10. Trigonometric Wavelets

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1. Introduction

Trigonometric polynomials and the approximation of periodic functions by polynomials play an important role in harmonic analysis. Here we are interested in constructing time-localized bases for certain spaces of trigonometric polynomials. Adapting basic ideas of wavelet theory, translation invariant and orthonormal bases are employed, both of which are constructed from equidistant shifts of a single polynomial. One of our main goals is the investigation of basis transformations in a form that facilitates fast algorithms. We observe that the corresponding transformation matrices have a circulant structure and can be diagonalized by Fourier matrices. The resulting diagonal matrices contain the eigenvalues, which are computed explicitly. So, the algorithms can be easily realized using the Fast Fourier Transform (FFT).

Further, we consider the nesting of the sample spaces to obtain multiresolution analyses (MRA’s). For the resulting orthogonal wavelet spaces, we proceed as above and find wavelet bases consisting of translates of a single polynomial. Again, translation invariant and orthonormal bases are constructed, which show the same time-frequency behaviour as the sample bases do. The basis transformations can be described analogously by circulant matrices.

Most important for practical reasons are the decomposition of signals in frequency bands, which correspond to the wavelet spaces, and their reconstruction. Focusing on basis transformations, the two-scale relations and decomposition formulas are also given in matrix notation suitable for the use of FFT methods. Then the transformations for signal data follow easily.

Let us start with the explanation of time localization and frequency localization for trigonometric polynomials. If we consider a polynomial $p_n \in \Pi_n$ then

$$p_n(x) = \sum_{k=-n}^{n} c_k(f)e^{ikx}$$
is a basis representation in terms of the naturally best frequency localized basis \( \{ e^{ikx} \} \). The coefficients \( c_k(f) \) turn out to be the Fourier coefficients

\[
c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)e^{-ikt} \, dt.
\]

On the other hand, if we want to write the polynomial \( p_n \) in a basis such that the coefficients give the most time localized information we end up with a representation of \( p_n \) as a Lagrange interpolation polynomial

\[
p_n(x) = \frac{1}{n} \sum_{k=-n+1}^{n} p_n \left( \frac{k\pi}{n} \right) D_n^* \left( x - \frac{k\pi}{n} \right).
\]

Here \( D_n^* \) is the modified Dirichlet kernel

\[
D_n^*(x) = \frac{1}{n} \sum_{k=-n+1}^{n-1} e^{ikx} + \frac{e^{inx} + e^{-inx}}{4} = \begin{cases} 
\sin nx, & \sin \frac{nx}{2}, \text{ if } x \not\in 2\pi\mathbb{Z}, \\
\frac{\sin \frac{nx}{2}}{n}, & \text{if } x \in 2\pi\mathbb{Z}.
\end{cases}
\]

Obviously, the coefficients \( p_n \left( \frac{kn}{n} \right) \) yield the most time localized information of \( p_n \).

These two bases namely

\[
\{ e^{ikx} \}_{k=-n}^{n-1} \quad \text{and} \quad \left\{ D_n^* \left( x - \frac{k\pi}{n} \right) \right\}_{k=-n+1}^{n},
\]

could be seen as extreme cases of time- and frequency localized bases in corresponding spaces. As one can assume there is no basis which is optimal in both senses. However, one could try to find a compromise although the different uncertainty principles behind are not the topic of this article (cf. [1], [2], [8], [12], [13]). Our aim is to construct a couple of bases consisting of equidistant translates of one underlying function with different properties.

The basic idea is the orthogonal decomposition

\[
L_{2\pi}^2 = W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots
\]

Hence, every \( f \in L_{2\pi}^2 \) can be uniquely decomposed into the sum

\[
f = \sum_{j=-1}^{\infty} f_j
\]
with $f_j \in W_j, j = -1, 0, 1, 2, \ldots$. How to choose now these spaces $W_j$?

The general idea is to compose $W_k$ from functions living mainly in a certain frequency band. However, to ensure also time localization one needs now basis elements $\psi_{j,s}$ in $W_j$ which are time-localized as much as possible.

A particular interesting constellation is given by

$$\psi_{j,s}(x) = \Psi_j \left(x - \frac{s \pi}{2^j}\right), \quad s = 0, \ldots, 2^{j+1} - 1,$$

and

$$W_j = \text{span}\{\psi_{j,s} : s = 0, \ldots, 2^{j+1} - 1\}.$$  \hspace{1cm} (1)

For different purposes it is also essential to put together some wavelet spaces to build new spaces. We write for $j = -1, 0, 1, \ldots$

$$V_{j+1} = W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \ldots \oplus W_j.$$

In applications one starts often with a projection of a given function into $V_j$. Therefore it is important to know bases for $V_j$. As usual we call a function $\phi_{j,s}$ scaling function if the

$$\phi_{j,s}(x) = \Phi_j \left(x - \frac{s \pi}{2^j}\right), \quad s = 0, \ldots, 2^{j+1} - 1,$$

constitute a basis for $V_j$, i.e.,

$$V_j = \text{span}\{\phi_{j,s} : s = 0, \ldots, 2^{j+1} - 1\}.$$  \hspace{1cm} (2)

Now we want to discuss the possibilities of decomposition and reconstruction. First we start with a function $\Phi_{j+1} \in L^2_{2\pi}$ spanning $V_{j+1}$. The question whether we can decompose $V_{j+1}$ into an orthogonal sum of $V_j$ and $W_j$ spanned by the translates of $\Phi_j$ and $\Psi_j$ we can answer in the following affirmative way.

