\pi i}{2}}x_2(1-x_2^2)^\beta & \ldots & e^{\frac{-\alpha\pi i}{2}}x_{n'+1}(1-x_{n'+1}^2)^\beta \\ \vdots & \vdots & \ddots & \vdots \\ e^{\frac{\alpha\pi i}{2}}x_1^{m'-1}(1-x_1^2)^\beta & e^{-\frac{\alpha\pi i}{2}}x_2^{m'-1}(1-x_2^2)^\beta & \ldots & e^{\frac{-\alpha\pi i}{2}}x_{n'+1}^{m'-1}(1-x_{n'+1}^2)^\beta \end{pmatrix} \tag{4.1}$$

where $n' = n-1$, $m' = 1, \ldots, n-1$ and $\beta = (n'+1)/2 - m'$. Thanks to the structure of the matrices $\mathbf{H}_m$ in (3.6) it is enough to study the cases of $m = n+2, \ldots, 3n/2+1$ or equivalently $m' = 1, \ldots, (n'+1)/2$. The following theorem establishes then the desired result. For simplicity of notation, we denote $n'=n$ and $m'=m$.

Theorem 2. *Let $n \in \mathbb{N}$ be odd. Furthermore, let $\alpha \in (0,2)$ be a fixed real number and let $-1 < x_{n+1} < x_n < \ldots < x_1 < 1$ be $n+1$ pairwise different points such that $x_{n+2-k} = -x_k$, $(k = 1, \ldots, (n+1)/2)$. The matrices $\mathbf{C}_m \in \mathbb{C}^{(n+1)\times(n+1)}$ $(m = 1, \ldots, (n+1)/2)$ defined as in (4.1) are nonsingular.*

Proof. For the sake of simplicity, let us start with the extreme case $m=(n+1)/2$.

(I) For $m = (n+1)/2$, we have $\beta = 0$ and the matrix $\mathbf{C}_m$ attains the form

$$\mathbf{C}_{(n+1)/2} = [\boldsymbol{g}_0, \boldsymbol{g}_1, \boldsymbol{g}_2, \ldots, \boldsymbol{g}_{(n-1)/2}, \boldsymbol{h}_0, \ldots, \boldsymbol{h}_{(n-1)/2}]^T,$$

where

$$\boldsymbol{g}_r := (x_1^r, x_2^r, \ldots, x_{n+1}^r)^T, \quad r = 0, \ldots, (n-1)/2,$$

and

$$\boldsymbol{h}_r := (e^{\frac{\alpha\pi i}{2}} x_1^r \,(1-x_1^2)^\beta, e^{\frac{-\alpha\pi i}{2}} x_2^r \,(1-x_2^2)^\beta, \ldots, e^{\frac{-\alpha\pi i}{2}} x_{n+1}^r \,(1-x_{n+1}^2)^\beta)^T.$$

Our task is to prove that the homogeneous linear system of equations associated to $\mathbf{C}_{(n+1)/2}^T$ only has the trivial solution.

For this purpose, let $\boldsymbol{x} = (a_0, \ldots, a_{(n-1)/2}, b_0, \ldots, b_{(n-1)/2})^T \in \mathbb{R}^{n+1}$. The assumption $\mathbf{C}_{(n+1)/2}^T\, \boldsymbol{x} = \mathbf{0}$ implies that

$$\begin{aligned}
\sum_{k=0}^{\frac{n-1}{2}} \left(a_k + e^{\frac{\alpha\pi i}{2}} b_k\right) x_i^k &= 0 \quad \text{for odd } \ i \in \{1, \ldots, n\}, \\
\sum_{k=0}^{\frac{n-1}{2}} \left(a_k + e^{-\frac{\alpha\pi i}{2}} b_k\right) x_i^k &= 0 \quad \text{for even } \ i \in \{2, \ldots, n+1\}.
\end{aligned}$$

As $\sum_{k=0}^{(n-1)/2} \left(a_k + e^{\frac{\alpha\pi i}{2}} b_k\right) x^k$ is a polynomial of degree $(n-1)/2$ with $(n+1)/2$ zeros in $x_1, x_3, \ldots, x_n$, it must be the zero polynomial, i.e.

$$a_k + e^{\frac{\alpha\pi i}{2}} b_k = 0 \qquad \text{for all } \ k = 0, \ldots, \frac{n-1}{2}.$$

Analogously, we can conclude that

$$a_k + e^{-\frac{\alpha\pi i}{2}} b_k = 0 \qquad \text{for all } \ k = 0, \ldots, \frac{n-1}{2}.$$

Subtracting these equations for $k = 0, \ldots, (n-1)/2$, we obtain that

$$\left(e^{\frac{\alpha\pi i}{2}} - e^{-\frac{\alpha\pi i}{2}}\right) b_k = \left(2i \sin\frac{\alpha\pi}{2}\right) b_k = 0, \quad k = 0, \ldots, \frac{n-1}{2}.$$

Considering that $\alpha \in (0, 2)$, we finally conclude that $b_k = 0$ and consequently also $a_k = 0$ for all $k = 0, \ldots, (n-1)/2$.

(II) Now, let $1 \le m \le (n-1)/2$. For this part of the proof we will exploit the symmetric distribution of the points $x_i = -x_{n+2-i}$ $(i = 1, \ldots, (n+1)/2)$. Since the cases of odd and even m are treated in the same way and lead to the same result, we restrict ourselves in this paper to the case of odd m.

