Multivariate periodic wavelet analysis

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Abstract

General multivariate periodic wavelets are an efficient tool for the approximation of multidimensional functions, which feature dominant directions of the periodicity.

One-dimensional shift invariant spaces and tensor-product wavelets are generalized to multivariate shift invariant spaces on non-tensor-product patterns. In particular, the algebraic properties of the automorphism group are investigated. Possible patterns are classified. By divisibility considerations, decompositions of shift invariant spaces are given.

The results are applied to construct multivariate orthogonal Dirichlet kernels and the respective wavelets. Furthermore a closure theorem is proven.

Key words: Periodic wavelet, multivariate patterns, Fourier techniques, Dirichlet kernel, decomposition and reconstruction formulas, shift invariant space

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1. Introduction

In the last decades, multiscale decomposition of functions and corresponding shift invariant spaces have shown to be a useful tool in image and signal processing. In this paper, we develop a framework for periodic multiscale analysis with the help of shift invariant spaces and thus a framework for wavelets for the approximation and decomposition of periodic multivariate functions on multidimensional patterns of grid points, which are not necessarily tensor products of one-dimensional grids in the co-ordinate directions.

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Therefore, we start with the investigation of shift invariant spaces where the vector-valued shifts are generated by any regular integer matrix $\mathbf{M}$, which is not necessarily diagonal. The periodic patterns bijectively map to the generating groups $\mathcal{G}(\mathbf{M})$ of congruence classes of integer vectors modulo the matrix $\mathbf{M}$. This bijection is the key to the structure of the patterns and the shift invariant spaces, e. g. the theorem on elementary divisors allows us to describe the structure of the Fourier matrix $\mathcal{F}(\mathbf{M})$ in a convincing new manner. Furthermore, all possible patterns of a fixed order $m = |\det \mathbf{M}|$ can be classified, and representatives of the classes are given.

On the one hand, our work is based on the investigation of one-dimensional periodic shift invariant spaces and wavelets in [17, 19, 20, 22, 8], where fundamental techniques for the investigation of periodic wavelets are given.

On the other hand, the multivariate non-periodic case has been studied in e.g. [2, 3, 4]. Here, the setting for the discussion of non-tensor-product wavelets was developed. The discussion of multivariate periodic shift invariant spaces and multivariate wavelets is based on [7, 12, 15, 14]. In particular, algorithms for reconstruction and decomposition are given in [13].

We follow this approach, and we combine it with fundamental Fourier techniques from the one-dimensionsal setting, which are generalized to the multivariate periodic situation.

Hence, the present paper starts with the discussion of the basic pattern properties in Subsections 2.2, 2.3 and 2.4 after some preliminaries and notations. In particular, the generalized Fourier matrix $\mathcal{F}(\mathbf{M})$ is discussed in Subsection 2.3, and a new proof for the orthogonality is given there. As new results, the patterns are classified, and representative matrices are chosen from the set of matrices generating identical patterns in Subsection 2.4.

Section 3 deals with shift invariant spaces, which are generated by the groups $\mathcal{G}(\mathbf{M})$ discussed before. Here, we recognize most of the one-dimensional results from [17, 20, 22] after a suitable generalization, e.g. of the divisibility concept to the multivariate situation. So, Section 3 presents basic theorems about multivariate shift invariant spaces in our new conception. Parts of the results can be found in [14], which concentrates on wavelet frames and potential sequences of integer matrices $\mathbf{M}$, and in [12], where solutions of periodic refinement equations are shown to be linearly independent.

Finally, the orthogonal decomposition of shift invariant spaces is discussed in Section 4 in the case that all subspaces have identical patterns. The decomposition of the shift invariant spaces is related to the matrix decomposition $\mathbf{M} = \mathbf{J}\mathbf{N}$ with a general regular integer matrix $\mathbf{J}$. Whereas in the one-dimensional case, identical pattern of the subspaces are stringent, we intend to generalize the decomposition technique to subspaces on different patterns in the multivariate case. That will enable us to encode directional information of a decomposed function in everyone of the subspaces, which are specific to the underlying pattern. The following Section 5 gives first decomposition results for non-identical patterns.

The paper finishes with extended examples of wavelet decompositions for the orthonormalized Dirichlet kernel in Section 6 with $\mathbf{M}_j = \mathbf{J}^{(j)}\mathbf{M}_{j-1}$. We find that a restricted number of matrices $\mathbf{J}^{(j)}$ can be determined in Lemma 6.5, that enable the construction of wavelets. This restricted number of matrix extensions of the frequency domain nevertheless allows a large variety of decomposition chains. We prove in Theorem 6.8, under which conditions the closure of the union of wavelet spaces is already the Hilbert space of quadratically integrable functions on the multivariate torus.

The results about the patterns or the generating groups respectively are the initial point for further investigations of multivariate periodic wavelets in the non-tensor-product case, e.g. their approximation properties. Further research is needed to adapt the theoretical basics in the present paper to realistically applicable algorithms.

The general periodic grids allow to highlight certain selected directions in the wavelet approximation. In particular the dominance of the co-ordinate directions is overcome in the multivariate case. Although the present work is theoretical and general, multivariate wavelets might serve for the detection of edges in images [10, 11]. Furthermore the approximation of structured functions with preferred directions [6, 18] is a possible field for the application of multivariate wavelets after a future development of the theory.

2. Preliminary investigations

2.1. Basic notations

The function space we have in mind is $L^2(\mathbb{T}^d)$ with the torus $\mathbb{T}^d \cong [0, 2\pi)^d$, which is the Hilbert space of 2π-periodic d-variate functions with the inner product

$$\langle \varphi, \psi \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \varphi(\mathbf{x}) \overline{\psi(\mathbf{x})} \, \mathrm{d}\mathbf{x}$$

and finite norm $\|\varphi\|^2 = \langle \varphi, \varphi \rangle$. Every function $\varphi \in L^2(\mathbb{T}^d)$ can be written as its Fourier series

$$\varphi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(\varphi)\mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}} \tag{1}$$

with Fourier coefficients

$$c_{\mathbf{k}}(\varphi) = \langle \varphi, \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ} \rangle = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} \varphi(\mathbf{x})\mathrm{e}^{-\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}}\,\mathrm{d}\mathbf{x}$$

indexed by integer vectors $\mathbf{k} \in \mathbb{Z}^d$. We exploit the relationship to the Hilbert space $\ell^2(\mathbb{Z}^d)$ of all generalized sequences $\mathbf{c} = (c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}$ with the inner product

$$\langle \mathbf{c}, \mathbf{d} \rangle_{\ell^2} = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}\overline{d}_{\mathbf{k}}$$

and the norm $\|\mathbf{c}\|_{\ell^2} = \langle \mathbf{c}, \mathbf{c} \rangle_{\ell^2}$. The isomorphism between $L^2(\mathbb{T}^d)$ and $\ell^2(\mathbb{Z}^d)$ is represented by Parseval's equation

$$\langle \varphi, \psi \rangle = \langle (c_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d}, (d_{\mathbf{k}})_{\mathbf{k}\in\mathbb{Z}^d} \rangle_{\ell^2} = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(\varphi)\overline{c_{\mathbf{k}}(\psi)} \quad \text{for all} \quad \varphi, \psi \in L^2(\mathbb{T}^d). \tag{2}$$

We denote the set of regular integer matrices by $\mathbb{Z}^{d\times d}_{\mathrm{reg}}$. In general, the modulus of the determinant of a matrix $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ is named $m = |\det \mathbf{M}|$. Every matrix $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ defines a pattern

$$\mathcal{P}(\mathbf{M}) = \{\mathbf{y} \in \mathbb{R}^d \,:\, \mathbf{M}\mathbf{y} \in \mathbb{Z}^d\} = \mathbf{M}^{-1}\mathbb{Z}^d.$$

Obviously, the pattern $\mathcal{P}(\mathbf{M})$ is 1-periodic, i. e., $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ implies $\mathbf{y} + \mathbf{z} \in \mathcal{P}(\mathbf{M})$ for all $\mathbf{z} \in \mathbb{Z}^d$.

A linear subspace $V \subseteq L^2(\mathbb{T}^d)$ is called $\mathbf{M}$-shift invariant [3, 12] or shortly $\mathbf{M}$-invariant if $\varphi \in V$ implies

$$T(\mathbf{y})\varphi = \varphi(\circ - 2\pi\mathbf{y}) \in V \quad \text{for all} \quad \mathbf{y} \in \mathcal{P}(\mathbf{M}).$$

Let us remark that a simple calculation gives

$$c_{\mathbf{k}}(T(\mathbf{y})\varphi) = \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{y}}c_{\mathbf{k}}(\varphi). \tag{3}$$

We need some preliminary considerations about the group properties of the pattern elements, the related Fourier matrix and the classification of patterns before we start to investigate shift invariant spaces in Section 3.

2.2. Generating group

We introduce the congruence relation between integer vectors $\mathbf{h}, \mathbf{k} \in \mathbb{Z}^d$ by

$$\mathbf{h} \equiv \mathbf{k} \bmod \mathbf{M} \;\Leftrightarrow\; \exists \mathbf{z} \in \mathbb{Z}^d \,:\, \mathbf{h} = \mathbf{k} + \mathbf{M}\mathbf{z}$$

and the respective congruence classes

$$\bar{\mathbf{k}} = \{\mathbf{h} \in \mathbb{Z}^d \,:\, \mathbf{h} \equiv \mathbf{k} \bmod \mathbf{M}\} = \{\mathbf{h} \in \mathbb{Z}^d \,:\, \mathbf{h} - \mathbf{k} \in \mathbf{M}\mathbb{Z}^d\}.$$

Analogously, we define the congruence relation between real vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ by $\mathbf{x} \equiv \mathbf{y} \bmod \mathbf{I}$ with respect to the unit matrix $\mathbf{I}$ if and only if there exists an integer vector $\mathbf{z} \in \mathbb{Z}^d$ with $\mathbf{x} = \mathbf{y} + \mathbf{I}\mathbf{z} = \mathbf{y} + \mathbf{z}$.

The generating group consists of the congruence classes of integer vectors modulo the matrix $\mathbf{M}$. We write

$$\mathcal{G}(\mathbf{M}) = \mathbb{Z}^d / \mathbf{M}\mathbb{Z}^d, \tag{4}$$

and we identify the congruence classes $\bar{\mathbf{g}} \in \mathcal{G}(\mathbf{M})$ with their representatives $\mathbf{g} \in \mathbb{Z}^d$. Obviously, we have $\mathbf{M}^{-1}\bar{\mathbf{k}} = \{\mathbf{y} \in \mathbb{R}^d \,:\, \mathbf{y} \equiv \mathbf{M}^{-1}\mathbf{k} \bmod \mathbf{I}\}$. Thus, every unit cube of $\mathbb{R}^d$ contains exactly one element of $\mathbf{M}^{-1}\bar{\mathbf{k}}$, i. e., in particular, we can write

$$|\mathbf{M}^{-1}\bar{\mathbf{k}} \cap [0,1)^d| = 1.$$

Hence, a suitable choice of representatives $\mathbf{g} \in \mathcal{G}(\mathbf{M})$ is described by

$$\mathbf{y} = \mathbf{M}^{-1}\mathbf{g} \in \mathcal{P}(\mathbf{M}) \cap [0,1)^d = \mathcal{P}_{\mathbf{I}}(\mathbf{M}), \tag{5}$$

which expresses the close relation between the generating group $\mathcal{G}(\mathbf{M})$ and the pattern $\mathcal{P}(\mathbf{M})$. All these representatives $\mathbf{g} \in \mathcal{G}(\mathbf{M})$ lie in the semi-open parallelepiped $\mathbf{M}[0,1)^d$. We remark that $\mathcal{P}_{\mathbf{I}}(\mathbf{M})$ equipped by the addition modulo the unit matrix $\mathbf{I}$ is isomorphic to the generating group $\mathcal{G}(\mathbf{M})$, i. e.,

$$(\mathcal{P}_{\mathbf{I}}(\mathbf{M}), + \bmod \mathbf{I}) \cong \mathcal{G}(\mathbf{M}).$$

The theorem on elementary divisors [16, Ch. 10] assures the decomposition

$$\mathbf{M} = \mathbf{Q}\mathbf{E}\mathbf{R} \;\text{ with }\; \mathbf{E} = \operatorname{diag}\left((\varepsilon_j)_{j=1}^d\right) \tag{6}$$

with two co-ordinate transformations $\mathbf{R}, \mathbf{Q} \in \mathbb{Z}^{d\times d}_{\text{reg}}$ of $\mathbb{Z}^d$, i. e., $|\det \mathbf{R}| = |\det \mathbf{Q}| = 1$ and elementary divisors $\varepsilon_j \in \mathbb{N}$ fulfill the property $\varepsilon_{j-1} | \varepsilon_j$ for $j = 2, \ldots, d$. The decomposition (6) is called Smith normal form [5, Ch. 5].

Since $\mathbf{R}$ and $\mathbf{Q}$ define co-ordinate transformations of $\mathbb{Z}^d$, the groups $\mathcal{G}(\mathbf{M})$ and $\mathcal{G}(\mathbf{E})$ are isomorphic. Hence, the generating group is a direct product of cyclic groups $\mathcal{C}(\varepsilon_j)$ of the order of the elementary divisors

$$\mathcal{G}(\mathbf{M}) \cong \mathcal{C}(\varepsilon_1) \otimes \ldots \otimes \mathcal{C}(\varepsilon_d). \tag{7}$$

Consequently, every element of the generating group can be addressed by d co-ordinates g_j with $0 \leq g_j < \varepsilon_j$, i.e., $\mathbf{g} = \mathbf{g}(g_1, \ldots, g_d)$ and the problem is reduced to the tensor-product case in the transformed co-ordinates. In practice, it is rather expensive to calculate the transformations $\mathbf{R}$ and $\mathbf{Q}$ and hence the addressing indices g_j of the elements $\mathbf{g}$ [1].

We remark that the order of the generating group is

$$|\mathcal{G}(\mathbf{M})| = \varepsilon_1 \cdot \ldots \cdot \varepsilon_d = |\det \mathbf{M}| = m. \tag{8}$$

The transposed matrix $\mathbf{M}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{E}\mathbf{Q}^{\mathrm{T}}$ has the same elementary divisors and the respective generating groups

$$\mathcal{G}(\mathbf{M}) \cong \mathcal{G}(\mathbf{M}^{\mathrm{T}})$$

are isomorphic. In particular, the elements of $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ can be addressed by the same index vectors as the ones of $\mathcal{G}(\mathbf{M})$.

2.3. Fourier matrix

The Fourier matrix with respect to the matrix $\mathbf{M}$ is defined [9] by

$$\mathcal{F}(\mathbf{M}) = \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi \mathrm{i} \mathbf{h}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{g}}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}}),\mathbf{g}\in\mathcal{G}(\mathbf{M})} \in \mathbb{C}^{m\times m}. \tag{9}$$

The definition (9) depends on the arrangement of the elements $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ indicating the rows of the Fourier matrix and on the arrangement of the elements $\mathbf{g} \in \mathcal{G}(\mathbf{M})$ indicating the columns. If not pointed out differently, we refer to an arbitrary but fixed arrangement of the elements. We additionally require that the elements of $\mathcal{P}_{\mathbf{I}}(\mathbf{M})$ are arranged in the same order as the elements $\mathbf{g} \in \mathcal{G}(\mathbf{M})$.