**Theorem 1.1** Let $\Phi_{j+1} \in L^2_{2\pi}$ be given.

a) The translates $\phi_{j+1,s}, s = 0, \ldots, 2^{j+2} - 1$ are linearly independent if and only if the Fourier coefficients

$$c_k = c_k(\Phi_{j+1})$$

satisfy

$$\sum_{m=-\infty}^{\infty} |c_{k+m2^{j+2}}|^2 > 0 \quad \text{for all} \quad k = 0, \ldots, 2^{j+2} - 1.$$  \hspace{1cm} (3)
b) If (3) is satisfied then the space \( V_{j+1} \) of dimension \( 2^{j+2} \) spanned by the \( 2^{j+2} \) equidistant translates can be decomposed into an orthogonal sum

\[
V_{j+1} = V_j \oplus W_j,
\]

where \( V_j \) and \( W_j \) are both of dimension \( 2^{j+1} \) and spanned by translates of some functions \( \Phi_j \) and \( \Psi_j \), respectively (cf. (1), (2)).

A proof can be found in [15] or [10]. Note that the decomposition is not uniquely determined.

Unfortunately, the converse of this decomposition theorem is not true. i.e., if one starts with two arbitrary spaces \( V_j \) and \( W_j \) of dimension \( 2^{j+1} \) and of the form (1) and (2) respectively, then \( V_j \oplus W_j \) is not necessarily spanned by the \( 2^{j+2} \) equidistant translates of one function. We give here the following counter example from [15]. Let \( j = 0 \) and

\[
\Phi_0(x) = \sum_{k=-3}^{0} e^{ikx}, \quad \Psi_0(x) = \sum_{k=1}^{3} e^{ikx}.
\]

Then

\[
V_0 = \text{span} \{ \Phi_0, \Phi_0(\cdot - \pi) \} = \text{span} \{ 1 + e^{-2i\pi}, e^{-i\pi} + e^{-3i\pi} \},
\]

\[
W_0 = \text{span} \{ \Psi_0, \Psi_0(\cdot - \pi) \} = \text{span} \{ e^{2i\pi}, e^{i\pi} + e^{3i\pi} \}.
\]

Easily one can see \( V_0 \perp W_0 \). Writing

\[
V_1 = V_0 \oplus W_0
\]

the question is whether \( V_1 \) is spanned by a function \( \Phi_1 \) and its 3 translates

\[
V_1 = \text{span} \left\{ \Phi_1, \Phi_1(\cdot - \pi), \Phi_1(\cdot - \pi), \Phi_1 \left( \cdot - \frac{2\pi}{2} \right) \right\}.
\]

Assume now that (4) is true. Then it follows for every \( f \in V_1 \) that the function \( f(\cdot - \frac{3\pi}{2}) \) is an element of \( V_1 \). In particular, \( f = e^{i\pi} + e^{3i\pi} \in W_0 \subset V_1 \) but

\[
f \left( \cdot - \frac{3\pi}{2} \right) = -i(e^{i\pi} - e^{3i\pi}) \notin V_1.
\]

Hence, (4) is not satisfied, which means that reconstruction is not always possible. Therefore it is not the usual way to start with wavelet functions and to construct scaling functions out of them.

2. Modified Fourier Sums of Spline Functions

In this section we present counter examples of bases for particular polynomial spaces \( V_j \) and \( W_j \). Other examples of trigonometric polynomial
wavelets are studied e.g. in [12], [13], [14]. More informations on the general construction principles of periodic wavelets can be found in [6], [8], [10], [15].

In the following we write \( N = 2^j \) for \( j \in \mathbb{N}_0 \). For simplicity we choose as a starting point the following definition of polynomial spaces of scaling functions

\[
V_j := \text{span} \{1, \cos x, \ldots, \cos(N-1)x, \sin x, \ldots, \sin Nx\}. \tag{5}
\]

Note that this definition is not of the form \( \mathcal{G}/2 \). However, we will show a representation of that kind in the sequel. Note further that the examples presented here depend essentially on the fact that \( \sin N\phi \) is included and \( \cos N\phi \) is excluded. One could exchange the frequency \( \sin N\phi \) by \( \cos N\phi \) in the definition of \( V_j \). Spaces of that kind are discussed in detail in [11], [13], [15].

Easily we obtain the following two properties which are satisfied for these spaces of polynomials

\[
V_j \subset V_{j+1}, \quad j \in \mathbb{N}_0, \\
L_2 \left( \mathbb{R} / 2 \pi \right) = \overline{\left\{ \bigcup_{j=0}^{\infty} V_j \right\}}.
\]

Analogously we define for \( j \in \mathbb{N}_0 \)

\[
W_j := \text{span} \{\cos Nx, \ldots, \cos(2N-1)x, \sin(N+1)x, \ldots, \sin 2Nx\}. \tag{6}
\]

The spaces \( V_j \) and \( W_j \) for \( j \in \mathbb{N}_0 \) have dimension \( 2N \) and it holds

\[
V_{j+1} = V_j \oplus W_j, \\
L_2 \left( \mathbb{R} / 2 \pi \right) = \overline{\left\{ V_0 \oplus \bigoplus_{j=0}^{\infty} W_j \right\}}.
\]

Now we want to construct scaling functions and wavelets with the properties (2) and (1), respectively. This approach goes back to the paper of Chui and Mhaskar [3] and the generalizations presented here are taken from Ihsberner [5].

For shortness we write in the sequel

\[
x_{j,k} := \frac{2\pi k}{2N}, \quad k = 0, 1, \ldots, 2N - 1.
\]

A typical candidate for the scaling function seems to be the modified Dirichlet kernel and its translates which is motivated by the localization properties

\[
D_N \left( \frac{2\pi k}{2N} \right) = \begin{cases} 
0, & \text{if } k \not\equiv 0 \mod 2N, \\
N, & \text{if } k \equiv 0 \mod 2N.
\end{cases} \tag{7}
\]
However, \(D_N^s\) is not an element of \(V_j\) as defined here. To overcome this difficulty we proceed here by integrating the modified Dirichlet kernel. Therefore we define scaling functions for \(s = 0, \ldots, 2N - 1\) by

\[
\phi_{j,s}(x) := \Phi_j(x - x_{j,s})
\]

with

\[
\Phi_j(x) := \frac{1}{\pi} \int_{-\pi}^\pi D_N(x - t) \, dt.
\]  \hspace{1cm} (8)

By the help of the convolution of periodic functions we can write

\[
\Phi_j(x) = \frac{1}{\pi}(h_j * D_N^s)(x),
\]

where the \(2\pi\)-periodic function \(h_j\) is given by

\[
h_j(x) := \begin{cases} 
1, & \text{if } x \in [0, x_{j,1}), \\
0, & \text{otherwise in } (-\pi, \pi).
\end{cases}
\]

Note that \(h_j\) is also a (Haar-)scaling function for a spline multiresolution.