So, let $m = 2k + 1$: In order to obtain a simpler representation of the matrix under consideration, we multiply from the right hand side by the regular matrix

$$\begin{aligned}
\mathbf{Q} \ :=\ & \Big[\mathbf{e}_1 + \mathbf{e}_{n+1}, \ldots, \mathbf{e}_j + \mathbf{e}_{n+2-j}, \ldots, \mathbf{e}_{(n+1)/2} + \mathbf{e}_{\frac{n+3}{2}}, \\
& \mathbf{e}_1 - \mathbf{e}_{n+1}, \ldots, \mathbf{e}_j - \mathbf{e}_{n+2-j}, \ldots, \mathbf{e}_{(n+1)/2} - \mathbf{e}_{\frac{n+3}{2}}\Big].
\end{aligned}$$

This matrix multiplication represents the operation of summing and subtracting the columns of $\mathbf{C}_m$ corresponding to the symmetric points x_j and

x_{n+2-j} for $j = 1, \ldots, (n+1)/2$. After multiplying from the left hand side by the permutation matrix

$$\mathbf{P} := [\mathbf{e}_1, \mathbf{e}_3, \ldots, \mathbf{e}_n, \mathbf{e}_2, \mathbf{e}_4, \ldots, \mathbf{e}_{n+1}]^T,$$

our matrix $\mathbf{C}_m$ finally attains the more convenient expression

$$\mathbf{P}\,\mathbf{C}_m\,\mathbf{Q} = 2 \left(\begin{array}{cc} \mathbf{C}_m^{11} & \mathbf{C}_m^{12} \\ \mathbf{C}_m^{21} & \mathbf{C}_m^{22} \end{array} \right). \tag{4.2}$$

These matrices $\mathbf{C}_m^{ij} \in \mathbb{C}^{\frac{(n+1)}{2} \times \frac{(n+1)}{2}}$ $(i, j = 1, 2)$ have the form

$$\begin{aligned}
\mathbf{C}_m^{11} &= \mathbf{D}_m^{11}\left[\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{(n-1)/2-k}, \tilde{\boldsymbol{w}}_0, \tilde{\boldsymbol{w}}_1, \tilde{\boldsymbol{w}}_{k-1}\right]^T, && (4.3)\\
\mathbf{C}_m^{22} &= \mathbf{D}_m^{22}\left[\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{(n-3)/2-k}, \tilde{\boldsymbol{v}}_0, \tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_k\right]^T, && (4.4)\\
\mathbf{C}_m^{12} &= \mathbf{D}_m^{12}\left[\mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{w}_0, \boldsymbol{w}_1, \ldots, \boldsymbol{w}_{k-1}\right]^T, && (4.5)\\
\mathbf{C}_m^{21} &= \mathbf{D}_m^{21}\left[\mathbf{0}, \ldots, \mathbf{0}, \boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_k\right]^T, && (4.6)
\end{aligned}$$

where

$$\begin{aligned}
\boldsymbol{u}_r &= \left(x_1^{2r}, x_2^{2r}, \ldots, x_{(n+1)/2}^{2r}\right)^T,\\
\boldsymbol{y}_r &= \left(x_1^{2r+1}, x_2^{2r+1}, \ldots, x_{(n+1)/2}^{2r+1}\right)^T,\\
\boldsymbol{v}_r &= \left(x_1^{2r}\,(1-x_1^2)^\beta, x_2^{2r}\,(1-x_2^2)^\beta, \ldots, x_{\frac{n+1}{2}}^{2r}\,(1-x_{\frac{n+1}{2}}^2)^\beta\right)^T,\\
\tilde{\boldsymbol{v}}_r &= \left(x_1^{2r}\,(1-x_1^2)^\beta, -x_2^{2r}\,(1-x_2^2)^\beta, \ldots, -x_{\frac{n+1}{2}}^{2r}\,(1-x_{\frac{n+1}{2}}^2)^\beta\right)^T,\\
\boldsymbol{w}_r &= \left(x_1^{2r+1}\,(1-x_1^2)^\beta, x_2^{2r+1}\,(1-x_2^2)^\beta, \ldots, x_{\frac{n+1}{2}}^{2r+1}\,(1-x_{\frac{n+1}{2}}^2)^\beta\right)^T,\\
\tilde{\boldsymbol{w}}_r &= \left(x_1^{2r+1}\,(1-x_1^2)^\beta, -x_2^{2r+1}\,(1-x_2^2)^\beta, \ldots, -x_{\frac{n+1}{2}}^{2r+1}\,(1-x_{\frac{n+1}{2}}^2)^\beta\right)^T
\end{aligned}$$

and $\mathbf{D}_m^{ij}$ $(i,j=1,2)$ are the diagonal matrices given by

$$\begin{aligned}
\mathbf{D}_m^{11} &= \operatorname{diag}\left(1,\ldots,1,\overbrace{i\sin\frac{\alpha\pi}{2},\ldots,i\sin\frac{\alpha\pi}{2}}^{k \text{ times}}\right),\\
\mathbf{D}_m^{12} &= \operatorname{diag}\left(1,\ldots,1,\overbrace{\cos\frac{\alpha\pi}{2},\ldots,\cos\frac{\alpha\pi}{2}}^{k \text{ times}}\right),\\
\mathbf{D}_m^{21} &= \operatorname{diag}\left(1,\ldots,1,\overbrace{\cos\frac{\alpha\pi}{2},\ldots,\cos\frac{\alpha\pi}{2}}^{(k+1) \text{ times}}\right),\\
\mathbf{D}_m^{22} &= \operatorname{diag}\left(1,\ldots,1,\overbrace{i\sin\frac{\alpha\pi}{2},\ldots,i\sin\frac{\alpha\pi}{2}}^{(k+1) \text{ times}}\right).
\end{aligned}$$

As we will show in Lemma 2, the matrices $\mathbf{C}_m^{11}$ and $\mathbf{C}_m^{22}$ are regular for $\alpha \in (0,2)$. Exploiting the block partition of $\mathbf{P}\mathbf{C}_m\mathbf{Q}$ in (4.2), we obtain

$$\det\left(\mathbf{P}\,\mathbf{C}_m\,\mathbf{Q}\right) = 2^{n+1}\det\mathbf{C}_m^{11}\det\left(\mathbf{C}_m^{22} - \mathbf{C}_m^{21}\,(\mathbf{C}_m^{11})^{-1}\,\mathbf{C}_m^{12}\right).$$

This so-called *Schur complement formula* may be verified via multiplication of

$$\begin{pmatrix} \mathbf{C}_m^{11} & \mathbf{C}_m^{12} \\ \mathbf{C}_m^{21} & \mathbf{C}_m^{22} \end{pmatrix} \quad \text{by} \quad \begin{pmatrix} \mathbf{I} & -(\mathbf{C}_m^{11})^{-1}\,\mathbf{C}_m^{12} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

In Lemma 3, we will show that $\mathbf{C}_m^{21}\,(\mathbf{C}_m^{11})^{-1}\,\mathbf{C}_m^{12} = \mathbf{0}$, so we can conclude that

$$\det\mathbf{P}\mathbf{C}_m\mathbf{Q} = 2^{n+1}\,\det\mathbf{C}_m^{11}\,\det\mathbf{C}_m^{22} \neq 0$$

and hence $\det\mathbf{C}_m \neq 0$, which completes the proof. ■

Let us now present the lemma, which establishes the regularity of the matrices $\mathbf{C}_m^{11}$ and $\mathbf{C}_m^{22}$ appearing in equations (4.3) and (4.4).