If $\mathbf{a} = (a_{\mathbf{g}})_{\mathbf{g}\in\mathcal{G}(\mathbf{M})} \in \mathbb{C}^m$ denotes a vector indexed by the elements of $\mathcal{G}(\mathbf{M})$, then its discrete Fourier transform

$$\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} = \sqrt{m}\,\mathcal{F}(\mathbf{M})\mathbf{a} \in \mathbb{C}^m \tag{10}$$

is a vector with elements $\hat{a}_{\mathbf{h}}$ indexed by the elements $\mathbf{h}$ of $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$. Of course, the indexing arrangements of $\mathcal{G}(\mathbf{M})$ respectively $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ used in $\mathbf{a}$ and in $\hat{\mathbf{a}}$ have to be identically used in $\mathcal{F}(\mathbf{M})$, too.

The following lemma proves the tensor product structure of the Fourier matrix. The consequent Corollary 2.2 already has been shown in [9] by a technical proof without referring to the tensor product structure.

Lemma 2.1. *Let* $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$. *Then there are standard Fourier matrices*

$$\mathcal{F}_{\varepsilon_j} = \frac{1}{\sqrt{\varepsilon_j}} \left(\mathrm{e}^{-2\pi \mathrm{i} h_j \varepsilon_j^{-1} g_j} \right)_{h_j, g_j = 0}^{\varepsilon_j - 1}$$

and permutation matrices $\mathbf{P}_{\mathbf{g}}$ *and* $\mathbf{P}_{\mathbf{h}}$ *so that*

$$\mathcal{F}(\mathbf{M}) = \mathbf{P}_{\mathbf{h}} \left(\mathcal{F}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{F}_{\varepsilon_d} \right) \mathbf{P}_{\mathbf{g}} \tag{11}$$

with the Kronecker product $\otimes$.

Proof. Due to (6) with the co-ordinate transformations $\mathbf{R}$ and $\mathbf{Q}$ of $\mathbb{Z}^d$, the representatives $\mathbf{g} = \mathbf{Q}\tilde{\mathbf{g}}$ and $\mathbf{h} = \mathbf{R}^{\mathrm{T}}\tilde{\mathbf{h}}$ can be chosen with

$$\tilde{\mathbf{g}} \in \mathbf{Q}^{-1}\mathcal{G}(\mathbf{M}) = \{0, \ldots, \varepsilon_1 - 1\} \times \ldots \times \{0, \ldots, \varepsilon_d - 1\}$$

and

$$\tilde{\mathbf{h}} \in \mathbf{R}^{-\mathrm{T}}\mathcal{G}(\mathbf{M}^{\mathrm{T}}) = \{0, \ldots, \varepsilon_1 - 1\} \times \ldots \times \{0, \ldots, \varepsilon_d - 1\},$$

and they can be arranged alpha-numerically by the index vector $(g_1, \ldots, g_d)$ respectively $(h_1, \ldots, h_d)$ with $0 \le g_j, h_j < \varepsilon_j$. Then, we find

$$\mathcal{F}(\mathbf{M}) = \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi \mathrm{i} (\mathbf{R}^{-\mathrm{T}}\mathbf{h})^{\mathrm{T}} \mathbf{E}^{-1} \mathbf{Q}^{-1} \mathbf{g}} \right)_{\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}), \mathbf{g} \in \mathcal{G}(\mathbf{M})}$$

and thus

$$\mathcal{F}(\mathbf{M}) = \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi \mathrm{i} \tilde{\mathbf{h}}^{\mathrm{T}} \mathbf{E}^{-1} \tilde{\mathbf{g}}} \right)_{\tilde{\mathbf{h}} \in \mathbf{R}^{-\mathrm{T}}\mathcal{G}(\mathbf{M}^{\mathrm{T}}), \tilde{\mathbf{g}} \in \mathbf{Q}^{-1}\mathcal{G}(\mathbf{M})}$$

and hence using the alpha-numerical arrangement

$$\mathcal{F}_{\mathrm{an}}(\mathbf{M}) = \frac{1}{\sqrt{m}} \left(\mathrm{e}^{-2\pi \mathrm{i} (h_1 \varepsilon_1^{-1} g_1 + \ldots + h_d \varepsilon_d^{-1} g_d)} \right)_{(h_1, \ldots, h_d), (g_1, \ldots, g_d) = (0, \ldots, 0)}^{(\varepsilon_1 - 1, \ldots, \varepsilon_d - 1)}. \tag{12}$$

We find that $\mathcal{F}_{\mathrm{an}}(\mathbf{M})$ is the Kronecker product of standard Fourier matrices

$$\mathcal{F}_{\mathrm{an}}(\mathbf{M}) = \mathcal{F}_{\varepsilon_1} \otimes \ldots \otimes \mathcal{F}_{\varepsilon_d}. \tag{13}$$

The arrangement of the group elements $\mathbf{g}$ and $\mathbf{h}$ results from the alphanumerical arrangement by the permutation matrices $\mathbf{P}_{\mathbf{h}}$ and $\mathbf{P}_{\mathbf{g}}$, and the Fourier matrix reads now $\mathcal{F}(\mathbf{M}) = \mathbf{P}_{\mathbf{h}}\mathcal{F}_{\mathrm{an}}(\mathbf{M})\mathbf{P}_{\mathbf{g}}^{\mathrm{T}}$ as proposed in (11). □

The Fourier matrix $\mathcal{F}(\mathbf{M})$ is not symmetric in general. The Kronecker product (13) is the starting point for the construction of fast Fourier algorithms for $\mathcal{F}(\mathbf{M})$, where $\mathcal{F}_{\varepsilon_j}$ can be further decomposed into a Kronecker product of smaller standard Fourier matrices whenever $\varepsilon_j \in \mathbb{N}$ is not prime.

Corollary 2.2. *Every integer matrix* $\mathbf{M} \in \mathbb{Z}_{\mathrm{reg}}^{d\times d}$ *generates a Fourier matrix* $\mathcal{F}(\mathbf{M})$ *with the orthogonality property* $\mathcal{F}(\mathbf{M})\overline{\mathcal{F}(\mathbf{M})}^{\mathrm{T}} = \mathbf{I}$.

Proof. We start with (11) and (13), and we calculate

$$\mathcal{F}(\mathbf{M})\overline{\mathcal{F}(\mathbf{M})}^{\mathrm{T}} = \mathbf{P}_{\mathbf{h}}\mathcal{F}_{\mathrm{an}}(\mathbf{M})\mathbf{P}_{\mathbf{g}}^{\mathrm{T}}\,\mathbf{P}_{\mathbf{g}}\overline{\mathcal{F}_{\mathrm{an}}(\mathbf{M})}^{\mathrm{T}}\mathbf{P}_{\mathbf{h}}^{\mathrm{T}} = \mathbf{P}_{\mathbf{h}}\mathcal{F}_{\mathrm{an}}(\mathbf{M})\overline{\mathcal{F}_{\mathrm{an}}(\mathbf{M})}^{\mathrm{T}}\mathbf{P}_{\mathbf{h}}^{\mathrm{T}},$$

with

$$\mathcal{F}_{\mathrm{an}}(\mathbf{M})\overline{\mathcal{F}_{\mathrm{an}}(\mathbf{M})}^{\mathrm{T}} = \mathcal{F}_{\varepsilon_1}\overline{\mathcal{F}_{\varepsilon_1}}^{\mathrm{T}} \otimes \ldots \otimes \mathcal{F}_{\varepsilon_d}\overline{\mathcal{F}_{\varepsilon_d}}^{\mathrm{T}} = \mathbf{I} \in \mathbb{C}^{m\times m},$$

where we have reduced the assertion to the respective property of standard matrices. □

The following lemma is an example for a well-known one-dimensional result on circulant matrices, which can be straightforward transformed into the multivariate setting.

Lemma 2.3. *For every vector* $\mathbf{a} = (a_{\mathbf{g}})_{\mathbf{g}\in\mathcal{G}(\mathbf{M})}$*, the identity*

$$\operatorname{circ}\mathbf{a} = (a_{\mathbf{g}-\mathbf{l}})_{\mathbf{g},\mathbf{l}} = \mathcal{F}(\mathbf{M})^{\mathrm{T}}\operatorname{diag}\hat{\mathbf{a}}\,\overline{\mathcal{F}(\mathbf{M})}$$

is satisfied.

Proof. We write the matrix $\mathbf{B} = \overline{\mathcal{F}(\mathbf{M})}\operatorname{circ}\mathbf{a}\,\mathcal{F}(\mathbf{M})^{\mathrm{T}}$ in the form

$$\mathbf{B} = (b_{\mathbf{hk}})_{\mathbf{h},\mathbf{k}} = \frac{1}{m}\left(\mathrm{e}^{2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{g}}\right)_{\mathbf{h},\mathbf{g}}(a_{\mathbf{l}-\mathbf{g}})_{\mathbf{g},\mathbf{l}}\left(\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{l}}\right)_{\mathbf{l},\mathbf{k}}$$

with $\mathbf{g},\mathbf{l} \in \mathcal{G}(\mathbf{M})$ and $\mathbf{h},\mathbf{k} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$. The entry $b_{\mathbf{hk}}$ is

$$b_{\mathbf{hk}} = \frac{1}{m}\sum_{\mathbf{g}\in\mathcal{G}(\mathbf{M})}\sum_{\mathbf{l}\in\mathcal{G}(\mathbf{M})}\mathrm{e}^{2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{g}}\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{l}}a_{\mathbf{l}-\mathbf{g}},$$

which after an index shift $\mathbf{l}-\mathbf{g} \to \mathbf{g}$ reads as

$$b_{\mathbf{hk}} = \frac{1}{m} \sum_{\mathbf{g}\in\mathcal{G}(\mathbf{M})} a_{\mathbf{g}} \mathrm{e}^{-2\pi \mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{g}} \sum_{\mathbf{l}\in\mathcal{G}(\mathbf{M})} \mathrm{e}^{-2\pi \mathrm{i}(\mathbf{h}-\mathbf{k})^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{l}}.$$

Now, the first sum is just $\hat{a}_{\mathbf{h}}$, and the second sum is m for $\mathbf{h}=\mathbf{k}$ in $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and vanishes otherwise because it is the entry $(\mathbf{h}-\mathbf{k})$ of $m\mathcal{F}(\mathbf{M})\mathcal{F}(\mathbf{M})^{\mathrm{T}}$, cf. Corollary 2.2. That proves the lemma. □

2.4. Pattern classification

Matrices $\mathbf{M},\mathbf{N} \in \mathbb{Z}^{d\times d}$ are called equivalent $\mathbf{M} \cong \mathbf{N}$ if they generate identical pattern $\mathcal{P}(\mathbf{M}) = \mathcal{P}(\mathbf{N})$.

Lemma 2.4. *Matrices $\mathbf{M} \cong \mathbf{N}$ are equivalent if and only if there exists a matrix $\mathbf{Q} \in \mathbb{Z}^{d\times d}$ with $|\det \mathbf{Q}| = 1$ and $\mathbf{N} = \mathbf{QM}$.*

Proof. First, $\mathbf{M} \cong \mathbf{N}$ implies $\mathcal{P}_{\mathbf{I}}(\mathbf{M}) = \mathcal{P}_{\mathbf{I}}(\mathbf{N})$ and by (5) and (8) it follows $|\det \mathbf{M}| = |\det \mathbf{N}|$. The equivalence $\mathbf{M} \cong \mathbf{N}$ implies that $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ leads to $\mathbf{y} \in \mathcal{P}(\mathbf{N})$. The pattern elements $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ are related to $\mathbf{z} \in \mathbb{Z}^d$ by the definition $\mathbf{y} = \mathbf{M}^{-1}\mathbf{z}$. The property $\mathbf{y} \in \mathcal{P}(\mathbf{N})$ is expressed by

$$\mathbf{NM}^{-1}\mathbf{z} \in \mathbb{Z}^d \;\text{ for all }\; \mathbf{z} \in \mathbb{Z}^d.$$

Hence, we conclude $\mathbf{NM}^{-1} \in \mathbb{Z}^{d\times d}$ and $|\det(\mathbf{NM}^{-1})| = 1$. Then $\mathbf{Q} = \mathbf{NM}^{-1}$.

Second, $\mathcal{P}(\mathbf{M}) = \mathcal{P}(\mathbf{N})$ follows from $\mathbf{N} = \mathbf{QM}$ by

$$\mathcal{P}(\mathbf{QM}) = \{\mathbf{y} : \mathbf{QMy} \in \mathbb{Z}^d\} = \{\mathbf{y} : \mathbf{My} \in \mathbf{Q}^{-1}\mathbb{Z}^d = \mathbb{Z}^d\} = \mathcal{P}(\mathbf{M}). \;\square$$

In the following lemma we choose a matrix from every equivalence class in a representative form.

Lemma 2.5. *The right upper triangular matrices $\mathbf{M} = (m_{jk})_{j,k=1}^d$ with $0 \le m_{jk} < m_{kk}$ for $j < k$ and all $k = 1,\ldots,d$ are representatives of the classes of equivalent matrices.*

Proof. First, we argue that every integer matrix $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ can be transformed into a right upper triangular matrix by multiplications with matrices of determinant 1. By appropriate addition matrices and permutation matrices the first column of $\mathbf{M}$ can be transformed into $m_1\mathbf{e}_1$ with the unit vector $\mathbf{e}_1 = (1,0,\ldots,0)^{\mathrm{T}} \in \mathbb{Z}^d$ and the greatest common divisor

$m_1 = \gcd\{m_{11}, \ldots, m_{d1}\}$, cf. the Euclidean algorithm. The step results in a matrix $\mathbf{M}'$ with a multiple of the unit vector in the first column. The procedure is continued by the same operation for the first column of the minor $(m'_{jk})_{j,k=2}^{d}$ of $\mathbf{M}'$ and so on with smaller and smaller minors.

Second, the renamed right upper triangular matrix $\mathbf{M}$ is transformed by adding an integer multiple of the second row to the first one assuring $0 \leq m_{12} < m_{22}$. That is proceeded by adding integer multiples of the third row to the first two rows assuring $0 \leq m_{13} < m_{33}$ and $0 \leq m_{23} < m_{33}$ and further on until the claimed structure is received.

Now, we regard two matrices $\mathbf{M} = (m_{jk})_{j,k=1}^{d}$ and $\mathbf{N} = (n_{jk})_{j,k=1}^{d}$ of the given structure and we assume that they generate identical patterns. The equality $|\det \mathbf{M}| = |\det \mathbf{N}|$ implies $m_{11} \cdot \ldots \cdot m_{dd} = n_{11} \cdot \ldots \cdot n_{dd}$. We write $\mathbf{M} = \tilde{\mathbf{M}} \operatorname{diag}(m_{kk})_{k=1}^{d}$ and $\mathbf{N} = \tilde{\mathbf{N}} \operatorname{diag}(n_{kk})_{k=1}^{d}$ with the left upper triangular matrices $\tilde{\mathbf{M}}$ and $\tilde{\mathbf{N}}$ with units at the diagonal. We multiply $\mathbf{N}\mathbf{M}^{-1}$ with the kth unit vector $\mathbf{e}_k$ and find the kth component

$$\left(\mathbf{N}\mathbf{M}^{-1}\mathbf{e}_k\right)_k = \frac{n_{kk}}{m_{kk}} \in \mathbb{Z}, \tag{14}$$

which is an integer because of Lemma 2.4. The assumption of identical patterns leads to $n_{kk} = m_{kk}$ for $k = 1, \ldots, d$. Now, $\mathbf{N}\mathbf{M}^{-1} = \tilde{\mathbf{N}}\tilde{\mathbf{M}}^{-1}$ is valid. The non-diagonal entries of $\tilde{\mathbf{N}}$ and $\tilde{\mathbf{M}}$ lie in the interval $[0, 1)$. The condition $\tilde{\mathbf{N}} = \mathbf{Q}\tilde{\mathbf{M}}$ with $\mathbf{Q} \in \mathbb{Z}_{\text{reg}}^{d \times d}$ leads to $\mathbf{Q} = \mathbf{I}$ by checking successively the resulting conditions for the entries of $\mathbf{Q}$ in each row. We conclude, that every matrix $\mathbf{M}$ can be transformed into a representative form, and that two different matrices in representative form generate different patterns. □

This result restricts the possible patterns for a given matrix determinant and group order $m = |\mathcal{P}_{\mathbf{I}}(\mathbf{M})|$, cf. (5).