Analogously we define wavelets for \(s = 0, \ldots, 2N - 1\) by

\[
\psi_{j,s}(x) := \Psi_j(x - x_{j,s}),
\]

where

\[
\Psi_j(x) := \frac{1}{\pi} \int_{-\pi}^\pi (D_N^s(x - t) - D_N^s(x - t - \pi)) \, dt
\]

\[
- \frac{1}{\pi} \int_{-\pi}^\pi (D_N^s(x - t) - D_N^s(x - t - \pi)) \, dt.
\]  \hspace{1cm} (9)

Again we rewrite this definition by introducing the corresponding \(2\pi\)-periodic (Haar-)spline wavelet

\[
g_j(x) := \begin{cases} 
1, & \text{if } x \in [0, x_{j+1,1}), \\
-1, & \text{if } x \in [x_{j+1,1}, x_{j,1}), \\
0, & \text{otherwise in } (-\pi, \pi).
\end{cases}
\]

We obtain

\[
\Psi_j(x) = \frac{1}{\pi}(g_j * (D_N^s - D_N^s))(x).
\]

These scaling functions and wavelets can also be seen as modified Fourier sums of the Haar functions \(h_j\) and \(g_j\), respectively.
Writing the modified Fourier sum for $f \in L^2_{2\pi}$ and $m \in \mathbb{N}$ as

$$S'_m(f, x) := \frac{a_0}{2} + \sum_{k=1}^{m-1} (a_k \cos kx + b_k \sin kx) + \frac{a_m}{2} \cos mx + \frac{b_m}{2} \sin mx,$$

where

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \, dt, \quad b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \, dt,$$

we derive immediately

$$\Phi_j(x) = S'_{N_j}(h_j, x), \quad \Psi_j(x) = S'_{2N_j}(g_j, x) - S'_{N_j}(g_j, x).$$

Consequently we obtain the following imbedding.

**Lemma 2.1** For $s = 0, \ldots, 2N-1$ the scaling functions $\phi_{j,s}$ are elements of $V_j$ and the wavelets $\psi_{j,s}$ are elements of $W_j$.

**Proof** Representing the scaling functions $\phi_{j,s}$ in terms of linear combinations of the sine and cosine functions we have

$$\phi_{j,s}(x) = \frac{1}{2N} + \frac{1}{N} \sum_{k=1}^{N-1} \frac{1}{\pi k} \left( (\sin x_{j,k(s+1)} - \sin x_{j,ks}) \cos kx + (\cos x_{j,ks} - \cos x_{j,k(s+1)}) \sin kx \right) + \frac{(-1)^s}{N\pi} \sin N x$$

which means $\phi_{j,s} \in V_j$. For the wavelets $\psi_{j,s}$ we write

$$\psi_{j,s}(x) = \frac{(-1)^s}{N\pi} \cos N x + \frac{1}{N\pi} \sin 2N x + \sum_{k=N+1}^{2N-1} \frac{1}{\pi k} \left( (2 \sin x_{j+1,k(2s+1)} - \sin x_{j,k(s+1)} - \sin x_{j,ks}) \cos kx + (\cos x_{j,ks} - 2 \cos x_{j+1,k(2s+1)} + \cos x_{j,k(s+1)}) \sin kx \right)$$

and hence $\psi_{j,s} \in W_j$.

It remains to prove the linear independence of the translates so that the scaling functions and wavelets really constitute a basis. Using (3) we will
show that later.
Let us mention here the lack of localization as a particular drawback of the
scaling functions and wavelets defined above. As it is known for Dirichlet
kernels there are large oscillations also away from the point of amplitude.
This can be illustrated by different measures. The most famous is the $L_2$-
norm which includes an additional term of logarithmic order. Another one
is an uncertainty product for periodic functions given by Breitenberger (cf.
[2], [8], [12]). A third one is the ability of detection of singularities with
such wavelets (cf. [7]). We do not go here into the details of these time-
frequency localization properties.
Our aim here is now to introduce better localized scaling functions and
wavelets for the spaces $V_j$ and $W_j$ defined in (5) and (6), respectively.
Therefore we generalize Haar functions to splines of higher order. They are
more smooth which yields scaling functions and wavelets which are more lo-
calized in the sense that they have less oscillations. In other words one could
also say they have a smaller $L_2$-norm, or they have a smaller uncertainty-
product or they are better suited for the detection of singularities of higher
order.
As usual we define these splines by convolution. Therefore for $0 < b < \pi$
let
\[ \tilde{h} := \frac{1}{2b} \chi_{[-b, b]}, \quad \text{in } (-\pi, \pi) \]
and $2\pi$-periodically extended. For $l \in \mathbb{N}$ and $b l < \pi$ let
\[ \tilde{H}_l := \begin{cases} \tilde{h}, & \text{if } l = 1, \\ \tilde{h} \ast \ldots \ast \tilde{h}, & \text{if } l > 1. \end{cases} \]
Let us note that $\tilde{H}_l$ is zero in $(-\pi, \pi) \setminus (-lb, lb)$ and for $l > 1$ the spline
$\tilde{H}_l$ is $l - 2$ times continuously differentiable. Furthermore, in the intervals
$[-lb + 2b(k - 1), -lb + 2bk]$ for $k = 0, \ldots, l$ the spline is an algebraic
polynomial of degree $(l - 1)$. To simplify our further considerations we
assume that $b \sim \frac{1}{N}$ where $N$ is the highest order of sampling we start
with and $b$ is not a rational multiple of $\pi$.
After these preliminaries the following definitions are very straightforward.