Lemma 2. *Let $n, m \in \mathbb{N}$ be odd integers with $1 \leq m = 2k+1 \leq (n+1)/2$. The matrices $\mathbf{C}_m^{11}$ and $\mathbf{C}_m^{22}$, introduced in* (4.3) *and* (4.4) *are nonsingular.*

Proof. We start with the matrix

$$\mathbf{C}_m^{11} = \mathbf{D}_m^{11}\,\left[\boldsymbol{u}_0, \boldsymbol{u}_1, \ldots, \boldsymbol{u}_{(n-1)/2-k}, \tilde{\boldsymbol{w}}_0, \tilde{\boldsymbol{w}}_1, \tilde{\boldsymbol{w}}_{k-1}\right]^T.$$

Since $\alpha \in (0,2)$, the first factor of this decomposition is regular and hence it simply remains to establish the non-singularity of the second factor. For this purpose, we show that the homogeneous linear system of equations associated to the transpose of $(\mathbf{D}_m^{11})^{-1}\mathbf{C}_m^{11}$ only possesses the trivial solution. So, let $(a_0, a_1, \ldots, a_{(n-m)/2}, b_0, b_1, \ldots, b_{(m-3)/2})^T \in \mathbb{R}^{(n+1)/2}$ be a solution of the homogeneous system associated to the matrix under consideration. Then there exists an even polynomial $p(x) = \sum_{k=0}^{(n-m)/2} a_k x^{2k}$ of degree less or equal to $n-m$ and an odd polynomial $q(x) = \sum_{s=0}^{(m-3)/2} b_s x^{2s+1}$ of degree at most $m-2$ such that

$$\begin{aligned} \sum_{k=0}^{\frac{n-m}{2}} a_k x^{2k} &= (1-x^2)^\beta \sum_{s=0}^{\frac{m-3}{2}} b_s x^{2s+1} \qquad \text{for } x \in \{x_2, x_4, \ldots, x_{\frac{n+1}{2}}\}, \\ \sum_{k=0}^{\frac{n-m}{2}} a_k x^{2k} &= -(1-x^2)^\beta \sum_{s=0}^{\frac{m-3}{2}} b_s x^{2s+1} \qquad \text{for } x \in \{x_1, x_3, \ldots, x_{\frac{n-1}{2}}\}. \end{aligned}$$

Some lengthy but elementary calculations involving the analysis of the changes of sign of the presented polynomials p and q yield now that $a_k = b_k = 0$ for all k (for more details cf. Chapter 2 in [6]).

Now, let us establish the regularity of

$$\mathbf{C}_m^{22} = \mathbf{D}_m^{22} \left[\boldsymbol{y}_0, \boldsymbol{y}_1, \ldots, \boldsymbol{y}_{(n-3)/2-k}, \tilde{\boldsymbol{v}}_0, \tilde{\boldsymbol{v}}_1, \tilde{\boldsymbol{v}}_k\right]^T.$$

Let $\boldsymbol{a} := (a_0, a_1, \ldots, a_{(n-2-m)/2}, b_0, \ldots, b_{(m-1)/2})^T \in \mathbb{R}^{(n+1)/2}$ be a nonzero solution of the homogeneous system $(\mathbf{D}_m^{22})^{-1}(\mathbf{C}_m^{22})^T\boldsymbol{a} = \mathbf{0}$. Then there exists an odd polynomial $p(x) := \sum_{k=0}^{(n-2-m)/2} a_k x^{2k+1}$ of degree less or equal to $n-m-1$ and an even polynomial $q(x) := \sum_{s=0}^{(m-1)/2} b_s x^{2s}$ of degree at most $m-1$ such that

$$\begin{aligned} \sum_{k=0}^{\frac{n-2-m}{2}} a_k x^{2k+1} &= (1-x^2)^\beta \sum_{s=0}^{\frac{m-1}{2}} b_s x^{2s} \qquad \text{for } x \in \{x_2, x_4, \ldots, x_{\frac{n+1}{2}}\}, \\ \sum_{k=0}^{\frac{n-2-m}{2}} a_k x^{2k+1} &= -(1-x^2)^\beta \sum_{s=0}^{\frac{m-1}{2}} b_s x^{2s} \qquad \text{for } x \in \{x_1, x_3, \ldots, x_{\frac{n-1}{2}}\}. \end{aligned}$$

Analyzing now the degrees of these polynomials and considering their necessary changes of sign implied by the former equations, it is again not difficult to draw the conclusion that p and q are the zero polynomials, which completes the proof of this theorem. ■

Finally, we concentrate on the following result which has proven helpful in the proof of Theorem 2 and thereby complete the proof of Theorem 1.