Lemma 2.6. *The number of possible patterns of a given order m is*

$$\sum_{m = t_1 \cdot \ldots \cdot t_d} t_2 t_3^2 \cdot \ldots \cdot t_d^{d-1} = \frac{1}{m} \sum_{m = t_1 \cdot \ldots \cdot t_d} t_1 t_2^2 t_3^3 \cdot \ldots \cdot t_d^d$$

where the sum runs over all decompositions of m into d positive integers with consideration of the succession.

Proof. We count the representative forms of possible matrices $\mathbf{M}$. Every decomposition $m = t_1 \cdot \ldots \cdot t_d$ generates a possible diagonal, and the right upper triangle can be filled up with $k - 1$ entries $0 \leq m_{jk} < m_{kk}$. □

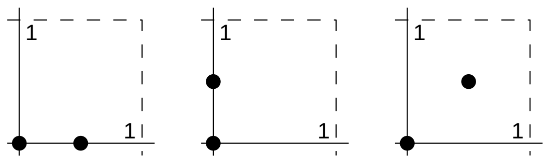

Figure 1: All possible patterns for $d = 2$ and $m = 2$. From left to right: $\mathcal{P}_{\mathbf{I}}(\mathbf{A}_1)$, $\mathcal{P}_{\mathbf{I}}(\mathbf{A}_2)$, $\mathcal{P}_{\mathbf{I}}(\mathbf{A}_3)$, cf. Example 2.8.

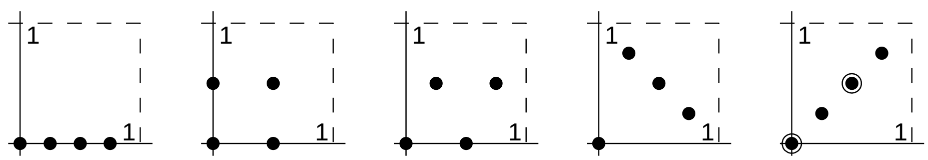

Figure 2: Five of the seven possible patterns for $d = 2$ and $m = 4$. From left to right: $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_1)$, $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_2)$, $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_3)$, $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_5)$, $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_7)$, compare Example 2.8. The pattern $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_4)$ is the mirrored $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_1)$ at the diagonal, and the mirrored $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_3)$ is $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_6)$. The inclusion is demonstrated in $\mathcal{P}_{\mathbf{I}}(\mathbf{B}_7)$, which contains $\mathcal{P}_{\mathbf{I}}(\mathbf{A}_3)$.

In particular, the practically relevant cases $d = 2$ and $d = 3$ give a manageable number of possible patterns. For instance with $m = 2^j$, we find $2^{j+1} - 1$ different patterns for $d = 2$ and $(2^{j+1} - 1)(2^{j+2} - 1)/3$ for $d = 3$.

Lemma 2.7. *Let be* $\mathbf{M}, \mathbf{N} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$. *The inclusion* $\mathcal{P}(\mathbf{N}) \subseteq \mathcal{P}(\mathbf{M})$ *is equivalent to the existence of a matrix* $\mathbf{J} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ *with* $\mathbf{M} = \mathbf{J}\mathbf{N}$.

Proof. The existence of $\mathbf{J}$ assures that $\mathbf{y} \in \mathcal{P}(\mathbf{N})$ implies $\mathbf{N}\mathbf{y} \in \mathbb{Z}^d$ and hence $\mathbf{J}\mathbf{N}\mathbf{y} \in \mathbb{Z}^d$, too. We find $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ for all $\mathbf{y} \in \mathcal{P}(\mathbf{N})$.

On the other hand for $\mathcal{P}(\mathbf{N}) \subseteq \mathcal{P}(\mathbf{M})$, we argue as in Lemma 2.4 that $\mathbf{M}\mathbf{N}^{-1}\mathbf{z} \in \mathbb{Z}^d$ for all $\mathbf{z} \in \mathbb{Z}^d$ and we find $\mathbf{J} = \mathbf{M}\mathbf{N}^{-1} \in \mathbb{Z}^{d\times d}$. □

We remark that due to non-commutativity of the product $\mathbf{M} = \mathbf{J}\mathbf{N}$, the pattern $\mathcal{P}(\mathbf{J})$ is not a sub-pattern of $\mathcal{P}(\mathbf{M})$ in general.

Example 2.8. *For $d = 2$ and $m = 2$, we find the representatives*

$$\mathbf{A}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{A}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \ \textit{and} \ \mathbf{A}_3 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$$

for the classes of different patterns. The respective patterns are given in Fig. 1. Since the generating group is of order 2 and the cyclic group $\mathcal{C}_2$ is the only one of order 2, we obtain $\mathcal{G}(\mathbf{A}_j) \cong \mathcal{C}_2$ for $j \in \{1, 2, 3\}$.

Fig. 2 gives the possible patterns for $d = 2$ and $m = 4$. The representatives of equivalent patterns are $\mathbf{B}_1, \ldots, \mathbf{B}_7$ with e. g.

$$\mathbf{B}_1 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{B}_2 = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \ \mathbf{B}_3 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \ \mathbf{B}_4 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix},$$

and

$$\mathbf{B}_5 = \begin{pmatrix} 1 & 1 \\ 0 & 4 \end{pmatrix}, \ \mathbf{B}_6 = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}, \ \mathbf{B}_7 = \begin{pmatrix} 1 & 3 \\ 0 & 4 \end{pmatrix}.$$

We remark the product decomposition $\mathbf{B}_1 = \mathbf{A}_1\mathbf{A}_1$, $\mathbf{B}_2 = \mathbf{A}_1\mathbf{A}_2 = \mathbf{A}_2\mathbf{A}_1$, $\mathbf{B}_3 = \mathbf{A}_3\mathbf{A}_1$, $\mathbf{B}_5 = \mathbf{A}_2\mathbf{A}_3$ and $\mathbf{B}_7 = \mathbf{A}_3\mathbf{A}_3$ and the related inclusion relations e. g. $\mathcal{P}(\mathbf{A}_3) \subset \mathcal{P}(\mathbf{B}_5)$, cf. Fig. 2. We find $\mathcal{G}(\mathbf{B}_j) \cong \mathcal{C}_4$ for $j \in \{1, 3, 4, 5, 6, 7\}$ because all these groups are of order 4 and contain an element of order 4. But already here, we find $\mathcal{G}(\mathbf{B}_2) \cong \mathcal{C}_2 \otimes \mathcal{C}_2$, which is the Klein four-group and hence different from $\mathcal{C}_4$.

An immediate consequence of Lemma 2.7 is the inclusion

$$\mathcal{P}(\mathbf{I}) \subseteq \mathcal{P}(\mathbf{M}) \subseteq \mathcal{P}(m\mathbf{I}), \tag{15}$$

and hence all vectors $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ fulfill $m\mathbf{y} \in \mathbb{Z}^d$, i. e., the components of $\mathbf{y}$ are multiples of $1/m$.

Since $\mathcal{P}(\mathbf{I}) = \mathbb{Z}^d$, we can write

$$\mathcal{P}(\mathbf{J}) = \bigcup_{\mathbf{y} \in \mathcal{P}_\mathbf{I}(\mathbf{J})} \mathcal{P}(\mathbf{I}) + \mathbf{y}$$

and hence

$$\mathcal{P}(\mathbf{JN}) = \bigcup_{\mathbf{y} \in \mathcal{P}_\mathbf{I}(\mathbf{J})} \mathbf{N}^{-1}\mathcal{P}(\mathbf{I}) + \mathbf{N}^{-1}\mathbf{y} = \bigcup_{\mathbf{y} \in \mathcal{P}_\mathbf{I}(\mathbf{J})} \mathcal{P}(\mathbf{N}) + \mathbf{N}^{-1}\mathbf{y}.$$

All elements of $\mathbf{u} \in \mathcal{P}_\mathbf{I}(\mathbf{JN})$ can be given in the form

$$\mathbf{u} = \mathbf{x} + \mathbf{N}^{-1}\mathbf{y} \bmod \mathbf{I} \ \text{ with } \ \mathbf{x} \in \mathcal{P}_\mathbf{I}(\mathbf{N}), \, \mathbf{y} \in \mathcal{P}_\mathbf{I}(\mathbf{J}), \tag{16}$$

and from $\mathcal{P}_\mathbf{I}(\mathbf{J}) \cap \mathbb{Z}^d = \{\mathbf{0}\}$ follows $|\mathcal{P}_\mathbf{I}(\mathbf{JN})| = |\mathcal{P}_\mathbf{I}(\mathbf{N})| \cdot |\mathcal{P}_\mathbf{I}(\mathbf{J})|$. Thus the decomposition (16) of $\mathbf{u} \in \mathcal{P}_\mathbf{I}(\mathbf{JN})$ is unique.

Let us remark that $|\det \mathbf{Q}| = 1$ implies $\mathcal{P}(\mathbf{N}) = \mathcal{P}(\mathbf{QN})$, but in general it does not imply the equivalence of the patterns $\mathcal{P}(\mathbf{JN})$ and $\mathcal{P}(\mathbf{JQN})$ because $\mathcal{P}(\mathbf{J})$ and $\mathcal{P}(\mathbf{JQ})$ may differ.

By Lemma 2.7 all possible patterns are arranged in a pattern tree with half-group properties. Following the tree in the construction of $\mathbf{M}$ as a product of matrices with smaller determinants generates an indexing system of the elements of $\mathcal{G}(\mathbf{M})$ and the related Fourier matrix. Now, we show how patterns can be factorized.

Lemma 2.9. *For all divisors n of m, there is a matrix $\mathbf{N}$ with $|\det \mathbf{N}| = n$ providing a sub-pattern $\mathcal{P}(\mathbf{N}) \subseteq \mathcal{P}(\mathbf{M})$.*

Proof. In (6), the diagonal matrix $\mathbf{E}$ can be easily factorized $\mathbf{E} = \mathbf{E}_{m/n}\mathbf{E}_n$ with $\det \mathbf{E}_n = n$. Now, $\mathbf{N} = \mathbf{E}_n\mathbf{R}$ together with Lemma 2.7 yield the proposition. □

As an immediate consequence of Lemma 2.9, every matrix $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ can be factorized in a product $\mathbf{M} = \mathbf{M}_1 \cdot \ldots \cdot \mathbf{M}_k$, where $|\det \mathbf{M}_j| = p_j$ are the prime factors of $m = |\det \mathbf{M}|$ providing sub-patterns

$$\mathcal{P}(\mathbf{M}_k) \subset \mathcal{P}(\mathbf{M}_{k-1}\mathbf{M}_k) \subset \ldots \subset \mathcal{P}(\mathbf{M}).$$

Next, we investigate the sum of different patterns

$$\mathcal{P} = \mathcal{P}(\mathbf{M}_1) + \mathcal{P}(\mathbf{M}_2) = \{\mathbf{x} + \mathbf{y} \,:\, \mathbf{x} \in \mathcal{P}(\mathbf{M}_1),\, \mathbf{y} \in \mathcal{P}(\mathbf{M}_2)\}. \tag{17}$$

Lemma 2.10. *Let be $\mathbf{M}_1, \mathbf{M}_2 \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ arbitrarily given. The sum of the patterns $\mathcal{P}(\mathbf{M}_1)$ and $\mathcal{P}(\mathbf{M}_2)$ with $m_1 = |\det \mathbf{M}_1|$ and $m_2 = |\det \mathbf{M}_2|$ is a pattern $\mathcal{P}(\mathbf{K})$ for some matrix $\mathbf{K} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$. The intersection $\mathcal{P}(\mathbf{M}_1) \cap \mathcal{P}(\mathbf{M}_2)$ is a common sub-pattern of $\mathcal{P}(\mathbf{M}_1)$ and $\mathcal{P}(\mathbf{M}_2)$. If $\mathbf{N} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ is a matrix with $\mathcal{P}(\mathbf{N}) = \mathcal{P}(\mathbf{M}_1) \cap \mathcal{P}(\mathbf{M}_2)$, then there is the divisibility relation*

$$|\det \mathbf{K}| = \frac{m_1 m_2}{|\det \mathbf{N}|}.$$

Proof. Let $\mathcal{U} = \mathcal{P}(\mathbf{M}_1) \cap \mathcal{P}(\mathbf{M}_2)$. Now, $\mathbf{x}, \mathbf{y} \in \mathcal{U}$ implies $\mathbf{x} - \mathbf{y} \in \mathcal{U}$, and due to $\mathcal{U} \subseteq \mathcal{P}(\mathbf{M}_1)$ and $\mathcal{U} \subseteq \mathcal{P}(\mathbf{M}_2)$, the set $\mathcal{U}$ is a sub-pattern generated by a matrix $\mathbf{N} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$, i. e. $\mathcal{U} = \mathcal{P}(\mathbf{N})$, where the matrix $\mathbf{N}$ is not yet explicitly known. The sum

$$\mathcal{P}_{\mathbf{I}} = \{\mathbf{x} + \mathbf{y} \bmod \mathbf{I} \,:\, \mathbf{x} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M}_1),\, \mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M}_2)\}$$

is an Abelian group. Furthermore, we know $\mathcal{P}_{\mathbf{I}} \subset \mathcal{P}_{\mathbf{I}}(m_1 m_2 \mathbf{I})$ due to (15). Hence, there is a matrix $\mathbf{K}$ producing the pattern $\mathcal{P} = \mathcal{P}(\mathbf{K})$. The generating group $\mathcal{G}(\mathbf{K})$ fulfills

$$\mathcal{G}(\mathbf{K}) \cong \mathcal{G}(\mathbf{M}_1) \times (\mathcal{G}(\mathbf{M}_2)/(\mathcal{G}(\mathbf{M}_1) \cap \mathcal{G}(\mathbf{M}_2))),$$

and the assertion is proven. $\square$

In general, the simple union $\mathcal{P}(\mathbf{M}_1) \cup \mathcal{P}(\mathbf{M}_2)$ is not a pattern, which can be easily seen from Figs. 1 and 2.

The matrix $\mathbf{K}$ can be reconstructed by finding d elements $\mathbf{y}_1, \ldots, \mathbf{y}_d$ of $\mathcal{P}(\mathbf{K})$ spanning whole $\mathcal{P}(\mathbf{K})$ by its integer linear combinations, cf. (7). The first element is found by looking for an element of highest possible order in $(\mathcal{P}_{\mathbf{I}}, + \bmod \mathbf{I}) \cong \mathcal{G}(\mathbf{K})$, called $\mathbf{y}_d$. Then, $\mathbf{y}_{d-1}$ is an element of highest possible order in the factorial group $(\mathcal{P}_{\mathbf{I}}, + \bmod I)/\langle \mathbf{y}_1 \rangle$ and so on. Finally, the matrix $\mathbf{K} \in \mathbb{Z}^{d \times d}$ and $\mathbf{K}\mathbf{y}_j \in \mathbb{Z}^d$ for $j = 1, \ldots, d$ is found. The matrix $\mathbf{K}$ is unique among the representatives of equivalence classes of matrices producing identical patterns.