**Definition 2.1** For $l \in \mathbb{N}$ and $s = 0, \ldots, 2N - 1$ we write
\[ \phi_{j,s}^l(x) = \Phi_j^l(x - x_{j,s}) := S_N^l(\tilde{H}_l \ast h_j, x - x_{j,s}) \]
\[ = (H_l \ast \phi_{j,s})(x) \]
and
\[ \psi_{j,s}^l(x) = \Psi_j^l(x - x_{j,s}) := S_{2N}^*(\tilde{H}_l * g_{j,x} - x_{j,s}) - S_{N}^*(\tilde{H}_l * g_{j,x} - x_{j,s}) \]
\[ = (\tilde{H}_l * \psi_{j,s})(x). \] (13)

For \( l = 0 \) we write analogously
\[ \phi_{j,s}^0(x) = \Phi_j^0(x - x_{j,s}) := \phi_{j,s}(x) \]
and
\[ \psi_{j,s}^0(x) = \Psi_j^0(x - x_{j,s}) := \psi_{j,s}(x). \]

Again we have easily the following imbedding.

**Theorem 2.1** For \( s = 0, \ldots, 2N - 1 \) the scaling functions \( \phi_{j,s} \) are elements of \( V_j \) and the wavelets \( \psi_{j,s} \) are elements of \( W_j \).

**Proof** By Lemma 2.1 the functions \( \phi_{j,s} \) are elements of \( V_j \). Then a convolution with the even function \( \tilde{H}_l \) remains in \( V_j \). The same holds true for the wavelets. \( \square \)

To show basis properties of these scaling functions and wavelets we compute the complex Fourier coefficients. Therefore, for \( f, g \in L_{2\pi}^1 \), \( k \in \mathbb{Z} \) and \( y \in \mathbb{R} \) we make extensive use of
\[ c_k(f * g) = 2\pi c_k(f) c_k(g), \]
\[ c_k(f(-y)) = c_k(f) e^{-iky}. \]

Moreover, the Fourier coefficients of the characteristic function \( \chi_{[-b, b]} \) for \( k \in \mathbb{Z} \) can be calculated by
\[ c_k(\chi_{[-b, b]}) = \frac{1}{2\pi} \int_{-b}^{b} e^{-ikt} dt = \frac{2b}{2\pi} \text{sinc} kb, \]
where the sinus cardinalis is defined by
\[ \text{sinc} x = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases} \]

With \( x_{j,1} = \pi/N \) and the observation \( \chi_{(-b/2, b/2)}(-b/2) = \chi_{(0, b)} \) we
obtain
\[ c_k(\Phi_{j_l}^0) = \frac{1}{\pi} c_k \left( \chi_{[-\pi, \pi]} \left( \cdot - \frac{\pi}{2N} \right) * D_N^j \right) \]
\[ = 2 \ c_k \left( \chi_{[-\pi, \pi]} \right) e^{\frac{-i \pi k}{N}} c_k(D_N^j) \]
\[ = \frac{1}{N} \text{sinc} \frac{k \pi}{2N} e^{\frac{-i \pi k}{N}} c_k(D_N^j) \]
\[ = \begin{cases} 
\frac{1}{2N}, & \text{if } k = 0, \\
\frac{i}{2N \pi} \left( e^{\frac{-i \pi k}{N}} - 1 \right), & \text{if } 0 < |k| < N, \\
\frac{-i}{2N \pi}, & \text{if } |k| = N, \\
0, & \text{if } |k| > N.
\end{cases} \]

With
\[ c_k(\tilde{H}_l) = \frac{1}{2\pi} \text{sinc}^l kb \]
we obtain
\[ c_k(\Phi_{j_l}^l) = c_k(\Phi_{j_l}^0 * \tilde{H}_l) \]
\[ = 2 \pi c_k(\Phi_{j_l}^0) c_k(\tilde{H}_l) \]
\[ = c_k(\Phi_{j_l}^0) \text{sinc}^l kb. \]

Hence the scaling functions \( \Phi_{j_l}^l \) possess for \( l \in \mathbb{N}_0 \) the following representation
\[ \Phi_{j_l}(x) = \sum_{k=-N}^{N} c_k(\Phi_{j_l}^l) e^{ikx}, \]
where
\[ c_k(\Phi_{j_l}^l) = c_k(\Phi_{j_l}^0) \text{sinc}^l kb \]
\[ = \begin{cases} 
\frac{1}{2N}, & \text{if } k = 0, \\
\frac{i}{2N \pi} \left( e^{\frac{-i \pi k}{N}} - 1 \right) \text{sinc}^l kb, & \text{if } |k| = 1, \ldots, N - 1, \\
\frac{-i}{2N \pi} \text{sinc}^l Nb, & \text{if } |k| = N, \\
0, & \text{otherwise.}
\end{cases} \]

Analogously, for \( l \in \mathbb{N}_0 \) the wavelet functions \( \Psi_{j_l}^l \) satisfy
\[ \Psi_{j_l}(x) = \sum_{|k|=-N}^{2N} c_k(\Psi_{j_l}^l) e^{ikx}, \]
where
\[ c_k(\Psi^j_y) = c_k(\Psi^j_y) \sin^j kb \]
\[ = \begin{cases} 
\frac{1}{2N\pi} \sin^j N, & \text{if } |k| = N, \\
\frac{i}{2N\pi} (1 - \cos x_{j+1, k}) e^{-\frac{2\pi i k}{N}} \sin^j kb, & \text{if } N < |k| < 2N, \\
\frac{i}{2N\pi} \sin^j 2N, & \text{if } |k| = 2N, \\
0, & \text{otherwise}. 
\end{cases} \]

Now we are enabled to show the basis properties of these functions.