Lemma 3. *Let* $\mathbf{C}_m^{11}$, $\mathbf{C}_m^{12}$ *and* $\mathbf{C}_m^{21}$ $(m = 1, \dots, (n-1)/2)$ *be defined as in* (4.3)*,* (4.5) *and* (4.6)*. Then*

$$\mathbf{C}_m^{21}\,(\mathbf{C}_m^{11})^{-1}\,\mathbf{C}_m^{12} = \mathbf{0}.$$

Proof. Since the cases of even and odd m are handled in the same way, we assume again without loss of generality, that $m = 2k+1$ is an odd integer. Let $\boldsymbol{a}_k$ $(k = (n+1)/2,\ (n-1)/2, \dots,\ 1)$ denote the columns of the matrix $(\mathbf{C}_m^{11})^{-1}\,\mathbf{D}_m^{11}$, i.e.

$$(\mathbf{C}_m^{11})^{-1}\,\mathbf{D}_m^{11} = \left[\boldsymbol{a}_{\frac{n+1}{2}}, \dots, \boldsymbol{a}_k, \dots, \boldsymbol{a}_1\right],$$

and for $r = 0, \dots, k$ consider the column vectors

$$\boldsymbol{v}_r = \left(x_1^{2r}\,(1-x_1^2)^\beta, x_2^{2r}\,(1-x_2^2)^\beta, \dots, x_{\frac{n+1}{2}}^{2r}\,(1-x_{\frac{n+1}{2}}^2)^\beta\right)^T \in \mathbb{R}^{\frac{n+1}{2}}$$

constituted by the nonzero entries of the columns of the second factor of the matrix $\mathbf{C}_m^{21}$ in (4.6). The multiplication of $\mathbf{C}_m^{21}$ and $(\mathbf{C}_m^{11})^{-1}$ yields

$$\mathbf{C}_m^{21}\,\left(\mathbf{C}_m^{11}\right)^{-1} = \mathbf{D}_m^1\,\underbrace{[\overbrace{\mathbf{0}, \dots, \mathbf{0}}^{\frac{n+1}{2}-k}, \overbrace{\boldsymbol{v}_0, \dots, \boldsymbol{v}_k}^{k+1}]^T\,[\boldsymbol{a}_{\frac{n+1}{2}}, \dots, \boldsymbol{a}_k, \dots, \boldsymbol{a}_1]}_{(*)}\,(\mathbf{D}_m^2)^{-1}.$$

Note that the product in $(*)$ is equal to

$$[\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{\frac{n+1}{2}-k \text{ times}}, \boldsymbol{b}_0, \boldsymbol{b}_1, \dots, \boldsymbol{b}_k]^T,$$

where the row vector $\boldsymbol{b}_r^T := \boldsymbol{v}_r^T\,\left(\mathbf{C}_m^{11}\right)^{-1}$ has the entries

$$\boldsymbol{b}_r\left(\frac{n+1}{2} - s + 1\right) = \boldsymbol{v}_r^T\,\boldsymbol{a}_s, \quad r = 0, \dots, k,\ s = 1, \dots, \frac{n+1}{2}.$$

Since the first $(n+1)/2 - k$ rows of the matrix $\mathbf{C}_m^{12}$ are zero, the assertion is proved if also the last k columns of $(*)$ vanish, i.e.

$$\boldsymbol{b}_r\left(\frac{n+1}{2} - s + 1\right) = \boldsymbol{v}_r^T\,\boldsymbol{a}_s = 0 \quad \text{for all } r = 1, \dots, k,\ s = 1, \dots, k.$$

To show this, let $r \in \{1, \dots, k\}$ and $s \in \{1, \dots, k\}$ be fixed. On the one hand, observe that by definition

$$\left(\mathbf{D}_m^2\right)^{-1}\,\mathbf{C}_m^{11}\,\boldsymbol{a}_s = \mathbf{e}_{\frac{n+1}{2}-s+1}. \tag{4.7}$$

Since $(n+1)/2-k \leq (n+1)/2-s$, equation (4.7) yields the following homogeneous system of $(n+1)/2-k$ equations

$$\begin{aligned}
0 &= \boldsymbol{a}_s(1)+\cdots+\boldsymbol{a}_s\left((n+1)/2\right), \\
&\vdots \\
0 &= \boldsymbol{a}_s(1)\; x_1^{2r}+\cdots+\boldsymbol{a}_s\;\left((n+1)/2)\right) x_{(n+1)/2}^{2r}, \qquad (4.8)\\
&\vdots \\
0 &= \boldsymbol{a}_s(1)\; x_1^{2j}+\cdots+\boldsymbol{a}_s\;\left((n+1)/2\right)\; x_{(n+1)/2}^{2j}, \\
&\vdots \\
0 &= \boldsymbol{a}_s(1)\; x_1^{n-1-4k+2r}+\cdots+\boldsymbol{a}_s\;\left((n+1)/2\right)\; x_{(n+1)/2}^{n-1-4k+2r}, \qquad (4.9)\\
&\vdots \\
0 &= \boldsymbol{a}_s(1)\; x_1^{n+1-4k+2r}+\cdots+\boldsymbol{a}_s\;\left((n+1)/2\right)\; x_{(n+1)/2}^{n+1-4k+2r}, \\
&\vdots \\
0 &= \boldsymbol{a}_s(1)\; x_1^{n-2k-1}+\cdots+\boldsymbol{a}_s\;\left((n+1)/2\right) x_{(n+1)/2}^{n-2k-1},
\end{aligned}$$

where j ranges from $r+1$ to $(n+1)/2-2k+r$. On the other hand, using that $\beta=(n+1)/2-(2k+1)$, we obtain that

$$\begin{aligned}
x^{2r}(1-x^2)^{\beta} &= x^{2r} \sum_{j=0}^{\frac{n+1}{2}-(2k+1)} \binom{\frac{n+1}{2}-(2k+1)}{j}(-1)^j\, x^{2j} \\
&= \sum_{j=0}^{\frac{n+1}{2}-(2k+1)} \binom{\frac{n+1}{2}-(2k+1)}{j}(-1)^j\, x^{2(j+r)}. \qquad (4.10)
\end{aligned}$$