Analogously, the matrix $\mathbf{N}$ can be reconstructed from the intersection $\mathcal{U}$.

3. Shift invariant spaces

The span of all translates $T(\mathbf{y})$, $\mathbf{y} \in \mathcal{P}(\mathbf{M})$ of a function $\varphi \in L^2(\mathbb{T}^d)$ is a $\mathbf{M}$-shift invariant subspace. It is denoted by

$$V_{\mathbf{M}}^{\varphi} = \operatorname{span}\left\{T(\mathbf{y})\varphi \,:\, \mathbf{y} \in \mathcal{P}(\mathbf{M})\right\} = \operatorname{span}\left\{T(\mathbf{y})\varphi \,:\, \mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})\right\}.$$

Several theorems from the one-dimensional case $d = 1$ [20, 17, 19, 22] can be formulated in the present general investigation as well. Some results of Section 3 are given for completeness although they are already found in the literature. So, Theorem 3.1 and 3.3 together with Corollary 3.2 are consequences of results in [14], which concentrates on wavelet frames and which assumes the matrix $\mathbf{M}$ to be a potential of an integer matrix. Similar proofs are found there. Similarly, Corollaries 3.5 and 3.6 can be found in [12] where the linear independence is shown for solutions of periodic refinement equations.

The following Theorem 3.1 shows that the translates $T(\mathbf{y})\varphi$ span the same subspace as the so-called orthogonal splines [20]

$$f_{\mathbf{h}}^{\varphi}(\mathbf{x}) = \sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi) \mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{k}^{\mathrm{T}}\mathbf{M})\mathbf{x}} \tag{18}$$

for $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ do.

Theorem 3.1. *Let be* $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ *and* $\varphi \in L^2(\mathbb{T}^d)$*. The elements* $\mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})$ *and* $\mathbf{g} \in \mathcal{G}(\mathbf{M})$ *in indexing the Fourier matrix* $\mathcal{F}(\mathbf{M}^{\mathrm{T}})$ *are related by (5). Then, the vector of translates of* φ *and the vector of orthogonal splines* $(f^{\varphi}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})}$ *fulfill the equation*

$$(T(\mathbf{y})\varphi)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} = \sqrt{m}\,\mathcal{F}(\mathbf{M})^{\mathrm{T}}\,(f^{\varphi}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})}\,, \tag{19}$$

Proof. The equations (3) and (1) verify the relation

$$T(\mathbf{y})\varphi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{y}} c_{\mathbf{k}}(\varphi)\mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}}.$$

Now, we address all integer vectors by $\mathbf{h} + \mathbf{M}^{\mathrm{T}}\mathbf{k}$ with $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and $\mathbf{k} \in \mathbb{Z}^d$, cf. (4). We use $\mathbf{k}^{\mathrm{T}}\mathbf{M}\mathbf{y} \in \mathbb{Z}$ because of $\mathbf{y} \in \mathcal{P}(\mathbf{M})$, and we get the linear combination

$$T(\mathbf{y})\varphi(\mathbf{x}) = \sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} f^{\varphi}_{\mathbf{h}}(\mathbf{x}).$$

This equation contains an arbitrary row of the required relation (19) what finishes the proof. □

The following corollary is an immediate consequence of Theorem 3.1.

Corollary 3.2. *The shift invariant space of every* $\varphi \in L^2(\mathbb{T}^d)$ *with respect to a matrix* $\mathbf{M} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ *fulfills*

$$V^{\varphi}_{\mathbf{M}} = \operatorname{span}\left\{f^{\varphi}_{\mathbf{h}} \,:\, \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})\right\}.$$

Theorem 3.3. *It holds* $\xi \in V^{\varphi}_{\mathbf{M}}$ *if and only if there exists a vector* $\mathbf{a} = (a_{\mathbf{y}})_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})}$ *with the discrete Fourier transform* $\hat{\mathbf{a}} = (\hat{a}_{\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})}$ *fulfilling*

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi) = \hat{a}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi) \;\; \textit{for all} \;\; \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}), \mathbf{k} \in \mathbb{Z}^d. \tag{20}$$

Then, the function ξ *is the linear combination of translates*

$$\xi = \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} a_{\mathbf{y}} T(\mathbf{y})\varphi. \tag{21}$$

Proof. If $\xi \in V_{\mathbf{M}}^{\varphi}$ and thus (21) is valid, then

$$c_{\mathbf{k}}(\xi) = \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} a_{\mathbf{y}} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{y}} c_{\mathbf{k}}(\varphi).$$

Using the notation $\mathbf{h} + \mathbf{M}^{\mathrm{T}}\mathbf{k}$ with $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and $\mathbf{k} \in \mathbb{Z}^d$ for the integer vectors, we find

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi) = \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} a_{\mathbf{y}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi) = \hat{a}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi),$$

what actually is (20).

If otherwise, we start with (20), we can note

$$\xi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(\xi)\mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}} = \sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \sum_{\mathbf{k}\in\mathbb{Z}^d} \hat{a}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)\mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{k}^{\mathrm{T}}\mathbf{M})\mathbf{x}}$$

and thus

$$\xi(\mathbf{x}) = \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} a_{\mathbf{y}} \sum_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)\mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{k}^{\mathrm{T}}\mathbf{M})\mathbf{x}}.$$

The multiplications with $\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{M}\mathbf{y}} = 1$ points out that the double sum over $\mathbf{h}$ and $\mathbf{k}$ is a sum over all integer vectors, and we obtain (21). □

Theorem 3.4. *Let $\varphi, \psi \in L^2(\mathbb{T}^d)$ be two given functions. Then, the Gram matrix $\mathbf{G} = (\langle T(\mathbf{x})\varphi, T(\mathbf{y})\psi\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})}$ is circulant, and it is true that*

$$\mathbf{G} = \mathcal{F}(\mathbf{M})^{\mathrm{T}} \operatorname{diag}\left(m \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)\overline{c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\psi)} \right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \overline{\mathcal{F}(\mathbf{M})}.$$

Proof. The matrix is circulant, $\mathbf{G} = \operatorname{circ}\mathbf{a}$, because

$$(\langle T(\mathbf{x})\varphi, T(\mathbf{y})\psi\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} = (\langle \varphi, T(\mathbf{y}-\mathbf{x})\psi\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})}$$

with the vector $\mathbf{a}$ and $a_{\mathbf{y}} = \langle\varphi, T(\mathbf{y})\psi\rangle$. Using Lemma 2.3 and Parseval's equation (2), we get

$$\hat{a}_{\mathbf{h}} = \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} \mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(\varphi)\overline{c_{\mathbf{k}}(T(\mathbf{y})\psi)} = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(\varphi)\overline{c_{\mathbf{k}}(\psi)} \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} \mathrm{e}^{-2\pi\mathrm{i}(\mathbf{h}-\mathbf{k})^{\mathrm{T}}\mathbf{y}}$$

for $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$. Since the sum over $\mathbf{y}$ is m for $\mathbf{k} \equiv \mathbf{h} \bmod \mathbf{M}^{\mathrm{T}}$ in $\mathbb{Z}^d$ and vanishes else, the assertion is proven. □

Corollary 3.5. *The set of translates* $\{T(\mathbf{y})\varphi : \mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})\}$ *is linearly independent if and only if*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)|^2 > 0 \quad \textit{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}). \tag{22}$$

Proof. The equation (22) assures that the Gram matrix [20], which here is $\mathbf{G} = (\langle T(\mathbf{x})\varphi, T(\mathbf{y})\varphi\rangle)_{\mathbf{x},\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})}$ is regular by Theorem 3.4. $\square$

Consequently, the condition (22) yields to $\dim V_{\mathbf{M}}^{\varphi} = m$.

Corollary 3.6. *The functions* $\{T(\mathbf{y})\varphi : \mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})\}$ *are orthonormal if and only if*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)|^2 = \frac{1}{m} \quad \textit{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}).$$

Proof. The Gram matrix of a set of orthonormal functions is the unit matrix, i. e., all its eigenvalues are 1, and the asserted corollary immediately follows by Theorem 3.4. $\square$

Furthermore, we can state the next corollary in complete analogy to [22], and the proof evolves respectively, too.

Corollary 3.7. *If the relation (22) is satisfied, then*

$$(T(\mathbf{y})\varphi_{\mathrm{on}})_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} = \mathcal{F}(\mathbf{M})^{\mathrm{T}} \operatorname{diag}\left(m \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)|^2\right)^{-\frac{1}{2}}_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \overline{\mathcal{F}(\mathbf{M})}(T(\mathbf{y})\varphi)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})}$$

is a vector of orthonormal translates of φ_{on} *with* $V_{\mathbf{M}}^{\varphi_{\mathrm{on}}} = V_{\mathbf{M}}^{\varphi}$*. The Fourier coefficients of* φ_{on} *are*

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi_{\mathrm{on}}) = \frac{c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)}{\left(m\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi)|^2\right)^{\frac{1}{2}}} \quad \textit{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}),\, \mathbf{k} \in \mathbb{Z}^d.$$

4. Orthogonal decomposition on fixed patterns

This section deals with the decomposition of the $\mathbf{M}$-shift invariant subspace $V_{\mathbf{M}}^{\zeta}$ for a function ζ into an orthogonal direct sum of $\mathbf{N}$-shift invariant subspaces in the case that $\mathbf{N}$ generates a sub-pattern $\mathcal{P}(\mathbf{N}) \subset \mathcal{P}(\mathbf{M})$ and we know $\mathbf{M} = \mathbf{J}\mathbf{N}$ with an integer matrix $\mathbf{J} \in \mathbb{Z}^{d\times d}$. More general decomposition results are given in Section 5.

Theorem 4.1. *Let the function* $\zeta \in L^2(\mathbb{T}^d)$ *generate the* m*-dimensional* $\mathbf{M}$*-shift invariant space* $V_{\mathbf{M}}^{\zeta}$ *with* $m = |\det \mathbf{M}|$ *and* $\mathbf{M} = \mathbf{JN}$ *for* $\mathbf{J}, \mathbf{N} \in \mathbb{Z}_{\text{reg}}^{d\times d}$*. Then there are functions* $\xi_{\mathbf{g}} \in L^2(\mathbb{T}^d)$*,* $\mathbf{g} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})$ *with*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{k}}(\xi_{\mathbf{g}})|^2 > 0 \quad \text{for all } \mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}}),$$

which produce the orthogonal decomposition

$$V_{\mathbf{M}}^{\zeta} = \bigoplus_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} V_{\mathbf{N}}^{\xi_{\mathbf{g}}}.$$

Proof. Since ζ generates an m-dimensional shift invariant space, the translates $T(\mathbf{y})\zeta$ with $\mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})$ are linearly independent, and the function ζ fulfills condition (22), cf. Corollary 3.5. We regard the generalized sequences $\mathbf{a}^{\mathbf{h}} = (a_{\mathbf{k}}^{\mathbf{h}})_{\mathbf{k}\in\mathbb{Z}^d}$ for $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ with

$$a_{\mathbf{k}}^{\mathbf{h}} = \begin{cases} c_{\mathbf{k}}(\zeta) & \text{if } \mathbf{k} \equiv \mathbf{h} \bmod \mathbf{M}^{\mathrm{T}} \text{ in } \mathbb{Z}^d, \\ 0 & \text{else.} \end{cases}$$

By Corollary 3.2, the set

$$\left\{ \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}^{\mathbf{h}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ} \,:\, \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}) \right\}$$

is a basis of $V_{\mathbf{M}}^{\zeta}$. The basis is orthogonal because $\langle \mathrm{e}^{\mathrm{i}\mathbf{k}_1^{\mathrm{T}}\circ}, \mathrm{e}^{\mathrm{i}\mathbf{k}_2^{\mathrm{T}}\circ} \rangle$ vanishes for $\mathbf{k}_1 \neq \mathbf{k}_2$, and $a_{\mathbf{k}}^{\mathbf{h}_1} \overline{a}_{\mathbf{k}}^{\mathbf{h}_2}$ vanishes for $\mathbf{h}_1 \neq \mathbf{h}_2$.

We refer to (16) and see that every element $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ can be decomposed as $\mathbf{h} = \mathbf{N}^{\mathrm{T}}\mathbf{g} + \tilde{\mathbf{h}}$ with $\mathbf{g} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})$ and $\tilde{\mathbf{h}} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}})$. Hence, the span of the basis can be decomposed into

$$\operatorname{span} \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}^{\mathbf{h}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ} \,:\, \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}) \right\} =$$
$$\bigoplus_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} \operatorname{span} \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^d} a_{\mathbf{k}}^{\mathbf{h}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ} \,:\, \mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}}) + \mathbf{N}^{\mathrm{T}}\mathbf{g} \right\},$$

which is formulated as

$$V_{\mathbf{M}}^{\zeta} = \bigoplus_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} \operatorname{span} \left\{ \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta) \mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{g}^{\mathrm{T}}\mathbf{N}+\mathbf{k}^{\mathrm{T}}\mathbf{M})\circ} \,:\, \mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}}) \right\}. \tag{23}$$

Now, (23) yields the Fourier coefficients of $\xi_{\mathbf{g}}$ which are $c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi_{\mathbf{g}}) = c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)$ for all $\mathbf{h} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})$, $\mathbf{k} \in \mathbb{Z}^d$ and vanishing for other indices. $\square$

Lemma 4.2. *Let $\zeta \in L^2(\mathbb{T}^d)$ generate the m-dimensional $\mathbf{M}$-shift invariant space $V_{\mathbf{M}}^{\zeta}$ with $m = |\det \mathbf{M}|$. Furthermore, a number $J = |\det \mathbf{J}|$ of functions*

$$\xi_j = \sum_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} a_{j,\mathbf{y}} T(\mathbf{y})\zeta$$

for $j = 1, \ldots, J$ may be given. Let $\mathbf{M} = \mathbf{J}\mathbf{N}$. Then, the following statements are valid:

(i) The set $\{T(\mathbf{y})\xi_j : \mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{N})\}$ is linearly independent if and only if

$$\sum_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} |\hat{a}_{j,\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}|^2 > 0 \quad \text{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}}).$$

(ii) Let $j_1 \neq j_2$. The translates are orthogonal, $\langle T(\mathbf{x})\xi_{j_1}, T(\mathbf{y})\xi_{j_2}\rangle = 0$ for all $\mathbf{x}, \mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{N})$, if and only if

$$\sum_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} \hat{a}_{j_1,\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \overline{\hat{a}}_{j_2,\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)|^2 = 0$$

is fulfilled for all $\mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}})$.

(iii) The vector of translates of ξ_j can be given by

$$\overline{\mathcal{F}(\mathbf{N})}\,(T(\mathbf{y})\xi_j)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{N})} = \sqrt{\frac{n}{m}}\,\mathbf{A}\,\overline{\mathcal{F}(\mathbf{M})}\,(T(\mathbf{y})\zeta)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})}$$

with $n = |\det \mathbf{N}|$ and the row vector of diagonal matrices

$$\mathbf{A} = \Big(\big(\operatorname{diag}(\hat{a}_{j,\mathbf{h}+\mathbf{J}^{\mathrm{T}}\mathbf{g}})_{\mathbf{h}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})}\big)_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})}\Big) \in \mathbb{C}^{n\times m}. \tag{24}$$

Proof. The Fourier coefficients fulfill $c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi_j) = \hat{a}_{j,\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)$ for all $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and $\mathbf{k} \in \mathbb{Z}^d$ due to Theorem 3.3. The assertion (i) immediately follows from Corollary 3.5 and the linear independence of the translates of ζ. Similarly, assertion (ii) is a reformulation of a vanishing Gram matrix in Theorem 3.4.