**Theorem 2.2** For \( l = 0, 1, 2, \ldots \) the system
\[ B_j(\Phi^j_y) := \{ \Phi^j_y(x - x_{j, s}) \}_{s=0}^{2N-1} = \{ \phi^j_{j,s}(x) \}_{s=0}^{2N-1} \]
is a basis for \( V_j \) and
\[ B_j(\Psi^j_y) := \{ \Psi^j_y(x - x_{j, s}) \}_{s=0}^{2N-1} = \{ \psi^j_{j,s}(x) \}_{s=0}^{2N-1} \]
is a basis for \( W_j \).

**Proof** We use Theorem 1.1 which states that \( B_j(\Phi^j_y) \) is a basis of
\[ \text{span} \{ \Phi^j_y(x - x_{j, s}) : s = 0, \ldots, 2N - 1 \}, \]
iff
\[ \sum_{m=-\infty}^{\infty} |c_{s+m2N}(\Phi^j_y)|^2 > 0 \text{ for } s = 0, \ldots, 2N - 1. \]
The nonvanishing Fourier coefficients of \( \Psi^j_y \) are exactly
\[ c_{-N}(\Phi^j_y), \ldots, c_1(\Phi^j_y), c_0(\Phi^j_y), c_1(\Phi^j_y), \ldots, c_N(\Phi^j_y). \]
Hence, the sum for \( s = 0, \ldots, 2N - 1 \) can be written as
\[ \sum_{m=-\infty}^{\infty} |c_{s+m2N}(\Phi^j_y)|^2 = |c_s(\Phi^j_y)|^2 + |c_{s-2N}(\Phi^j_y)|^2 > 0. \]
From Lemma 2.1 and Theorem 2.1 we have the imbedding of these linear independent functions into \( V_j \). The space \( V_j \) has dimension \( 2N \). Thus,
\[ V_j = \text{span} \{ \Phi^j_y(x - x_{j, s}) : s = 0, \ldots, 2N - 1 \} \]
and \( B_j(\Phi^j_y) \) is a basis of \( V_j \).
Analogously we prove the second part. For arbitrary \( l \in \mathbb{N}_0 \) the non-vanishing Fourier coefficients are

\[
c_{-2N}(\Psi_j^l), \ldots, c_{-N}(\Psi_j^l), \ c_N(\Psi_j^l), \ldots, \ c_{2N}(\Psi_j^l)
\]

and we obtain

\[
\sum_{m=-\infty}^{\infty} |f_{s+m2N}(\Psi_j^l)|^2 = \sum_{m=-1}^{1} |c_{s+m2N}(\Psi_j^l)|^2 > 0 \text{ for } s = 0, \ldots, 2N - 1.
\]

This implies that \( B_j(\Psi_j^l) \) is a basis of span \( \{ \Psi_j^l(x-x_{j,s}) : s = 0, \ldots, 2N - 1 \} \). Because of the dimension of \( W_j \) is \( 2N \) and Theorem 2.1 we obtain that \( B_j(\Psi_j^l) \) is a basis for \( W_j \).

Note that by definition of \( V_j \) and \( W_k \) the orthogonality

\[
\phi_{j,s}^{l} \perp \psi_{k,r}^{m}
\]

for \( k \geq j \) and arbitrary \( r, s, l, m \in \mathbb{N}_0 \) is easily seen.

However, these bases \( B_j(\Phi_j^l) \) and \( B_j(\Psi_j^l) \) are not orthogonal bases in the sense that the translates itself are orthogonal to each other. But there is a simple orthonormalization principle available. Particularly, one is interested in a procedure which ends up again in a translation invariant basis. Also for this purpose the sums of the form (3) play a very important role. In [10, 15] the following result is proved.

**Lemma 2.2** If \( \Phi_j^* \) and \( \Psi_j^* \) are \( 2\pi \)-periodic functions with Fourier coefficients

\[
c_k(\Phi_j^*) = c_k(\Phi_j^0) \frac{\sin^2 kb}{|\sin^2 kb|}
\]

\[
= \begin{cases} 
\frac{1}{\sqrt{2N}}, & \text{if } k = 0, \\
\frac{-\mu_k}{\sqrt{2N} |\sin^3 kb|}, & \text{if } |k| = 1, \ldots, N - 1, \\
\frac{-\mu_k}{2\sqrt{N} |\sin^3 Nk|}, & \text{if } |k| = N, \\
0, & \text{otherwise},
\end{cases}
\]

...
and
\[ c_k(\psi_j^*) = c_k(\psi_j^0) \frac{\sin^j kb}{\sin^j kb} \]
\[ = \begin{cases} \frac{\sin^j N b}{2 \sqrt{N} \sin^j N b}, & \text{if } |k| = N, \\ \frac{-i \sin^j N b}{2 \sqrt{N} \sin^j N b}, & \text{if } |k| = N + 1, \ldots, 2N - 1, \\ \frac{-i \sin^j 2N b}{2 \sqrt{N} \sin^j 2N b}, & \text{if } |k| = 2N, \\ 0, & \text{otherwise}, \end{cases} \]
respectively, then
\[ B_j(\Psi_j^*) := \{\phi_j^s(x - x_j,s)\}_{s=0}^{2N-1} = \{\psi_j^s(x)\}_{s=0}^{2N-1} \]
is an orthonormal basis for \( V_j \) and
\[ B_j(\Psi_j^*) := \{\phi_j^s(x - x_j,s)\}_{s=0}^{2N-1} = \{\psi_j^s(x)\}_{s=0}^{2N-1} \]
is an orthonormal basis for \( W_j \).