Since

$$0 \leq 2\;(j+r) \leq 2\left(\frac{n+1}{2}-2k-1+r\right) \leq 2\left(\frac{n+1}{2}-k-1\right) = n-2k-1,$$

all the monomials involved in (4.10) also appear in the homogeneous system of equations presented above. Let us now multiply the equations between (4.8) and (4.9), i.e. for $j=r,\ldots,r+(n+1)/2-2\,(k+1)$ by

$$\binom{\frac{n+1}{2}-(2k+1)}{j-r}(-1)^{j-r}.$$

By summing up the modified equations, it finally follows

$$\begin{aligned}
0 &= x_1^{2r}(1-x_1^2)^{\beta}\boldsymbol{a}_s(1)+\cdots+x_{\frac{n+1}{2}}^{2r}\left(1-x_{\frac{n+1}{2}}^2\right)^{\beta}\boldsymbol{a}_s\left((n+1)/2\right) \\
&= \boldsymbol{v}_r^T\;\boldsymbol{a}_s=\boldsymbol{b}_r^T\left((n+1)/2-s+1\right),
\end{aligned}$$

which completes the proof of Lemma 3. ■

4.2 The case of $m = 0, \ldots, n+1$

Before presenting Theorem 3 which fills the gap in the proof of Theorem 1 and establishes the regularity of the matrices $\mathbf{H}_m$ $(m = 0, \ldots, n+1)$, let us recall the Gauss-Kronrod quadrature formulae.

Starting with the so-called *Stieltjes* polynomials $E_{n+1} \in \Pi_{n+1}[-1,1]$ defined up to a multiplicative constant by

$$\int_{-1}^{1} P_n(x)\, E_{n+1}(x)\, x^{\kappa}\, dx = 0, \qquad \kappa = 0, 1, \ldots, n,$$

one can construct the so-called Gauss-Kronrod quadrature formulae, which are based on the union of the Gaussian nodes $\{x_j\}_{j=1,\ldots,n}$, i.e. the zeros of the Legendre polynomial P_n, and on the Kronrod nodes $\{y_k\}_{k=1,\ldots,n+1}$, which are the zeros of the Stieltjes polynomial E_{n+1}. Apart from being real and lying in $(-1,1)$, the Gauss and Kronrod nodes interlace.

With this notation the Gauss-Kronrod quadrature formulae, which are exact for polynomials of degree $3n+1$ for even n and for polynomials of degree $3n+2$ for odd n, read

$$\int_{-1}^{1} P(x)\, dx = \sum_{j=1}^{n} \gamma_j\, P(x_j) + \sum_{k=1}^{n+1} \lambda_k\, P(y_k), \quad P \in \Pi_{3n+1}[-1,1]. \tag{4.11}$$

It should also be mentioned that the weights $\{\gamma_j\}_{j=1,\ldots,n}$ and $\{\lambda_k\}_{k=1,\ldots,n+1}$ are positive. For a detailed description of the approximation properties of these quadrature formulae, we refer to Ehrich [1].

As we will see, the next auxiliary lemma plays a crucial role in the proof of Theorem 3.

Lemma 4. *Let $n \in \mathbb{N}$ be even and let $m \in \mathbb{N}$ be such that $1 \le m \le n$. Furthermore, let p_{m-1}, q_{n-m} and r_{m-1} be polynomials of degrees $m-1$, $n-m$ and $m-1$, respectively, and let E_{n+1} denote the Stieltjes polynomial of degree $n+1$. If*

$$p_{m-1}(x) + q_{n-m}(x)\,(1-x^2)^m = E_{n+1}(x)\, r_{m-1}(x) \quad \text{for all} \;\; x \in \mathbb{R},$$

then $p_{m-1} = q_{n-m} = r_{m-1} = 0$ are the zero polynomials.

Proof. First we observe that E_{n+1}, as an odd polynomial, can be factorized as

$$E_{n+1}(x) = C\, x \prod_{j=1}^{n/2} (x^2 - y_j^2) \qquad \text{for all } x \in \mathbb{R},$$

where $0 =: y_0 < y_1 < \cdots < y_{n/2} < 1$ denote the nonnegative zeros of E_{n+1}. Separating now the even and odd terms of the polynomials p_{m-1}, q_{n-m} and r_{m-1}, and denoting them with p^e_{m-1}, q^e_{n-m}, r^e_{m-1} and p^o_{m-1}, q^o_{n-m}, r^o_{m-1}, respectively, we come up with the following two equations

$$p^e_{m-1}(x) + q^e_{n-m}(x)\,(1-x^2)^m = r^o_{m-1}(x)\,x\prod_{j=1}^{n/2}(x^2-y_j^2) \quad \text{for all } x\in\mathbb{R}$$

and

$$p^o_{m-1}(x) + q^o_{n-m}(x)\,(1-x^2)^m = r^e_{m-1}(x)\,x\prod_{j=1}^{n/2}(x^2-y_j^2) \quad \text{for all } x\in\mathbb{R},$$

where the constant C has been moved into r^o_{m-1} and r^e_{m-1}. Our goal is to show that $p^e_{m-1}, q^e_{n-m} r^e_{m-1}$ and p^o_{m-1}, q^o_{n-m} and r^o_{m-1} are the zero polynomials. Since both equations are treated in the same way, we restrict here to the first case. For more details we refer to Chapter 4 in [6].

(I) If $m = 2k+1$ is **odd**, then equation (4.2) reads

$$p^e_{2k}(x) + q^e_{n-2k-2}(x)\,(1-x^2)^{2k+1} = r^o_{2k-1}(x)\,x\prod_{j=1}^{n/2}(x^2-y_j^2) \quad \text{for all } x\in\mathbb{R}.$$

Substituting now $y = x^2$, we obtain the equation

$$\tilde{p}_k(y)+\tilde{q}_{n/2-k-1}(y)\,(1-y)^{2k+1} = \tilde{r}_k(y)\prod_{j=1}^{n/2}(y-y_j^2) \quad \text{for all } y\in[0,\infty), \tag{4.12}$$

where $\tilde{p}_k$, $\tilde{q}_{n/2-k-1}$ and $\tilde{r}_k$ are polynomials of degrees k, $n/2-k-1$ and k, respectively. Furthermore, we know that $\tilde{r}_k(0)=0$. For the sake of notation, let us denote

$$Q_{n/2+k}(y) := \tilde{p}_k(y) + \tilde{q}_{n/2-k-1}(y)\,(1-y)^{2k+1} \in \Pi_{n/2+k}\,[-1,1].$$