Now, (19) reads as

$$\overline{\mathcal{F}(\mathbf{M})}\,(T(\mathbf{y})\zeta)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})} = \sqrt{m}\left(\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)\mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{k}^{\mathrm{T}}\mathbf{M})\circ}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})}.$$

Regarding the translates of ξ_j, it reads

$$\overline{\mathcal{F}(\mathbf{N})}\,(T(\mathbf{y})\xi_j)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{N})} = \sqrt{n}\left(\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{k}}(\xi_j)\mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{k}^{\mathrm{T}}\mathbf{N})\circ}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})},$$

which can be written as

$$\begin{aligned}
&\overline{\mathcal{F}(\mathbf{N})}\,(T(\mathbf{y})\xi_j)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{N})} = \dots\\
&= \sqrt{n}\left(\sum_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} \hat{a}_{j,\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}\sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)\mathrm{e}^{\mathrm{i}(\mathbf{h}^{\mathrm{T}}+\mathbf{g}^{\mathrm{T}}\mathbf{N}+\mathbf{k}^{\mathrm{T}}\mathbf{M})\circ}\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})}\\
&= \sqrt{\frac{n}{m}}\mathbf{A}\,\overline{\mathcal{F}(\mathbf{M})}\,(T(\mathbf{y})\zeta)_{\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})},
\end{aligned}$$

with the matrix

$$\mathbf{A} = \big(\operatorname{diag}(\hat{a}_{j,\mathbf{h}})_{\mathbf{h}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})},\dots,\operatorname{diag}(\hat{a}_{j,\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}})_{\mathbf{h}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})}\big)$$

given in (24), and the proof is completed. □

Assertion (iii) of Lemma 4.2 allows us to express a basis transformation from the translates of ζ into the translates of all ξ_j, $j = 1,\dots,J = m/n$ in the case $|\det\mathbf{J}| = J$. We get

$$\begin{aligned}
&\Big((T(\mathbf{y})\xi_1)_{\mathbf{y}},\dots,(T(\mathbf{y})\xi_J)_{\mathbf{y}}\Big)^{\mathrm{T}} = \qquad (25)\\
&\sqrt{\frac{n}{m}}\begin{pmatrix}\mathcal{F}(\mathbf{N})^{\mathrm{T}} & & \mathbf{0}\\ & \ddots & \\ \mathbf{0} & & \mathcal{F}(\mathbf{N})^{\mathrm{T}}\end{pmatrix}\begin{pmatrix}\mathbf{A}_{1,\mathbf{g}_1} & & \mathbf{A}_{1,\mathbf{g}_J}\\ & \ddots & \\ \mathbf{A}_{J,\mathbf{g}_1} & & \mathbf{A}_{J,\mathbf{g}_J}\end{pmatrix}\overline{\mathcal{F}(\mathbf{M})}\,(T(\mathbf{z})\zeta)_{\mathbf{z}}
\end{aligned}$$

In (25), it is $\mathbf{y}\in\mathcal{P}_{\mathbf{I}}(\mathbf{N})$ and $\mathbf{g}_j\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})$, and the elements of $\mathbf{z}\in\mathcal{P}_{\mathbf{I}}(\mathbf{M})$ are in the appropriate arrangement given by (16). The matrices $\mathbf{A}_{J,\mathbf{g}_1}$ are the diagonal matrices already used in Lemma 4.2, i. e.,

$$\mathbf{A}_{j,\mathbf{g}} = \operatorname{diag}(\hat{a}_{j,\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}})_{\mathbf{h}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})}.$$

The inverse of the matrix $\mathbf{A} = (\mathbf{A}_{j,\mathbf{g}})_{j=1,\ldots,J,\mathbf{g}\in\mathcal{G}(\mathbf{N}^{\mathrm{T}})}$ is easily found due to its rather simple structure, and (25) can be used as transformation equation in both directions.

The following Theorem 4.3 deals with the orthogonality of two subspaces. Whereas Theorem 4.1 states the existence of an orthogonal decomposition of $V_{\mathbf{M}}^{\zeta}$, the following one gives a condition to check whether a space $V_{\mathbf{N}}^{\eta}$ is the orthogonal complement of $V_{\mathbf{N}}^{\xi}$ in $V_{\mathbf{M}}^{\zeta}$ for the most common case of $J = |\det \mathbf{J}| = 2$.

Theorem 4.3. *Let be* $\mathbf{M} = \mathbf{J}\mathbf{N}$ *integer matrices* $\mathbf{M}, \mathbf{N}, \mathbf{J} \in \mathbb{Z}_{\mathrm{reg}}^{d\times d}$ *with* $|\det \mathbf{M}| = m$, $|\det \mathbf{N}| = n$ *and* $m = 2n$. *The function* ζ *has the* m*-dimensional* $\mathbf{M}$*-shift invariant space* $V_{\mathbf{M}}^{\zeta}$, *and the function* ξ *generates the* n*-dimensional* $\mathbf{N}$*-shift invariant space* $V_{\mathbf{N}}^{\xi}$. *Let* $\xi \in V_{\mathbf{M}}^{\zeta}$ *with*

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi) = \hat{a}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta) \quad \textit{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}),\ \mathbf{k} \in \mathbb{Z}^d.$$

Then, the space $V_{\mathbf{N}}^{\eta}$ *is the orthogonal complement of* $V_{\mathbf{N}}^{\xi}$ *in* $V_{\mathbf{M}}^{\zeta}$ *if and only if there are numbers* $\sigma_{\mathbf{h}} \in \mathbb{C}\backslash\{0\}$, $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ *with*

$$\sigma_{\mathbf{h}} = -\sigma_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \quad \textit{for} \quad \mathbf{g} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})\backslash\{\mathbf{0}\} \tag{26}$$

fulfilling

$$c_{\mathbf{k}}(\eta) = \frac{\sigma_{\mathbf{k} \bmod \mathbf{M}^{\mathrm{T}}} \overline{\hat{a}}_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g} \bmod \mathbf{M}^{\mathrm{T}}}}{\sum\limits_{\mathbf{l}\in\mathbb{Z}^d} |c_{\mathbf{k}+\mathbf{M}^{\mathrm{T}}\mathbf{l}}(\zeta)|^2} c_{\mathbf{k}}(\zeta) \tag{27}$$

for all $\mathbf{k} \in \mathbb{Z}^d$.

Proof. From $V_{\mathbf{N}}^{\eta} \oplus V_{\mathbf{N}}^{\xi} = V_{\mathbf{M}}^{\zeta}$ follows the existence of the vector $(\hat{b})_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})}$ with

$$c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\eta) = \hat{b}_{\mathbf{h}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta) \quad \text{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}),\ \mathbf{k} \in \mathbb{Z}^d$$

by Theorem 3.3. Since the translates of η are orthogonal to $V_{\mathbf{N}}^{\eta}$, Lemma 4.2 (ii) yields

$$\sum_{\mathbf{g}\in\mathcal{G}(\mathbf{J}^{\mathrm{T}})} \hat{a}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \overline{\hat{b}}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)|^2 = 0 \tag{28}$$

for all $\mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}})$. Since $|\mathcal{G}(\mathbf{J}^{\mathrm{T}})| = 2$, the element $\mathbf{g} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})\backslash\{\mathbf{0}\}$ is unique, and (28) can be written as

$$\hat{a}_{\mathbf{h}} \overline{\hat{b}}_{\mathbf{h}} \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)|^2 + \hat{a}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \overline{\hat{b}}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)|^2 = 0$$

for all $\mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}})$. If now $\hat{a}_{\mathbf{h}} \neq 0$ and $\hat{a}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \neq 0$, then we choose

$$\sigma_{\mathbf{h}} = \frac{\hat{b}_{\mathbf{h}}}{\hat{a}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}} \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)|^2 \text{ and } \sigma_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} = \frac{\hat{b}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}}{\hat{a}_{\mathbf{h}}} \sum_{\mathbf{k}\in\mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\zeta)|^2 \tag{29}$$

fulfilling all required conditions. If one of the named Fourier coefficients is vanishing, the appropriate one of the formulas in (29) is used to compute $\sigma_{\mathbf{h}}$ or $\sigma_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}$. The lacking value $\sigma_{\mathbf{h}}$ or respectively $\sigma_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}$ is chosen so that it obeys condition (26). Since the space $V_{\mathbf{N}}^{\xi}$ has full rank, the case that both Fourier coefficients vanish for all $\mathbf{k} \in \mathbb{Z}^d$ is excluded.

The opposite direction of the proof starting from the existence of numbers $\sigma_{\mathbf{h}}$ is a simple calculation following the reverse steps. □

Corollary 4.4. *The orthogonal complement of $V_{\mathbf{N}}^{\xi}$ in $V_{\mathbf{M}}^{\zeta}$ with integer matrices $\mathbf{M} = \mathbf{JN}$ and $|\det \mathbf{J}| = 2$ is an $\mathbf{N}$-shift invariant space, and there is a function $\eta \in L^2(\mathbb{T}^d)$, the translates of which form a basis of $V_{\mathbf{M}}^{\zeta} \ominus V_{\mathbf{N}}^{\xi}$.*

Proof. The assertion follows directly from Theorem 4.3 by choosing

$$\sigma_{\mathbf{h}} = 1 \text{ for } \mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}}) \text{ and } \sigma_{\mathbf{h}} = -1 \text{ for } \mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}}) + \mathbf{N}^{\mathrm{T}}\mathbf{g}$$

with $\mathbf{g} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})\backslash\{\mathbf{0}\}$. □

Using the notation from Theorem 4.3, we specify (25) for the practically most relevant case $|\det \mathbf{J}| = 2$. We get for $\mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{N})$, $\mathbf{z} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})$ and $\mathbf{h} \in \mathcal{G}(\mathbf{N}^{\mathrm{T}})$ the transformation formula

$$\begin{pmatrix} \overline{\mathcal{F}(\mathbf{N})}\,(T(\mathbf{y})\xi)_{\mathbf{y}} \\ \overline{\mathcal{F}(\mathbf{N})}\,(T(\mathbf{y})\eta)_{\mathbf{y}} \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \operatorname{diag}(\hat{\alpha}_{\mathbf{h}})_{\mathbf{h}} & \operatorname{diag}(\hat{\alpha}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}})_{\mathbf{h}} \\ \operatorname{diag}(\hat{\beta}_{\mathbf{h}})_{\mathbf{h}} & \operatorname{diag}(\hat{\beta}_{\mathbf{h}+\mathbf{N}^{\mathrm{T}}\mathbf{g}})_{\mathbf{h}} \end{pmatrix} \overline{\mathcal{F}(\mathbf{M})}\,(T(\mathbf{z})\zeta)_{\mathbf{z}}\,. \tag{30}$$

Again, the elements are arranged with respect to the decomposition (16).

Since the matrix occurring in (30) contains four diagonal blocks, its inversion is straightforward, and (25) can be inverted respectively, too.

5. Decomposition results on general patterns

After having discussed the decomposition of a shift invariant space $V_{\mathbf{M}}^{\zeta}$ into shift invariant spaces $V_{\mathbf{N}}^{\xi_{\mathbf{g}}}$ with a single common pattern $\mathcal{P}(\mathbf{N})$, we will discuss the decomposition of a shift invariant space into more general subspaces. The reason lies in the idea that every subspace encodes some properties which are related to the underlying pattern. Hence, different subspaces can encode different properties of a function in $V^{\zeta}_{\mathbf{M}}$.

The investigation becomes much more difficult when shift invariant spaces over independent patterns are considered. We present some basic ideas.

We regard two patterns $\mathcal{P}(\mathbf{M}_1)$ and $\mathcal{P}(\mathbf{M}_2)$. Their sum, defined in (17), is a pattern, too, cf. Lemma 2.10. We denote a matrix $\mathbf{K}$, which fulfills $\mathcal{P}(\mathbf{K}) = \mathcal{P}(\mathbf{M}_1)+\mathcal{P}(\mathbf{M}_2)$ as a least common multiple of $\mathbf{M}_1$ and $\mathbf{M}_2$, because there are matrices $\mathbf{J}_1$ and $\mathbf{J}_2$ with $\mathbf{K} = \mathbf{J}_1\mathbf{M}_1 = \mathbf{J}_2\mathbf{M}_2$, cf. Lemma 2.7. A matrix $\mathbf{K}'$ with $|\det \mathbf{K}'| < |\det \mathbf{K}|$ cannot have the same property.

Theorem 5.1. *Let $V^{\varphi}_{\mathbf{M}_1}$ and $V^{\psi}_{\mathbf{M}_2}$ be shift invariant spaces of full rank. The matrix $\mathbf{K}$ is a least common multiple of $\mathbf{M}_1$ and $\mathbf{M}_2$. Then, the sum $V^{\varphi}_{\mathbf{M}_1} + V^{\psi}_{\mathbf{M}_2}$ is direct if and only if the identity*

$$\hat{a}_{\mathbf{h} \bmod \mathbf{M}_1^{\mathrm{T}}} c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\varphi) = \hat{b}_{\mathbf{h} \bmod \mathbf{M}_2^{\mathrm{T}}} c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\psi) \quad \text{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{K}^{\mathrm{T}}),\, \mathbf{k} \in \mathbb{Z}^d \tag{31}$$

implies $\hat{a}_{\mathbf{h}_1} = 0$ for all $\mathbf{h}_1 \in \mathcal{G}(\mathbf{M}_1^{\mathrm{T}})$ and $\hat{b}_{\mathbf{h}_2} = 0$ for all $\mathbf{h}_2 \in \mathcal{G}(\mathbf{M}_2^{\mathrm{T}})$.

Proof. The sum is direct if $\xi \in V^{\varphi}_{\mathbf{M}_1} \cap V^{\psi}_{\mathbf{M}_2}$ implies $\xi = 0$. We denote $m_1 = |\det \mathbf{M}_1|$ and $m_2 = |\det \mathbf{M}_2|$. Let us assume that ξ lies in the intersection of both spaces. By Theorem 3.3, it follows the existence of two vectors $\mathbf{a} \in \mathbb{C}^{m_1}$ and $\mathbf{b} \in \mathbb{C}^{m_2}$ with

$$c_{\mathbf{h}_1+\mathbf{M}_1^{\mathrm{T}}\mathbf{k}_1}(\xi) = \hat{a}_{\mathbf{h}_1} c_{\mathbf{h}_1+\mathbf{M}_1^{\mathrm{T}}\mathbf{k}_1}(\varphi) \quad \text{for all} \quad \mathbf{h}_1 \in \mathcal{G}(\mathbf{M}_1^{\mathrm{T}}),\, \mathbf{k}_1 \in \mathbb{Z}^d$$

and

$$c_{\mathbf{h}_2+\mathbf{M}_2^{\mathrm{T}}\mathbf{k}_2}(\xi) = \hat{b}_{\mathbf{h}_2} c_{\mathbf{h}_2+\mathbf{M}_2^{\mathrm{T}}\mathbf{k}_2}(\psi) \quad \text{for all} \quad \mathbf{h}_2 \in \mathcal{G}(\mathbf{M}_2^{\mathrm{T}}),\, \mathbf{k}_2 \in \mathbb{Z}^d\,.$$

The elements $\mathbf{h} \in \mathcal{G}(\mathbf{K})$ can be converted into $\mathbf{h}_1 = \mathbf{h} \bmod \mathbf{M}_1^{\mathrm{T}} \in \mathcal{G}(\mathbf{M}_1^{\mathrm{T}})$ and $\mathbf{h}_2 = \mathbf{h} \bmod \mathbf{M}_2^{\mathrm{T}} \in \mathcal{G}(\mathbf{M}_2^{\mathrm{T}})$. The Fourier coefficients of ξ are identified, and the condition $\xi \in V^{\varphi}_{\mathbf{M}_1} \cap V^{\psi}_{\mathbf{M}_2}$ transforms into (31).