**Proof** In [10, 15] the following result is proved. If
\[ \sum_{m=-\infty}^{\infty} |\xi_{s+m2N}(\Phi_j^s)|^2 > 0 \quad \text{for } s = 0, \ldots, 2N - 1, \]
i.e., if the translates are linearly independent, then the orthonormalization procedure consists in putting
\[ c_k(\psi_j^*) := \frac{c_k(\Psi_j^s)}{\sqrt{2N} \sqrt{\sum_{m=-\infty}^{\infty} |\xi_{k+m2N}(\Phi_j^s)|^2}} \tag{14} \]
(\( k \in \mathbb{Z}, k_j = k \mod 2N \)), to construct \( \Phi_j^* \). Out of (14) the assertions can be easily deduced. \( \blacksquare \)

### 3. Decomposition and Reconstruction

A cornerstone for the development of reconstruction and decomposition algorithms for the trigonometric multiresolution analysis described in Section 2 is the knowledge of the corresponding reconstruction and decomposition sequences for the spaces \( V_j \) and \( W_j \). This section is devoted to the computation of these two-scale sequences for reconstruction and decomposition. As \( V_j \subset V_{j+1} \), there must be coefficients \( p_{j,n,s} \) such that
\[ \phi_{j,n} = \sum_{s=0}^{4N-1} p_{j,n,s} \phi_{j+1,s}. \]
The Theorem 3.1 establishes the precise values of these \( p_{j,n,s} \).

The other pair of important sequences for our approach are the so-called decomposition sequences. As \( V_{j+1} = V_j \oplus W_j \) for \( j \in \mathbb{N}_0 \), any \( \phi_{j+1,s} \in V_{j+1} \) can be written as a linear combination of the basis functions of \( V_j \) and \( W_j \), i.e. \( \phi_{j,k} \) and \( \psi_{j,k} \).

Note that we simplify our investigations by using the same \( b \) in (10) for any space \( V_j \) and \( W_j \). The general case can be obtained again by simple basis transformations of the same kind as described in the sequel.

An important part of the following observations is based on interpolation type representations of trigonometric polynomials (cf. [16], Chap. 10). The essential point is that

\[
\frac{1}{m} D_m^* \left( x - \frac{2k + 1}{2m} \pi \right) \quad \text{for} \quad k = 0, \ldots, 2m - 1
\]

can be used as fundamental polynomials of Lagrange interpolation (see (7)). More precisely, we obtain for trigonometric polynomials of the form

\[
\tilde{T}(x) = \frac{1}{2} a_0 + \sum_{k=1}^{m-1} \left( a_k \cos kx + b_k \sin kx \right) + b_m \sin mx
\]

that

\[
\tilde{T}(x) = \frac{1}{m} \sum_{k=0}^{2m-1} T \left( \frac{2k + 1}{2m} \pi \right) D_m^* \left( x - \frac{2k + 1}{2m} \pi \right).
\]

Analogous statements could be made for the fundamental polynomials

\[
\frac{1}{m} D_m^* \left( x - \frac{2k}{2m} \pi \right) \quad \text{for} \quad k = 0, \ldots, 2m - 1.
\]

Hence, for arbitrary polynomials

\[
T(x) = \frac{1}{2} a_0 + \sum_{k=1}^{m} \left( a_k \cos kx + b_k \sin kx \right)
\]

we have the two different representations

\[
T(x) = a_m \cos mx + \frac{1}{m} \sum_{k=0}^{2m-1} T \left( \frac{2k + 1}{2m} \pi \right) D_m^* \left( x - \frac{2k + 1}{2m} \pi \right) \quad \quad (15)
\]

\[
= b_m \sin mx + \frac{1}{m} \sum_{k=0}^{2m-1} T \left( \frac{2k}{2m} \pi \right) D_m^* \left( x - \frac{2k}{2m} \pi \right). \quad \quad (16)
\]

**Theorem 3.1 (Two-scale relations)** For \( l \in \mathbb{N}_0 \) and \( k = 0, \ldots, 2N - 1 \) it holds

\[
\phi_{j,k}(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_{j+1,s}) \phi_{j+1,2k,s}(x) \quad \quad (17)
\]
and

$$\psi_{j,k}^{l}(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} \tilde{R}_j(x_{j+1,s}) \phi_{j+1,2k+s}^l(x),$$  \hspace{1cm} (18)$$

where the coefficients are point evaluations of the trigonometric polynomials

$$R_j(x) := D_N^j(x) + D_N^j(x - x_{j+1,1})$$  \hspace{1cm} (19)$$

and

$$\tilde{R}_j(x) := D_N^*_j(x) - D_N^*_j(x - x_{j+1,1}) - D_N^j(x) + D_N^j(x - x_{j+1,1}).$$  \hspace{1cm} (20)$$

**Proof**  With (19), (20) and substitution we obtain for \(l = 0\)

$$\phi_{j,k}^0(x) = \frac{1}{\pi} \int_0^{2\pi} R_j(x - x_{j,k} - t) \, dt$$  \hspace{1cm} (21)$$

and

$$\psi_{j,k}^0(x) = \frac{1}{\pi} \int_0^{2\pi} \tilde{R}_j(x - x_{j,k} - t) \, dt.$$  \hspace{1cm} (22)$$

Note that \(R_j\) and \(\tilde{R}_j\) are trigonometric polynomials of degree \(2N\) without \(2N\)th sine frequency. Applying (16), namely

$$R_j(x - x_{j,k} - t) = \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_{j+1,s}) D_{2N}^j(x - x_{j+1,2k+s} - t)$$  \hspace{1cm} (23)$$

and

$$\tilde{R}_j(x - x_{j,k} - t) = \frac{1}{2N} \sum_{s=0}^{4N-1} \tilde{R}_j(x_{j+1,s}) D_{2N}^j(x - x_{j+1,2k+s} - t)$$  \hspace{1cm} (24)$$

we deduce from (21) and (23) that

$$\phi_{j,k}^0(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_{j+1,s}) \phi_{j+1,2k+s}^0(x).$$