Note that the expansion of the polynomial

$$Q_{n/2+k}(y) = \tilde{p}_k(y) + \tilde{q}_{n/2-k-1}(y)\,(1-y)^{2k+1} = \sum_{l=0}^{n/2+k}\alpha_l\,(1-y)^l$$

around $y = 1$ yields that $\alpha_{k+1} = \alpha_{k+2}\cdots = \alpha_{2k} = 0$. Combining now the fact that

$$Q^{(s)}_{n/2+k}(1) = s!\,\alpha_s = 0, \qquad \text{for } s = k+1,\ldots,2k,$$

with equation (4.12), we come up with the following Birkhoff interpolation problem

$$\begin{aligned} Q_{n/2+k}(y_j^2) &= 0, \qquad j = 0, 1, \ldots, n/2, \\ Q^{(s)}_{n/2+k}(1) &= 0, \qquad s = k+1, \ldots, 2k, \end{aligned}$$

which according to Chapter 1.5 of [5] only admits the trivial solution. Consequently, $Q_{n/2+k}$ is the zero polynomial.

(II) If $m = 2k$ is **even**, an argumentation along the same lines as in the case of odd m leads us to the equation

$$p^e_{2k-2}(x) + q^e_{n-2k}(x)\,(1-x^2)^{2k} = r^o_{2k-1}(x)\, x \prod_{j=1}^{n/2} (x^2 - y_j^2) \quad \text{for all } x \in \mathbb{R}.$$

Following now the same steps as in the case of odd m, we come up with the Birkhoff interpolation problem

$$\begin{aligned} Q_{n/2+k}(y_j^2) &= 0, \qquad j = 0, 1, \ldots, n/2, \\ Q^{(s)}_{n/2+k}(1) &= 0, \qquad s = k, \ldots, 2k-1, \end{aligned}$$

which again by the theorems of Chapter 1.5 in [5] only admits the trivial solution.

Using the same techniques as in the analysis of the first equation, we can show that also the odd part of $p_{m-1}(x) - q_{n-m}(x)\,(1-x^2)^m$ is zero, which finally completes the proof of Lemma 4. ∎

Now that we have collected the necessary ingredients, we are ready to state the promised result concerning the regularity of the matrices $\mathbf{H}_m$ for $m = 0, \ldots, n+1$.

Theorem 3. *Let $n \in \mathbb{N}$ be even and let $-1 < x_n < x_{n-1} < \cdots < x_1 < 1$ denote the zeros of the Legendre polynomial P_n. The matrices $\mathbf{H}_m$ ($m = 0, \ldots, n+1$), defined as in* (3.5)*, are regular.*

Proof. For the sake of simplicity, let us analyze the cases $m = 0$, $m = n+1$ and $1 \le m \le n$ separately.

(I) For $m = 0$, the matrix $\mathbf{H}_0$ attains the form

$$\mathbf{H}_0 = \begin{pmatrix} P_{n+1}(x_1) & P_{n+1}(x_2) & \cdots & P_{n+1}(x_n) \\ P_{n+2}(x_1) & P_{n+2}(x_2) & \cdots & P_{n+2}(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n}(x_1) & P_{2n}(x_2) & \cdots & P_{2n}(x_n) \end{pmatrix}.$$

According to Theorem 2.6 and Corollary 3.4 in [2], the matrix $\mathbf{H}_0$ is regular. This is one of the parts of the proof for which we have to require that $\{x_j\}_{j=1,\dots,n}$ are the zeros of P_n.

(II) Let $m = n+1$. The singularity of $\mathbf{H}_{n+1}$ would imply the nodes $\{x_j\}_{j=1,\dots,n}$ to be the zeros of a polynomial $\sum_{k=n+1}^{2n} \alpha_k P_k^{n+1}$ of degree at most $n-1$. Consequently, this polynomial has to be the zero polynomial. Since the functions $\{P_k^{(n+1)}\}_{k=n+1,\dots,2n}$ constitute a basis of $\Pi_{n-1}[-1,1]$, we can draw the conclusion that $\alpha_k = 0$ for all $k = n+1,\dots,2n$.

(III) Assume now that $1 \le m \le n$. The singularity of the matrix

$$\begin{pmatrix} P_{n+1}^m(x_1) & P_{n+1}^m(x_2) & \cdots & P_{n+1}^m(x_n) \\ P_{n+2}^m(x_1) & P_{n+2}^m(x_2) & \cdots & P_{n+2}^m(x_n) \\ \vdots & \vdots & \ddots & \vdots \\ P_{2n}^m(x_1) & P_{2n}^m(x_2) & \cdots & P_{2n}^m(x_n) \end{pmatrix}$$

would mean that the nodes x_j $(j = 1,\dots,n)$ are zeros of a polynomial of the form $\sum_{k=n+1}^{2n} \alpha_k P_k^m$, and consequently we would be able to factorize this function as

$$\sum_{k=n+1}^{2n} \alpha_k P_k^m(x) = P_n(x)\,(1-x^2)^{m/2}\,Q_{n-m}(x), \tag{4.13}$$

where $Q_{n-m}(x) = \sum_{k=0}^{n-m} \beta_k x^k$ is a polynomial of degree at most $n-m$. We want to show that all coefficients α_k $(k = n+1,\dots,2n)$ or accordingly, all coefficients β_k $(k = 0,\dots,n-m)$ are zero, thereby allowing us to conclude the regularity of the matrix $\mathbf{H}_m$.