If now, the sum $V^{\varphi}_{\mathbf{M}_1} + V^{\psi}_{\mathbf{M}_2}$ is direct, then $\xi = 0$, and we have

$$\hat{a}_{\mathbf{h} \bmod \mathbf{M}_1^{\mathrm{T}}} c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\varphi) = 0 \quad \text{for all } \mathbf{h} \in \mathcal{G}(\mathbf{K}^{\mathrm{T}}),\, \mathbf{k} \in \mathbb{Z}^d$$

and hence

$$\hat{a}_{\mathbf{h}_1} c_{\mathbf{h}_1+\mathbf{M}_1^{\mathrm{T}}\mathbf{k}_1}(\varphi) = 0 \quad \text{for all } \mathbf{h}_1 \in \mathcal{G}(\mathbf{M}_1^{\mathrm{T}}),\, \mathbf{k}_1 \in \mathbb{Z}^d.$$

The full rank condition of $V_{\mathbf{M}_1}^{\varphi}$ leads to the equivalence with $\hat{a}_{\mathbf{h}_1} = 0$ for all $\mathbf{h}_1 \in \mathcal{G}(\mathbf{M}_1^{\mathrm{T}})$. The respective consideration works for $b_{\mathbf{h}_2}$, $\mathbf{h}_2 \in \mathcal{G}(\mathbf{M}_2^{\mathrm{T}})$ and the Fourier coefficients of ψ.

If otherwise, the sum $V_{\mathbf{M}_1}^{\varphi} + V_{\mathbf{M}_2}^{\psi}$ is not direct, then there is a $\xi \neq 0$ with $\xi \in V_{\mathbf{M}_1}^{\varphi} \cap V_{\mathbf{M}_2}^{\psi}$, and the equivalence (31) need not to imply vanishing $\hat{a}_{\mathbf{h}_1}$ and $\hat{b}_{\mathbf{h}_2}$. $\square$

Theorem 5.2. *Let* $V_{\mathbf{M}_1}^{\varphi}$ *and* $V_{\mathbf{M}_2}^{\psi}$ *be shift invariant spaces with* $\dim V_{\mathbf{M}_1}^{\varphi} = |\det \mathbf{M}_1|$ *and* $\dim V_{\mathbf{M}_2}^{\psi} = |\det \mathbf{M}_2|$. *The matrix* $\mathbf{K}$ *is a least common multiple of* $\mathbf{M}_1$ *and* $\mathbf{M}_2$. *Then* $V_{\mathbf{M}_1}^{\varphi}$ *is orthogonal to* $V_{\mathbf{M}_2}^{\psi}$ *if and only if*

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\varphi)\overline{c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\psi)} = 0 \quad \textit{for all} \quad \mathbf{h} \in \mathcal{G}(\mathbf{K}^{\mathrm{T}}).$$

Proof. The orthogonality of the spaces is equivalent to the orthogonality of the translates of φ to the translates of ψ. Let be $\mathbf{y}_1 \in \mathcal{P}_{\mathbf{I}}(\mathbf{M}_1)$ and $\mathbf{y}_2 \in \mathcal{P}_{\mathbf{I}}(\mathbf{M}_2)$. Parseval's equation (2) and (3) imply

$$\langle T(\mathbf{y}_1)\varphi, T(\mathbf{y}_2)\psi\rangle = \sum_{\mathbf{k} \in \mathbb{Z}^d} \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}^{\mathrm{T}}(\mathbf{y}_1 - \mathbf{y}_2)} c_{\mathbf{k}}(\varphi)\overline{c_{\mathbf{k}}(\psi)}.$$

Since $\mathbf{K} = \mathbf{J}_1\mathbf{M}_1 = \mathbf{J}_2\mathbf{M}_2$ with integer matrices $\mathbf{J}_1$ and $\mathbf{J}_2$, we get

$$\langle T(\mathbf{y}_1)\varphi, T(\mathbf{y}_2)\psi\rangle = \sum_{\mathbf{h} \in \mathcal{G}(\mathbf{K}^{\mathrm{T}})} \mathrm{e}^{-2\pi \mathrm{i} \mathbf{h}^{\mathrm{T}}(\mathbf{y}_1 - \mathbf{y}_2)} \sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\varphi)\overline{c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\psi)}.$$

Due to Lemma 2.10, the terms $\mathbf{y}_1 - \mathbf{y}_2 \bmod \mathbf{I}$ cover the whole pattern $\mathcal{P}_{\mathbf{I}}(\mathbf{K})$, and the orthogonality condition reads

$$\mathbf{0} = \sqrt{m}\,\mathcal{F}(\mathbf{K})^{\mathrm{T}} \left(\sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\varphi)\overline{c_{\mathbf{h}+\mathbf{K}^{\mathrm{T}}\mathbf{k}}(\psi)} \right)_{\mathbf{h} \in \mathcal{G}(\mathbf{K}^{\mathrm{T}})}. \tag{32}$$

Inverse Fourier transform yields the assertion. $\square$

We remark that (32) analogously provides a biorthogonality result similar to the one in [21].

The next theorem gives a criterion to check whether the linear span of given functions is an $\mathbf{M}$-shift invariant space or not. This question occurs when a space is decomposed into more general subspaces over arbitrary patterns.

Theorem 5.3. *Let $\varphi_1, \dots, \varphi_m \in L^2(\mathbb{T}^d)$ be linearly independent and the matrix $\mathbf{M} \in \mathbb{Z}^{d\times d}$ with $|\det \mathbf{M}| = m$. There exists a function $\xi \in L^2(\mathbb{T}^d)$ with $V_{\mathbf{M}}^{\xi} = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}$ if and only if there is a matrix $\mathbf{A} \in \mathbb{C}^{m\times m}$ so that*

$$\mathbf{A}\left(c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi_j)\right)_{j,\mathbf{h}} \overline{\mathcal{F}(\mathbf{M})} \tag{33}$$

is a diagonal matrix that depends on $\mathbf{k} \in \mathbb{Z}^d$, and every diagonal entry is non-vanishing for at least one $\mathbf{k}$.

Proof. Since $V_{\mathbf{M}}^{\xi}$ has full rank, ξ fulfills condition (22). Let us assume at first that $V_{\mathbf{M}}^{\xi} = \operatorname{span}\{\varphi_1, \dots, \varphi_m\}$. Then, it is clear that

$$T(\mathbf{y})\xi = \sum_{j=1}^{m} a_j^{\mathbf{y}} \varphi_j$$

for every $\mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})$ with coefficients $a_j^{\mathbf{y}}$. Hence, we get

$$\mathrm{e}^{-2\pi\mathrm{i}\mathbf{h}^{\mathrm{T}}\mathbf{y}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi) = \sum_{j=1}^{m} a_j^{\mathbf{y}} c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi_j) \quad \text{for all } \ \mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}),\, \mathbf{k} \in \mathbb{Z}^d,$$

which can be written as

$$\sqrt{m}\,\operatorname{diag}\,\left(c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\xi)\right)_{\mathbf{h}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \mathcal{F}(\mathbf{M})^{\mathrm{T}} = \left(a_j^{\mathbf{y}}\right)_{\mathbf{y},j} \left(c_{h+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi_j)\right)_{j,\mathbf{h}}$$

for all $\mathbf{k} \in \mathbb{Z}^d$. Thus, the Fourier coefficients are on the diagonal of the matrix in (33), and from (22) follows, that every diagonal entry has to be non-vanishing at least once. The other direction of the proof works analogously. □

Condition (33) is a rather strong one if more than one matrix has to be checked, i. e., if more than one integer vector $\mathbf{k} \in \mathbb{Z}^d$ yields non-vanishing entries.

Theorem 5.3 can be easily generalized for more than m linearly independent functions φ_j and the question, if a shift invariant space is contained in its span. Then the occurring matrices $\mathbf{A}$ and $\left(c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(\varphi_j)\right)_{j,\mathbf{h}}$ are changed into rectangular matrices, and condition (33) remains.

Since Lemma 2.6 restricts the number of possible patterns, simple checking of all possible patterns can decide whether the span $\operatorname{span}\{\varphi_1, \dots, \varphi_m\}$ is a shift invariant space for some matrix $\mathbf{M}$ or not.

6. Example of a wavelet decomposition

6.1. Kernel functions

The elements of $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ play the role of discrete frequencies of functions defined on $\mathcal{G}(\mathbf{M})$, cf. (10). Such functions are vectors $\mathbf{a} = (a_{\mathbf{g}})_{\mathbf{g}\in\mathcal{G}(\mathbf{M})}$. Any set of representatives of $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and any choice of coefficients $\alpha_{\mathbf{k}} \neq 0$, $\mathbf{k} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}})$ generate a polynomial kernel function

$$D_{\mathbf{M}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{G}(\mathbf{M}^{\mathrm{T}})} \alpha_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}}. \tag{34}$$

In general, this kernel function is not real. We expand the set of frequencies with the aim to construct real kernels.

The shift of the cube $[0,1)^d$ by $-\frac{1}{2}(1,\ldots,1)^{\mathrm{T}} \in \mathbb{R}^d$ gives the set of representatives

$$\mathcal{G}(\mathbf{M}^{\mathrm{T}}) = \mathbb{Z}^d \cap \mathbf{M}^{\mathrm{T}}Q = \mathbf{M}^{\mathrm{T}}(\mathcal{P}(\mathbf{M}^{\mathrm{T}}) \cap Q) \;\text{ with }\; Q = \left[-\frac{1}{2}, \frac{1}{2}\right)^d \subset \mathbb{R}^d.$$

This set is expanded to the symmetric frequency domain

$$\mathcal{K}(\mathbf{M}^{\mathrm{T}}) = \mathbb{Z}^d \cap \mathbf{M}^{\mathrm{T}}\bar{Q} = \mathbf{M}^{\mathrm{T}}(\mathcal{P}(\mathbf{M}^{\mathrm{T}}) \cap \bar{Q}) \;\text{ with }\; \bar{Q} = \left[-\frac{1}{2}, \frac{1}{2}\right]^d \subset \mathbb{R}^d. \tag{35}$$

It is obvious that $\mathcal{G}(\mathbf{M}^{\mathrm{T}}) \subseteq \mathcal{K}(\mathbf{M}^{\mathrm{T}})$. We define the real kernel function

$$D^{\mathrm{re}}_{\mathbf{M}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}(\mathbf{M}^{\mathrm{T}})} \alpha_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}} \;\text{ with }\; \alpha_{\mathbf{k}} = \alpha_{-\mathbf{k}} \neq 0.$$

Due to the symmetry of $\mathcal{K}(\mathbf{M}^{\mathrm{T}})$, the frequency domain can be decomposed into the disjoint sets $\mathcal{K}(\mathbf{M}^{\mathrm{T}}) = \mathcal{K}^+(\mathbf{M}^{\mathrm{T}}) \cup \mathcal{K}^-(\mathbf{M}^{\mathrm{T}}) \cup \{\mathbf{0}\}$ so that from $\mathbf{k} \in \mathcal{K}^+(\mathbf{M}^{\mathrm{T}})$, it follows $-\mathbf{k} \in \mathcal{K}^-(\mathbf{M}^{\mathrm{T}})$ and vice versa. Hence, the real kernel function can be written as

$$D^{\mathrm{re}}_{\mathbf{M}}(\mathbf{x}) = \alpha_{\mathbf{0}} + 2\sum_{\mathbf{k}\in\mathcal{K}^+(\mathbf{M}^{\mathrm{T}})} \alpha_{\mathbf{k}} \cos(\mathbf{k}^{\mathrm{T}}\mathbf{x}) \in \mathbb{R}.$$

In the case $\alpha_{\mathbf{k}} = 1$ for all $\mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$, $D^{\mathrm{re}}_{\mathbf{M}}$ is called a Dirichlet kernel. We formulate two lemmas about real kernels. The proofs indicate that these lemmas are valid for the general kernel functions in (34), too.

Lemma 6.1. *The translates* $T(\mathbf{y})D^{\mathrm{re}}_{\mathbf{M}}$*,* $\mathbf{y} \in \mathcal{P}_{\mathbf{I}}(\mathbf{M})$ *of the real kernel* $D^{\mathrm{re}}_{\mathbf{M}}$ *span an* m*-dimensional space* $V_{\mathbf{M}}^{D^{\mathrm{re}}_{\mathbf{M}}}$*.*

Proof. The Fourier coefficients are

$$c_{\mathbf{k}}(D^{\mathrm{re}}_{\mathbf{M}}) = \alpha_{\mathbf{k}} \neq 0 \quad \text{for} \quad \mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}}).$$

Due to $\mathcal{G}(\mathbf{M}^{\mathrm{T}}) \subseteq \mathcal{K}(\mathbf{M}^{\mathrm{T}})$, Corollary 3.5 gives the proposition. □

Lemma 6.2. *If* $\mathbf{k} \in \mathbf{M}^{\mathrm{T}}\left(\mathcal{P}(\mathbf{M}^{\mathrm{T}}) \cap (Q \backslash \partial Q)\right)$ *then*

$$\mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ} \in V_{\mathbf{M}}^{D^{\mathrm{re}}_{\mathbf{M}}}.$$

Proof. The precondition $\mathbf{k} \in \mathbf{M}^{\mathrm{T}}(Q \backslash \partial Q)$ yields $\mathbf{k} + \mathbf{M}^{\mathrm{T}}\mathbf{z} \notin \mathcal{K}(\mathbf{M}^{\mathrm{T}})$ for all $\mathbf{z} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ because $\mathbf{M}^{-\mathrm{T}}(\mathbf{k}+\mathbf{M}^{\mathrm{T}}\mathbf{z}) = \mathbf{M}^{-\mathrm{T}}\mathbf{k}+\mathbf{z} \notin \bar{Q}$. Thus, Theorem 3.3 with

$$\hat{a}_{\mathbf{h}} = \begin{cases} 1 & \text{for} \quad \mathbf{h} = \mathbf{k}, \\ 0 & \text{else} \end{cases}$$

assures (20) with $\xi = \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\circ}$ and $\varphi = D^{\mathrm{re}}_{\mathbf{M}}$. □

We construct a kernel with orthonormal translates. Already Lemma 6.2 shows that the boundary $\partial\Omega$ and its corresponding rim of the frequency domain play a crucial role in the investigation of kernel functions. We define the number $r(\mathbf{k})$ of surficial planes of the parallelepiped $\mathbf{M}^{\mathrm{T}}\bar{Q}$ the point $\mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$ is lying on. This is the number of components of $\mathbf{y} = \mathbf{M}^{-\mathrm{T}}\mathbf{k}$, the modulus of which equals $\frac{1}{2}$, i. e.,

$$r(\mathbf{k}) = \left| \left\{ j \;:\; |\mathbf{k}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}_j| = \frac{1}{2} \right\} \right|. \tag{36}$$

In particular, inner points $\mathbf{k}$ of the parallelepiped fulfill $r(\mathbf{k}) = 0$. In the case $d = 3$, the property $r(\mathbf{k}) = 1$ denotes points inside one surficial plane, $r(\mathbf{k}) = 2$ means points on an edge and $r(\mathbf{k}) = d$ describes corner points. For completeness, we define $r(\mathbf{k}) = -1$ if $\mathbf{k} \notin \mathcal{K}(\mathbf{M}^{\mathrm{T}})$.