Analogously, from (22) and (24) it follows

$$\psi_{j,k}^0(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} \tilde{R}_j(x_{j+1,s}) \psi_{j+1,2k+s}^0(x).$$
Using \( l \)-times convolution yields for arbitrary \( l \in \mathbb{N}_0 \) (cf. (12) and (13)) that

\[
\phi^l_{j,k}(x) = \frac{1}{(2b)^l} \int_{-b}^{b} \cdots \int_{-b}^{b} \phi^0_{j,k}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
= \frac{1}{(2b)^l} \int_{-b}^{b} \cdots \int_{-b}^{b} \frac{1}{2N} \sum_{s=0}^{4N-1} (R_j(x_j+1,s)) \times
\]

\[
\phi^0_{j+1,2k+s}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
= \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_j+1,s) \frac{1}{(2b)^l} \times
\]

\[
\int_{-b}^{b} \cdots \int_{-b}^{b} \phi^0_{j+1,2k+s}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
= \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_j+1,s) \phi^l_{j+1,2k+s}(x)
\]

and

\[
\psi^l_{j,k}(x) = \frac{1}{(2b)^l} \int_{-b}^{b} \cdots \int_{-b}^{b} \psi^0_{j,k}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
= \frac{1}{(2b)^l} \int_{-b}^{b} \cdots \int_{-b}^{b} \frac{1}{2N} \times
\]

\[
\sum_{s=0}^{4N-1} \tilde{R}_j \, (x_j+1,s) \psi^0_{j+1,2k+s}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
= \frac{1}{2N} \sum_{s=0}^{4N-1} \tilde{R}_j \, (x_j+1,s) \frac{1}{(2b)^l} \times
\]

\[
\int_{-b}^{b} \cdots \int_{-b}^{b} \psi^0_{j+1,2k+s}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]
The two-scale relations allow a reconstruction of signals. To obtain formulas for the inverse operation we introduce some abbreviations. Let us define for all \( j \in \mathbb{N}_0 \) a \( 2N \)-periodic bi-infinite sequence \( \lambda_{j,s} \) by

\[
\lambda_{j,s} := s, \quad s = 1, \ldots, 2^{j+1}.
\]

Furthermore let

\[
\beta_{j,k} := \frac{1}{N} \sum_{n=1}^{N-1} \tan \frac{x_{j+1,n}}{2} \sin x_{j,sk},
\]

\[
\Delta_{j,k} := D_N^* \left( \frac{2k + 1}{2N} \pi \right),
\]

\[
c_{j,k} := \frac{1}{2} \delta_{k,0} + \frac{(-1)^k}{4N},
\]

\[
d_{j,k} := \frac{1}{2N} \left\{ \frac{1}{N} \sum_{s=0}^{2N-1} \left( \lambda_{j,k-s} - \frac{\lambda_{j,k} + \lambda_{j,k+1}}{2} \right) \Delta_{j,s} \right\}.
\]

**Theorem 3.2 (Decomposition formula)** For all \( l \in \mathbb{N}_0 \) and \( m < 2N \) we have

\[
d_{j+1,2m}(x) = \sum_{k=0}^{2N-1} \left( c_{j,k} + \frac{1}{2} \beta_{j,k} \right) \phi_{j+1,2k}(x) + \frac{1}{2} \psi_{j+1,2m-k}(x)
\]

\[
+ \frac{(-1)^m}{4N} \sum_{k=0}^{2N-1} (-1)^k \psi_{j,k}(x) + \sum_{k=0}^{2N-1} d_{j,k} \psi_{j+1,2m-k}(x)
\]

and

\[
d_{j+1,2m+1}(x) = \sum_{k=0}^{2N-1} \left( c_{j,k} + \frac{1}{2} \beta_{j,k} \right) \phi_{j+1,2k+1}(x) - \frac{1}{2} \psi_{j+1,2m-k}(x)
\]

\[
- \frac{(-1)^m}{4N} \sum_{k=0}^{2N-1} (-1)^k \psi_{j,k}(x) + \sum_{k=0}^{2N-1} d_{j,k} \psi_{j+1,2m-k}(x).
\]

To show this result we need some preliminary observations.

**Lemma 3.1** For \( N \in \mathbb{N} \) and \( l \in \mathbb{N}_0 \) we have

\[
\left( \frac{1}{Nb} \right)^l \sin^l N b \sin N x = \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \phi_{j,s}(x)
\]
and

\[
\left( \frac{1}{N b} \right)^l \sin^l N b \cos N x = \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}^l(x). \tag{30}
\]

**Proof** We start with

\[
\left( \frac{2}{N} \right)^l \sin^l N b \sin N x \\
= \int_b^{-b} \cdots \int_b^{-b} \sin N(x - s_1 - \ldots - s_l) ds_1 \ldots ds_l \\
= \frac{N}{2} \int_b^{-b} \cdots \int_b^{-b} \int_0^{\pi} \cos N(x - t - s_1 - \ldots - s_l) dt ds_1 \ldots ds_l.
\]

From (16) we deduce

\[
\cos N x = \frac{1}{N} \sum_{s=0}^{2N-1} \cos(Nx_{j,s}) D_N(x - x_{j,s}).
\]