As usual, we let $P_n^{(k)}$ denote the kth derivative of the polynomial P_n. Multiplying now both sides of equation (4.13) by P_l^m $(l = m,\dots,n)$, integrating over the interval $[-1,1]$ and exploiting the orthogonality of the associated Legendre functions P_l^m for $l = m, m+1,\dots,$ we obtain

$$\begin{aligned} 0 &= \sum_{k=n+1}^{2n} \alpha_k \int_{-1}^{1} P_k^m(x)\,P_l^m(x)\,dx = \int_{-1}^{1} P_n(x)\,(1-x^2)^{\frac{m}{2}}\,Q_{n-m}(x)\,P_l^m(x)\,dx \\ &= \sum_{k=0}^{n-m} \tilde{\beta}_k \int_{-1}^{1} P_n(x)\,(1-x^2)^m\,x^k\,P_l^{(m)}(x)\,dx, \end{aligned}$$

where the coefficients $\tilde{\beta}_k$ are given by

$$\tilde{\beta}_k = \left(\frac{(l-m)!}{(l+m)!}\right)^{1/2} \beta_k, \qquad k = 0,\dots,n-m.$$

In order to conclude that $\beta_k = \tilde{\beta}_k = 0$ $(k=0,\ldots,n-m)$, we have to establish the regularity of the $(n-m+1)\times(n-m+1)$-dimensional matrix

$$\left(\int_{-1}^{1} P_n(x)\,(1-x^2)^m\, x^k\, P_l^{(m)}(x)\,dx \right)_{k=0,\ldots,n-m,\ l=m,\ldots,n}.$$

Equivalently by performing a change of basis from $\{P_m^{(m)}(x),\ldots,P_n^{(m)}(x)\}$ to $\{1,x,\ldots,x^{n-m}\}$, we can examine the regularity of the matrix

$$\left(\int_{-1}^{1} P_n(x)\,(1-x^2)^m\, x^{r+s}\,dx \right)_{r,s=0,\ldots,n-m} \in \mathbb{R}^{(n-m+1)\times(n-m+1)}.$$

Since $P_n(x)\,(1-x^2)^m\,x^{r+s}$ $(r,s=0,\ldots,n-m)$ is a polynomial of degree at most $n+2m+2(n-m)=3n$, we can now make use of the Gauss-Kronrod quadrature formula (4.11)

$$\int_{-1}^{1} P_n(x)\,(1-x^2)^m\, x^{r+s}\,dx = \sum_{j=1}^{n} \gamma_j\, P_n(x_j)\,(1-x_j^2)^m\, x_j^{r+s}$$
$$+ \sum_{k=1}^{n+1} \lambda_k\, P_n(y_k)\,(1-y_k^2)^m\, y_k^{r+s} = \sum_{k=1}^{n+1} \lambda_k\, P_n(y_k)\,(1-y_k^2)^m\, y_k^{r+s}$$

and reduce our problem to studying the regularity of the matrix

$$\left(\sum_{k=1}^{n+1} \lambda_k\, P_n(y_k)\,(1-y_k^2)^m\, y_k^{r+s} \right)_{r,s=0,\ldots,n-m} \in \mathbb{R}^{(n-m+1)\times(n-m+1)}.$$

Note now that we can factorize the above matrix into a product $\mathbf{AB}$ with

$$\mathbf{A} = \left(\lambda_k\, P_n(y_k)\, y_k^r\right)_{r=0,\ldots,n-m,\ k=1,\ldots,n+1} \in \mathbb{R}^{(n-m+1)\times(n+1)}$$

and

$$\mathbf{B} = \left(y_k^s\,(1-y_k^2)^m\right)_{k=1,\ldots,n+1,\ s=0,\ldots,n-m} \in \mathbb{R}^{(n+1)\times(n-m+1)}.$$

It is clear, that $\mathbf{AB}$ is regular if and only if $\dim\,(\mathrm{Ker}(\mathbf{A}) \cap\ \mathrm{Im}(\mathbf{B})) = 0$. Consequently, it remains to prove that there does not exist any nonzero vector $\boldsymbol{z} \in \mathrm{Ker}(\mathbf{A}) \cap\ \mathrm{Im}(\mathbf{B})$. Assume that $\boldsymbol{z} = (z_k) \in \mathbb{R}^{n+1}$ lies in $\mathrm{Ker}(\mathbf{A}) \cap\ \mathrm{Im}(\mathbf{B})$. Then we can find a polynomial $q_n \in \Pi_n[-1,1]$ with

$$z_k = q_n(y_k), \quad k = 1,\ldots,n+1.$$

Since by assumption $\boldsymbol{z} \in \mathrm{Ker}(\mathbf{A})$, for $r = 0,\ldots,n-m$ we have

$$0 = (\mathbf{A}\,\boldsymbol{z})(r) = \sum_{k=1}^{n+1} \lambda_k\, P_n(y_k)\, y_k^r\, q_n(y_k) = \int_{-1}^{1} P_n(x)\, x^r\, q_n(x)\,dx, \qquad (4.14)$$

where in the last equality we have once more made use of the Gauss-Kronrod quadrature formula (4.11). Moreover, equation (4.14) can hold if and only if q_n is a polynomial of degree at most $m-1$. In fact, if $q_n \in \Pi_{m-1}[-1,1]$, it is straightforward to check that the integral in (4.14) vanishes since $x^r\, q_n(x)$ for $r = 0,\dots,n-m$ is a polynomial of degree at most $n-1$. On the contrary, assume that q_n is exactly of degree τ with $m \le \tau \le n$. Then we would obtain for $r = n-\tau$

$$\int_{-1}^{1} P_n(x)\, x^r\, q_n(x)\, dx \neq 0,$$

a fact which contradicts equation (4.14). So we can record that q_n is a polynomial of degree $m-1$ and we will denote it from now on with q_{m-1}.