If the property $r(\mathbf{k})$ is needed for different matrices $\mathbf{M}$, we write $r(\mathbf{k}) = r_{\mathbf{M}}(\mathbf{k})$.

Lemma 6.3. *With a point* $\mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$*, there are* $2^{r(\mathbf{k})}$ *points of the form* $\mathbf{k} + \mathbf{M}^{\mathrm{T}}\mathbf{z} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$*,* $\mathbf{z} \in \mathbb{Z}^d$ *and* $r(\mathbf{k} + \mathbf{M}^{\mathrm{T}}\mathbf{z}) = r(\mathbf{k})$*.*

Proof. From $\mathbf{k}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}_j = \pm\frac{1}{2}$, it follows

$$(\mathbf{k} \mp \mathbf{M}^{\mathrm{T}}\mathbf{e}_j)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}_j = \mathbf{k}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}_j \mp 1 = \mp\frac{1}{2}.$$

All other components $(\mathbf{k} \mp \mathbf{M}^{\mathrm{T}}\mathbf{e}_j)^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}_\ell = \mathbf{k}^{\mathrm{T}}\mathbf{M}^{-1}\mathbf{e}_l$ with $\ell \neq j$ are unchanged. This argumentation is valid for every possible $\mathbf{e}_j$ and all linear combinations of them. □

The definition (36) yields the symmetry $r(\mathbf{k}) = r(-\mathbf{k})$. Hence the choice

$$\alpha_k = \frac{1}{\sqrt{m}}2^{-\frac{r(\mathbf{k})}{2}} = \frac{1}{\sqrt{m}} \cdot \frac{1}{\sqrt{2}^{r(\mathbf{k})}}$$

generates a real kernel function. In analogy to the univariate setting, we call it orthonormalized Dirichlet kernel. We prove that its translates are orthonormal to each other.

Theorem 6.4. *The translates of the orthonormalized Dirichlet kernel*

$$D_{\mathbf{M}}^{\perp}(\mathbf{x}) = \frac{1}{\sqrt{m}} \sum_{k \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})} 2^{-\frac{r(\mathbf{k})}{2}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}\mathbf{x}} \tag{37}$$

with $m = |\det \mathbf{M}|$ *are orthonormal to each other.*

Proof. The Fourier coefficients are

$$c_{\mathbf{k}}(D_{\mathbf{M}}^{\perp}) = \frac{1}{\sqrt{m}}2^{-\frac{r(\mathbf{k})}{2}} \quad \text{for} \quad \mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}}) \tag{38}$$

and vanishing elsewhere. Lemma 6.3 yields

$$\sum_{\mathbf{h} \in \mathbb{Z}^d} |c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(D_{\mathbf{M}}^{\perp})|^2 = 2^{r(\mathbf{k})}c_{\mathbf{h}}(D_{\mathbf{M}}^{\perp})^2 = \frac{1}{m} \tag{39}$$

for $\mathbf{h} \in \mathcal{G}(\mathbf{M}^{\mathrm{T}}) \subseteq \mathcal{K}(\mathbf{M}^{\mathrm{T}})$ and thus, Corollary 3.6 leads to the assertion. □

6.2. Orthogonal wavelet decomposition

We concentrate on the case $d = 2$ in this subsection. Nevertheless, we show the connection to higher dimensional cases. Here, signature matrices are diagonal matrices, the entries of which are 1 or -1.

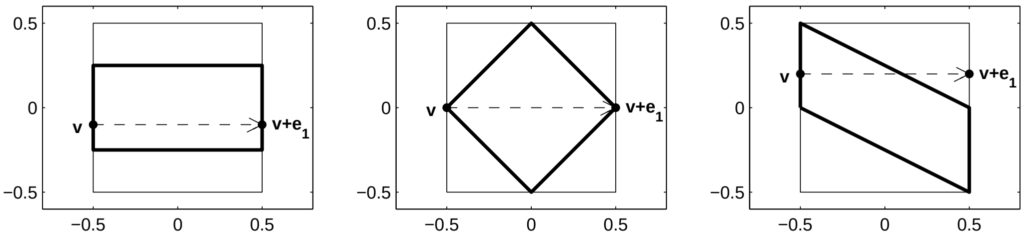

Figure 3: Square Q by solid lines and $\mathbf{J}^{-\mathrm{T}}Q$ by bold lines, exemplary $\mathbf{v}, \mathbf{v}+\mathbf{e}_1$ from Lemma 6.5 for $\mathbf{J}=\mathbf{J}_2$ (right), for $\mathbf{J}=\mathbf{J}_3$ (middle) and for $\mathbf{J}=\mathbf{J}_4$ (right).

Lemma 6.5. *If $d=2$ and $\mathbf{M}=\mathbf{J}\mathbf{N}$ where the matrix $\mathbf{J}$ is a product of signature matrices and a matrix*

$$\tilde{\mathbf{J}} \in \left\{ \mathbf{J}_1 = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \ \mathbf{J}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \ \mathbf{J}_3 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \right\}, \tag{40}$$

then the inclusion $D_{\mathbf{N}}^{\perp} \in V_{\mathbf{M}}^{D_{\mathbf{M}}^{\perp}}$ is valid.

Proof. A short calculation gives $\bar{Q} \subset \mathbf{J}^{\mathrm{T}}\bar{Q}$ and hence $\mathbf{N}^{\mathrm{T}}\bar{Q} \subset \mathbf{M}^{\mathrm{T}}\bar{Q}$ and $\mathcal{K}(\mathbf{N}^{\mathrm{T}}) \subset \mathcal{K}(\mathbf{M}^{\mathrm{T}})$ by (35). The Fourier coefficients $c_{\mathbf{k}}(D_{\mathbf{N}}^{\perp})$ and $c_{\mathbf{k}}(D_{\mathbf{M}}^{\perp})$ are found in (38), respectively. Lemma 6.3 gives $c_{\mathbf{h}}(D_{\mathbf{M}}^{\perp}) = c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(D_{\mathbf{M}}^{\perp})$ whenever $\mathbf{h}, \mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$. That is possible only if $r_{\mathbf{M}}(\mathbf{h}) > 0$, i. e., if $\mathbf{v} = \mathbf{M}^{-\mathrm{T}}\mathbf{h}$ has at least one component with the modulus $\frac{1}{2}$. We show that then $c_{\mathbf{h}}(D_{\mathbf{N}}^{\perp}) = c_{\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}}(D_{\mathbf{N}}^{\perp})$ is satisfied by proving the equality $r_{\mathbf{N}}(\mathbf{h}) = r_{\mathbf{N}}(\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k})$. We distinguish the different matrices $\mathbf{J}$.

First, we consider the case $\mathbf{J}=\mathbf{J}_2$. We start with an $\mathbf{v}$, the first component of which has the modulus $\frac{1}{2}$. Then $\mathbf{v}, \mathbf{v} \pm \mathbf{e}_1 \in \bar{Q}$ and $\mathbf{h}, \mathbf{h} \pm \mathbf{M}^{\mathrm{T}}\mathbf{e}_1 \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$. We calculate

$$\mathbf{h}^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{e}_j = \mathbf{v}^{\mathrm{T}}\mathbf{M}\mathbf{N}^{-1}\mathbf{e}_j = \mathbf{v}^{\mathrm{T}}\mathbf{J}\mathbf{e}_j \ \text{ and } \ (\mathbf{h}\pm\mathbf{M}^{\mathrm{T}}\mathbf{e}_1)^{\mathrm{T}}\mathbf{N}^{-1}\mathbf{e}_j = \mathbf{v}^{\mathrm{T}}\mathbf{J}\mathbf{e}_j \pm \mathbf{e}_1^{\mathrm{T}}\mathbf{J}\mathbf{e}_j.$$

Now, $j=1$ yields $\mathbf{e}_1^{\mathrm{T}}\mathbf{J}\mathbf{e}_1 = 1$, and $j=2$ leads to $\mathbf{e}_1^{\mathrm{T}}\mathbf{J}\mathbf{e}_2 = 0$. Thus, $\mathbf{h}, \mathbf{h} \pm \mathbf{M}^{\mathrm{T}}\mathbf{e}_1 \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$ yields $r_{\mathbf{N}}(\mathbf{h}) = r_{\mathbf{N}}(\mathbf{h} \pm \mathbf{M}^{\mathrm{T}}\mathbf{e}_1)$. If the second component of $\mathbf{v}$ has the modulus $\frac{1}{2}$, then $\mathbf{M}^{\mathrm{T}}\mathbf{v} \notin \mathcal{K}(\mathbf{N}^{\mathrm{T}})$, and $r_{\mathbf{N}}(\mathbf{h}+\mathbf{M}^{\mathrm{T}}\mathbf{k}) = -1$ for all $\mathbf{k} \in \mathbb{Z}^d$, cf. Fig. 3, left plot.

The case $\mathbf{J}=\mathbf{J}_1$ works in complete analogy with changed roles of $\mathbf{e}_1$ and $\mathbf{e}_2$.

In the case $\mathbf{J}=\mathbf{J}_3$, the only interesting points $\mathbf{v}$ are the corners of the rotated small square $\mathbf{J}^{-\mathrm{T}}\bar{Q}$, cf. Fig. 3, central plot. Then, one of the components of $\mathbf{v}$ has the modulus $\frac{1}{2}$ and the other one is vanishing. An analogous calculation with $\mathbf{e}_k^{\mathrm{T}}\mathbf{J}\mathbf{e}_j = \pm 1$ shows the assertion.

Hence

$$\hat{a}_{\mathbf{h}} = \begin{cases} \sqrt{\frac{m}{n}} 2^{\frac{1}{2}(r_{\mathbf{M}}(\mathbf{h}) - r_{\mathbf{N}}(\mathbf{h}))} & \text{if } \mathbf{h} \in \mathcal{K}(\mathbf{N}^{\mathrm{T}}) \cap \mathcal{G}(\mathbf{M}^{\mathrm{T}}), \\ 0 & \text{else} \end{cases} \tag{41}$$

defines the discrete Fourier transform of $\mathbf{a}$ in Theorem 3.3, which proves the assertion. $\square$

The matrix $\mathbf{J}_3$ is oftentimes called butterfly matrix. There are further matrices $\mathbf{J} \in \mathbb{Z}^{d\times d}$ with $|\det \mathbf{J}| = 2$ and the property $\bar{Q} \subset \mathbf{J}^{\mathrm{T}}\bar{Q}$. These are

$$\mathbf{J}_4 = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}, \quad \mathbf{J}_5 = \begin{pmatrix} 2 & 0 \\ 1 & 1 \end{pmatrix}$$

and their multiples with signature matrices. For instance the first of these matrices yields

$$\mathbf{J}_4^{-\mathrm{T}}\bar{Q} \cap \partial Q = \left\{-\frac{1}{2}\right\} \times \left[0, \frac{1}{2}\right] \cup \left\{\frac{1}{2}\right\} \times \left[-\frac{1}{2}, 0\right]$$

and $\mathbf{y} \in \mathbf{J}_4^{-\mathrm{T}}\bar{Q} \cap \partial Q$ implies $\mathbf{y} + \mathbf{e}_1 \notin \mathbf{J}_4^{-\mathrm{T}}\bar{Q} \cap \partial Q$ whenever the second component of $\mathbf{y}$ is non-vanishing, cf. Fig. 3, right plot.

If we omit the restriction to real kernel functions, then $\mathbf{J}_4$ and $\mathbf{J}_5$ and their multiples with signature matrices describe possible extensions of the frequency domain, which are now sets of representatives $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$.

The number of possible matrices does not essentially increase in higher dimensions. The matrices $\mathbf{J} \in \mathbb{Z}^{d\times d}_{\mathrm{reg}}$ with $|\det \mathbf{J}| = 2$ and the inclusion property in Lemma 6.5 contain a sub-matrix in the set (40).

Let $\mathbf{y}$ and $\mathbf{g}$ be the elements $\mathbf{y} = \mathcal{P}_{\mathbf{I}}(\mathbf{J})\backslash\{\mathbf{0}\}$ and $\mathbf{g} \in \mathcal{G}(\mathbf{J}^{\mathrm{T}})\backslash\{\mathbf{0}\}$. They are unique because of $|\det \mathbf{J}| = 2$. We define the wavelet ψ by

$$\psi(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{K}(\mathbf{M}^{\mathrm{T}})} \beta_{\mathbf{k}} \mathrm{e}^{\mathrm{i}\mathbf{k}^{\mathrm{T}}(\mathbf{x} - 2\pi\mathbf{N}^{-1}\mathbf{y})} \text{ with } \beta_{\mathbf{k}} = c_{\mathbf{k}}(D_{\mathbf{M}}^{\perp})\hat{a}_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g} \bmod \mathbf{M}^{\mathrm{T}}}, \tag{42}$$

where the entries $\hat{a}_{\mathbf{h}} \in \mathbb{R}$ are given in (41) and the Fourier coefficients $c_{\mathbf{k}}(D_{\mathbf{M}}^{\perp})$ of the orthonormalized Dirichlet kernel are known by (38).

Theorem 6.6. *Let* $\mathbf{M} = \mathbf{JN}$ *be given with an admissible matrix* $\mathbf{J}$ *from Lemma 6.5. The orthonormalized Dirichlet kernels* $D_{\mathbf{M}}^{\perp}$ *respectively* $D_{\mathbf{N}}^{\perp}$ *are defined in (37), and the wavelet* ψ *is given in (42). Then, we have the orthogonal decomposition*

$$V_{\mathbf{N}}^{D_{\mathbf{N}}^{\perp}} \oplus V_{\mathbf{N}}^{\psi} = V_{\mathbf{M}}^{D_{\mathbf{M}}^{\perp}}.$$

Proof. We show the existence of numbers $\sigma_{\mathbf{k}}$ for $\mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$, which fulfill the requirements of Theorem 4.3. The specific notations are $\zeta = D_{\mathbf{M}}^{\perp}$, $\xi = D_{\mathbf{N}}^{\perp}$ and $\eta = \psi$. The Fourier coefficients of the wavelet ψ are

$$c_{\mathbf{k}}(\psi) = \beta_{\mathbf{k}} \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}^{\mathrm{T}} \mathbf{N}^{-1} \mathbf{y}}.$$

The denominator in (27) is known as $1/m$ by (39). We distinguish the position of the point $\mathbf{k} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$.

First $\mathbf{k}$ may be an inner point of $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$, i. e., $r_{\mathbf{N}}(\mathbf{k}) = 0$. Then $\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g} \notin \mathcal{K}(\mathbf{N}^{\mathrm{T}})$ and thus $\hat{a}_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g} \bmod \mathbf{M}^{\mathrm{T}}} = 0$. Hence, $\beta_{\mathbf{k}} = 0$ by defining (42) and thus $c_{\mathbf{k}}(\psi) = 0$. Thus every choice of $\sigma_{\mathbf{k}}$ fulfills (27).