On the other hand, by assumption the vector $\boldsymbol{z}$ is a linear combination of the columns of $\mathbf{B}$. Hence, there exists a vector $\boldsymbol{v} = (v_k) \in \mathbb{R}^{n-m+1}$ such that for all $k = 1,\dots,n+1$

$$q_{m-1}(y_k) = z_k = (\mathbf{B}\,\boldsymbol{v})(k) = \sum_{s=0}^{n-m} v_s\; y_k^s\; (1-y_k^2)^m = (1-y_k^2)^m\, p_{n-m}(y_k),$$

where $p_{n-m}(x)$ is the polynomial $\sum_{s=0}^{n-m} v_s\, x^s$. Consequently, the polynomial $q_{m-1}(x) - p_{n-m}(x)\,(1-x^2)^m$ has $n+1$ zeros in $\{y_k\}_{k=1,\dots,n+1}$ and hence, we can factorize it in the form

$$q_{m-1}(x) - p_{n-m}(x)\,(1-x^2)^m = E_{n+1}(x)\, r_{m-1}(x),$$

where $r_{m-1} \in \Pi_{m-1}[-1,1]$ and E_{n+1} denotes the Stieltjes polynomial of degree $n+1$. On account of Lemma 4, q_{m-1} and p_{n-m} are the zero polynomials. Consequently $\mathrm{Ker}(\mathbf{A}) \cap\ \mathrm{Im}(\mathbf{B}) = \mathbf{0}$ and the matrix $\mathbf{AB}$ is regular, which completes the proof of Theorem 3. ■

5 Remarks

Two natural ways of distributing M_s points regularly on the sphere are either by choosing $s+2n+2$ symmetric latitudes and considering on each of them s equidistantly distributed points or by selecting s latitudinal circles, where each of them carries $s+2n+2$ equidistantly distributed nodes. For $s = n$, we have already seen that the second distribution strategy proves to be successful when we choose the heights of the n latitudinal circles as the zeros of the Legendre polynomial P_n. As it is shown in the next lemma, the first construction does not yield a FS for W_n^s ($s \in \mathbb{N}$).

Lemma 5. *Let $n, s \in \mathbb{N}$ and let ρ_j $(j = 1, \ldots, s+2n+2)$ denote $s+2n+2$ arbitrary latitudinal angles. Then the system of points*

$$\left\{\eta_{j,k} = \Psi\left(\rho_j, \theta_k^j\right) :\ j = 1, \ldots, s+2n+2,\ k = 1, \ldots, s\right\}$$

with

$$\theta_k^j = \begin{cases} \frac{2\pi k}{s} & \text{if } j \text{ is odd,} \\ \frac{(2(k-1)+\alpha)\pi}{s} & \text{if } j \text{ is even,} \end{cases}$$

and $\alpha = 0, 1$ does not constitute a FS for W_n^s.

Proof. Our goal is to establish the singularity of the interpolation matrix $\mathbf{B}_n^s$ (2.1) corresponding to the set of points $\{\eta_{j,k}\}$. After multiplying from the left hand side by a proper permutation matrix $\mathbf{P}$, we can reorder the rows of $\mathbf{B}_n^s$ in such a way that the interpolation matrix attains the form

$$\mathbf{P}\,\mathbf{B}_n^s = \left(\begin{array}{ccc} \mathbf{Y}_l^m & : & m \equiv 0 \bmod s,\ l \geq |m| \\ \hline \mathbf{Y}_l^m & : & m \equiv 1 \bmod s,\ l \geq |m| \\ \hline & \vdots & \\ \hline \mathbf{Y}_l^m & : & m \equiv (s-1) \bmod s,\ l \geq |m| \end{array}\right).$$

Here, $\mathbf{Y}_l^m$ again denotes the row vector containing the evaluation of the spherical harmonic Y_l^m at the nodes $\{\eta_{j,k}\}_{j=1,\ldots,2n+s+2,\ k=1,\ldots,s}$. Using the rank invariance of a matrix with respect to regular transformations, we transform $\mathbf{P}\,\mathbf{B}_n^s$ into a more accessible matrix by multiplying from the right hand side by the regular block diagonal matrix

$$\mathbf{F} = \operatorname{diag}\ \overbrace{(\mathbf{F}_s, \mathbf{F}_s, \ldots, \mathbf{F}_s)}^{2n+s+2 \text{ times}} \in \mathbb{C}^{M_s \times M_s}.$$

As usual, $\mathbf{F}_s$ stands for the $s \times s$-dimensional Fourier matrix with entries

$$\mathbf{F}_s(j,k) := \frac{1}{\sqrt{s}} \exp(-2\pi i (j-1)(k-1)/s).$$

Let $m \equiv 0 \bmod s$ and $j \in \{1, \ldots, s+2n+2\}$.
- For $1 \leq r \leq s$ and odd j it holds

$$\begin{aligned} (\mathbf{Y}_l^m\,\mathbf{F})(r + (j-1)\,s) &= \frac{1}{\sqrt{s}} \sum_{k=1}^{s} Y_l^m \left(\Psi\left(\rho_j, \frac{2\pi k}{s}\right)\right) e^{\frac{2\pi i}{s}(1-k)(r-1)} \\ &= \left(\frac{2l+1}{4\pi s}\right)^{1/2} P_l^{|m|}(\cos\rho_j)\, e^{\frac{2\pi i}{s}(r-1)} \sum_{k=1}^{s} e^{\frac{2\pi i}{s} k\,(m-r+1)} \\ &= \left(\frac{(2l+1)\,s}{4\pi}\right)^{1/2} P_l^{|m|}(\cos\rho_j)\,\delta_{r,1}. \end{aligned}$$

- Analogously, for even j we obtain

$$(\mathbf{Y}_l^m\,\mathbf{F})(r+(j-1)\,s) \;=\; \left(\frac{2l+1}{4\pi s}\right)^{1/2} P_l^{|m|}(\cos\rho_j)\,e^{\frac{\alpha\pi i}{s}m}\,\delta_{r,1}.$$

Since $e^{\frac{\alpha\pi i}{s}s}=e^{\frac{\alpha\pi i}{s}(-s)}$ for $\alpha=0,1$, for given $l_0\in\{n+1,\dots,n+s\}$, the row vectors $\mathbf{Y}_{l_0}^s\,\mathbf{F}$ and respectively $\mathbf{Y}_{l_0}^{-s}\,\mathbf{F}$ are linearly dependent. Consequently, the matrices $\mathbf{B}_n^s\,\mathbf{F}$ and accordingly $\mathbf{B}_n^s$ are singular and the underlying set of points cannot constitute a FS for W_n^s. ■

Acknowledgement. The first author was supported by the Graduate Program *Applied Algorithmic Mathematics* of the Munich University of Technology.

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