Second, $\mathbf{k}$ may be outside $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$. By the definition in (42), two points $\mathbf{k}, \mathbf{k} + \mathbf{M}^{\mathrm{T}}\mathbf{z} \in \mathcal{K}(\mathbf{M}^{\mathrm{T}})$ have identical $\beta_{\mathbf{k}} = \beta_{\mathbf{k}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}$. These two points lie at the rim of $\mathcal{K}(\mathbf{M}^{\mathrm{T}})$, i. e., $r_{\mathbf{M}}(\mathbf{k}) = r_{\mathbf{M}}(\mathbf{k} + \mathbf{M}^{\mathrm{T}}\mathbf{z})$, cf. the proof of Lemma 6.5, and we have $c_{\mathbf{k}}(D_{\mathbf{M}}^{\perp}) = c_{\mathbf{k}+\mathbf{M}^{\mathrm{T}}\mathbf{z}}(D_{\mathbf{M}}^{\perp})$. Since $\mathbf{k} + \mathbf{N}^{\mathrm{T}}\mathbf{g} \bmod \mathbf{M}^{\mathrm{T}}$ is a point in the inner of $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$, the number $\sigma_{\mathbf{k}}$ can be found by

$$\sigma_{\mathbf{k}} = \frac{c_{\mathbf{k}}(\psi)}{m c_{\mathbf{k}}(D_{\mathbf{M}}^{\perp}) \hat{a}_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g} \bmod \mathbf{M}^{\mathrm{T}}}}. \tag{43}$$

Equation (26) is satisfied because $\sigma_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}$ is not restricted by the first case.

Third, there are points $\mathbf{k}$ with $r_{\mathbf{N}}(\mathbf{k}) > 0$. If $\mathbf{k} + \mathbf{N}^{\mathrm{T}}\mathbf{g} \notin \mathcal{K}(\mathbf{N}^{\mathrm{T}})$, then the second case applies. If not, i. e., if $\mathbf{k}, \mathbf{k} + \mathbf{N}^{\mathrm{T}}\mathbf{g} \in \mathcal{K}(\mathbf{N}^{\mathrm{T}})$, we have to show that (43) leads to numbers $\sigma_{\mathbf{k}}$ and $\sigma_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}$ fulfilling (26). We calculate

$$c_{\mathbf{k}}(\psi) = \beta_{\mathbf{k}} \mathrm{e}^{-2\pi \mathrm{i} \mathbf{k}^{\mathrm{T}} \mathbf{N}^{-1} \mathbf{y}}$$

and

$$c_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}(\psi) = \beta_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g}} \mathrm{e}^{-2\pi \mathrm{i} (\mathbf{k}^{\mathrm{T}} + \mathbf{g}^{\mathrm{T}}\mathbf{N}) \mathbf{N}^{-1} \mathbf{y}} = c_{\mathbf{k}}(\psi) \mathrm{e}^{-2\pi \mathrm{i} \mathbf{g}^{\mathrm{T}} \mathbf{y}}.$$

Since, $\mathbf{g} = \mathbf{J}^{\mathrm{T}}\mathbf{z} + \mathbf{J}^{\mathrm{T}}\mathbf{l}$ with $\mathbf{z} \in \mathcal{P}_{\mathbf{I}}(\mathbf{J}^{\mathrm{T}}) \backslash \{\mathbf{0}\}$ and some $\mathbf{l} \in \mathbb{Z}^d$ and

$$\mathbf{z}^{\mathrm{T}} \mathbf{J} \mathbf{y} = \frac{1}{2}$$

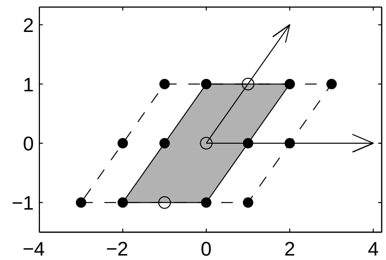

Figure 4: Frequency domains of Example 6.7 with column vectors of $\mathbf{M}^{\mathrm{T}}$. All marked points form $\mathcal{K}(\mathbf{M}^{\mathrm{T}})$. Elements of $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$ have a grey background. Full black points show the difference frequency domain $\mathcal{K}_{\mathrm{diff}}$.

for all admissible matrices $\mathbf{J}$, we find the condition $c_{\mathbf{k}}(\psi) = -c_{\mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g}}(\psi)$ if $\mathbf{k}, \mathbf{k}+\mathbf{N}^{\mathrm{T}}\mathbf{g} \in \mathcal{K}(\mathbf{N}^{\mathrm{T}})$. Since all other terms at the right-hand side of (43) stay constant, the proof is completed with this case. □

We remark, that the frequency domain of the wavelet ψ is not the whole $\mathcal{K}(\mathbf{M}^{\mathrm{T}})$ but only a difference domain $\mathcal{K}_{\mathrm{diff}}$ including the rim of $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$, i. e.,

$$\mathcal{K}_{\mathrm{diff}} = \mathcal{K}(\mathbf{M}^{\mathrm{T}}) \backslash \{\mathbf{k} : r_{\mathbf{N}}(\mathbf{k}) = \mathbf{0}\} = \mathcal{K}(\mathbf{M}^{\mathrm{T}}) \backslash \mathcal{K}(\mathbf{N}^{\mathrm{T}}) \cup \{\mathbf{k} : r_{\mathbf{N}}(\mathbf{k}) > 0\}.$$

The different domains are shown in Fig. 4 for the following example.

Example 6.7. *For $d = 2$ and with $\mathbf{J} = \mathbf{J}_1$, we regard the matrices*

$$\mathbf{M}^{\mathrm{T}} = \begin{pmatrix} 4 & 2 \\ 0 & 2 \end{pmatrix} \quad \text{and} \quad \mathbf{N}^{\mathrm{T}} = \begin{pmatrix} 2 & 2 \\ 0 & 2 \end{pmatrix}.$$

The edges of the parallelogram containing $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$ are $(2,0)^{\mathrm{T}}$ and $(2,2)^{\mathrm{T}}$, and the frequency domain is

$$\mathcal{K}(\mathbf{N}^{\mathrm{T}}) = \left\{ \begin{pmatrix} -2 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\},$$

The respective values of $r_{\mathbf{N}}(\mathbf{k})$ are $2, 1, 2, 1, 0, 1, 2, 1, 2$ in the order of the points $\mathbf{k} \in \mathcal{K}(\mathbf{N}^{\mathrm{T}})$. The difference frequency domain $\mathcal{K}_{\mathrm{diff}}$ is given in Fig. 4 by full black points.

Fig. 5 presents the orthogonalized Dirichlet kernel $D_{\mathbf{N}}^{\perp}$ for the matrix $\mathbf{N}$ from Example 6.7 and the respective wavelet function ψ. It is clearly remarkable that the Dirichlet kernel is not symmetric. i. e. in a certain sense, it prefers one diagonal direction.

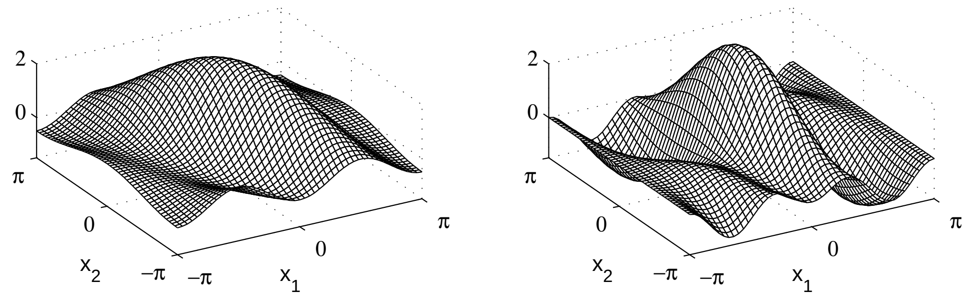

Figure 5: Left: Orthonormalized Dirichlet kernel $D_{\mathbf{N}}^{\perp}(\mathbf{x})$ for $d = 2$ and the matrix $\mathbf{N}$ from Example 6.7. Right: Respective wavelet function ψ.

6.3. Decomposition of the frequency space

Although the number of possible extensions of the frequency domain is rather restricted by Lemma 6.5, the variety of the possible frequency domains suffices to construct problem adapted decompositions of the frequency space. We consider a matrix

$$\mathbf{M}_j = \mathbf{J}^{(j)}\mathbf{J}^{(j-1)} \cdot \ldots \cdot \mathbf{J}^{(0)}\mathbf{N}$$

with $|\det \mathbf{J}^{(k)}| = 2$ for $k = 0, \ldots, j$ and the wavelet decomposition

$$V_{\mathbf{M}_j}^{D_{\mathbf{M}_j}^{\perp}} = V_{\mathbf{N}}^{D_{\mathbf{N}}^{\perp}} \oplus V_{\mathbf{N}}^{\psi_0} \oplus V_{\mathbf{J}^{(0)}\mathbf{N}}^{\psi_1} \oplus \ldots \oplus V_{\mathbf{J}^{(j-1)}\ldots\mathbf{J}^{(0)}\mathbf{N}}^{\psi_j}$$

with the wavelets ψ_k in the respective difference frequency domain. The space dimensions are

$$\dim V_{\mathbf{N}}^{D_{\mathbf{N}}^{\perp}} = n,\ \dim V_{\mathbf{J}^{(k-1)}\ldots\mathbf{J}^{(0)}\mathbf{N}}^{\psi_k} = 2^k n,\ \dim V_{\mathbf{M}_j}^{D_{\mathbf{M}_j}^{\perp}} = 2^{j+1}n.$$

Every matrix $\mathbf{J}^{(k)}$ increases the frequency domain by one of the geometrical extensions discussed in Lemma 6.5, cf. Fig. 6. In the case $d = 2$, there are three possible extension matrices $\mathbf{J}$ for orthogonalized Dirichlet kernels and seven possible extensions for general non-necessarily real kernels with frequencies in $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$. Of course, the matrix $\mathbf{N}$ determines the frequency domain, too. See Fig. 7 for an example of a non-standard decomposition of the frequency space.

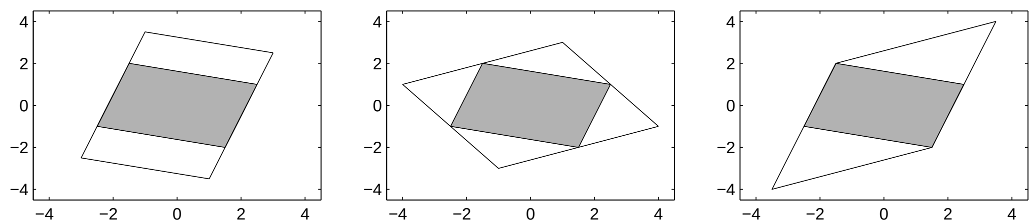

Figure 6: Left, middle: Possible extensions of the frequency domain $\mathcal{K}(\mathbf{N}^{\mathrm{T}})$ by different matrices $\mathbf{J} = \mathbf{J}_1$ and by the matrix $\mathbf{J} = \mathbf{J}_3$ (center). Right: The extension by $\mathbf{J} = \mathbf{J}_4$ is not possible for real kernels $D_{\mathbf{M}}^{\mathrm{re}}$, more general polynomial kernels (34) would allow this extension, too.

Theorem 6.8. *A sequence of matrix extensions $\mathbf{J}^{(\ell)} \in \{\mathbf{J}_1, \mathbf{J}_2, \mathbf{J}_3\}$, $\ell \in \mathbb{N}$ and a matrix $\mathbf{N} \in \mathbb{Z}_{\mathrm{reg}}^{d\times d}$ may be given so that for every $\mathbf{k} \in \mathbb{Z}^d$, there exists a $j \in \mathbb{N}$ with $\mathbf{k} \in \mathcal{K}((\mathbf{J}^{(j)}\mathbf{J}^{(j-1)} \cdot \ldots \cdot \mathbf{J}^{(0)}\mathbf{N})^{\mathrm{T}})$. Then the closure of the unification of all wavelet spaces and $V_{\mathbf{N}}^{D_{\mathbf{N}}^{\perp}}$ is the whole $L^2(\mathbb{T}^d)$, i. e.,*

$$\mathrm{clos}_{L^2(\mathbb{T}^d)} \left(V_{\mathbf{N}}^{D_{\mathbf{N}}^{\perp}} \oplus \bigoplus_{\ell\in\mathbb{N}} V_{\mathbf{J}^{(\ell)}\ldots\mathbf{J}^{(0)}\mathbf{N}}^{\psi_\ell} \right) = L^2(\mathbb{T}^d). \tag{44}$$

Proof. The proof follows immediately from Lemma 6.2 because the precondition assures that $\mathcal{K}(\mathbf{M}_j^{\mathrm{T}})$ tends to $\mathbb{Z}^d$ with increasing j. Therefore, every point $\mathbf{k} \in \mathbb{Z}^d$ lies in the inner $\mathcal{K}(\mathbf{M}_\ell^{\mathrm{T}})$ for all $\ell \geq j'$ with some $j' \in \mathbb{N}$. Hence, all frequencies occur in the unification, and the proposition (44) is proven [20]. □

The mentioned variety leads to the occurrence of anisotrop smoothness properties of the approximated functions in the wavelet analysis. Further investigations will be done in a forthcoming paper.

Furthermore, other frequency domains can be constructed by choosing other arbitrary sets of representatives $\mathcal{G}(\mathbf{M}^{\mathrm{T}})$ and by defining the frequency domain

$$\mathcal{K} = -\mathcal{G}(\mathbf{M}^{\mathrm{T}}) \cup \mathcal{G}(\mathbf{M}^{\mathrm{T}}).$$

The respective kernel over $\mathcal{K}$ is real, too. It need not contain any small frequencies and can be constructed specifically to the application problem, e. g. as a multidimensional sparse wavelet decomposition [11, 23].

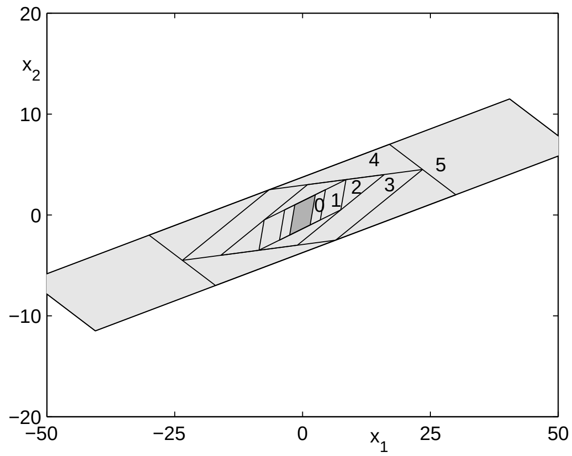

Figure 7: Decomposition of the frequency space by a sequence of frequency domains. Dark grey: smallest frequency domain, light grey: frequency domain of $V_{\mathbf{M}_5}^{D^{\perp}_{\mathbf{M}_5}}$. Numbers symbolize the difference frequency domains of the wavelet spaces.

7. Conclusion

The paper combines the investigation of integer matrices generating multivariate periodic patterns. These patterns are not necessarily tensor products of one-dimensional grids in the co-ordinate directions. They are isomorphic to Abelian groups which are classified by the theorem of elementary divisors. Furthermore, representative matrices among the class of all integer matrices are given, which determine identical patterns.

The generalization of one-dimensional results about periodic shift invariant spaces allows us to present a decomposition theorem of multivariate shift invariant spaces. Finally, the Dirichlet kernel and the respective wavelet are constructed in the multivariate case.

These investigations are the base for a broader investigation of general multivariate wavelets, of their approximation properties, of stable and fast numerical algorithms and of their ability in image processing.

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