Thus

\[
\left( \frac{1}{N b} \right)^l \sin^l N b \sin N x \\
= \frac{1}{2} \sum_{s=0}^{2N-1} \cos(Nx_{j,s}) \frac{1}{(2b)^l} \\
\int_b^{-b} \cdots \int_b^{-b} \int_0^{\pi} D_N(x - x_{j,s} - t - s_1 - \ldots - s_l) dt ds_1 \ldots ds_l \\
= \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}^l(x),
\]

which proves (29). To show (30) we note that

\[
\cos N x = \frac{N}{2} \left\{ \int_0^{\pi} \cos N(x - t) dt - \int_{2\pi}^{\pi} \cos N(x - t) dt \right\}. \tag{31}
\]
Rewriting the righthand side by using (16) we obtain

\[
\cos N(x-t) = 2 \cos N(x-t) - \cos N(x-t) \\
= \frac{2}{2N} \sum_{k=0}^{2N-1} \cos(Nx_{j+1,k})D_{2N}^*(x-t-x_{j+1,k}) \\
- \frac{1}{N} \sum_{k=0}^{2N-1} \cos(Nx_{j,k})D_N^*(x-t-x_{j,k}) \\
= \frac{1}{N} \sum_{k=0}^{2N-1} (-1)^k \left(D_{2N}^*(x-t-x_{j,k}) - D_N^*(x-t-x_{j,k}) \right). \tag{32}
\]

Inserting (32) in (31) and applying the definition of \( \psi_{j,s} \) we obtain

\[
\cos N x = \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}(x).
\]

Hence,

\[
\left(\frac{1}{N b}\right)^l \sin^l N b \cos N x \\
= \frac{1}{(2b)^l} \int_{-b}^{b} \cdots \int_{-b}^{b} \cos N(x-s_1 - \ldots - s_l) \, ds_1 \ldots ds_l \\
= \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \left(\frac{1}{(2b)^l} \int_{-b}^{b} \cdots \int_{-b}^{b} \psi_{j,s}(x-s_1 - \ldots - s_l) \, ds_1 \ldots ds_l \right) \\
= \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}(x).
\]

Lemma 3.2 For \( l \in \mathbb{N}_0 \) und \( m = 0, 1, \ldots, 2N-1 \) it holds

\[
\phi_{j+1,2m} - \phi_{j+1,2m+1} = - \sum_{k=0}^{2N-1} \beta_{j,s-m} \phi_{j,s} + \psi_{j,m} + \frac{(-1)^m}{2N} \sum_{k=0}^{2N-1} (-1)^s \psi_{j,s}.
\]

Proof In a first step we prove the Lemma for \( l = 0 \). Applying the
definitions (8) and (9) we obtain

\[
\phi_{j+1,2m}(x) - \phi_{j+1,2m+1}(x) = \Phi_{j+1}(x - x_j, m) - \Phi_{j+1}(x - x_j, m - x_{j+1,1}) - \Psi_j(x - x_j, m)
\]

\[
= \frac{1}{\pi} \int_0^{2\pi} (D_N^*(x - t - x_j, m) - D_N^*(x - t - x_j, m - x_{j+1,1})) \, dt
\]

\[
= \frac{1}{\pi} \sum_{k=1}^{N-1} \int_0^{2\pi} (\cos k(x - t - x_j, m)
\]

\[
- \cos k(x - t - x_j, m - x_{j+1,1}) \, dt
\]

\[
+ \frac{1}{2\pi} \int_0^{2\pi} (\cos N(x - t - x_j, m))
\]

\[
- \cos N(x - t - x_j, m - x_{j+1,1}) \, dt.
\]

Exploring the identity

\[
\int_0^{\frac{\pi}{N}} \sin k(x-t) \, dt = \frac{2}{k} \sin \frac{k\pi}{2N} \sin k \left( x - \frac{\pi}{2N} \right), \quad k \in \mathbb{N},
\]

we simplify for \( k = 1, \ldots, N \) the integral

\[
\int_0^{\frac{\pi}{N}} (\cos k(x-t) - \cos k(x - t - x_{j+1,1})) \, dt
\]

\[
= -2 \sin \frac{k\pi}{4N} \int_0^{\frac{\pi}{4N}} \sin k \left( x - t - \frac{\pi}{4N} \right) \, dt \quad (33)
\]

\[
= -\frac{4\sin^2 \left( \frac{k\pi}{4N} \right)}{k} \sin k \left( x - \frac{\pi}{2N} \right)
\]

\[
= -\frac{2\sin^2 \left( \frac{k\pi}{2N} \right)}{\sin \left( \frac{k\pi}{2N} \right)} \int_0^{\frac{\pi}{N}} \sin k(x-t) \, dt
\]

\[
= -\tan \left( \frac{k\pi}{4N} \right) \int_0^{\frac{\pi}{N}} \sin k(x-t) \, dt. \quad (34)
\]
Moreover, for \( k = N \) it follows from (33) and (30) that

\[
\int_0^{2\pi} (\cos N(x - t) - \cos N(x - t - x_{j+1,1})) \, dt = -2 \sin \frac{N\pi}{4N} \int_0^{2\pi} \sin N(x - t - \frac{\pi}{4N}) \, dt
\]

\[
= -\frac{\sqrt{2}}{N} \left( \cos \left( Nx - \frac{3\pi}{4} \right) - \cos \left( Nx - \frac{\pi}{4} \right) \right)
\]

\[
= \frac{2}{N} \cos Nx
\]

\[
= \frac{\pi}{N} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}(x).
\]

(35)

On the other hand, for \( k = 1, \ldots, N - 1 \) we apply (34) and (16) to obtain

\[
\int_0^{2\pi} (\cos k(x - t) - \cos k(x - t - x_{j+1,1})) \, dt = -\frac{1}{N} \tan \left( \frac{k\pi}{4N} \right) \sum_{s=0}^{2N-1} \sin kx_{j,s} \int_0^{2\pi} D_N^s(x - t - x_{j,s}) \, dt
\]

\[
= -\frac{\pi}{N} \tan \left( \frac{k\pi}{4N} \right) \sum_{s=0}^{2N-1} (\sin x_{j,s,k}) \phi_{j,s}(x).
\]

(36)

With (36), (35) and the definition of the \( \beta_{j,s} \) we argue

\[
\frac{1}{\pi} \sum_{k=1}^{N-1} \int_0^{2\pi} (\cos k(x - t - x_{j,m}) - \cos k(x - t - x_{j,m} - x_{j+1,1})) \, dt
\]

\[
+ \frac{1}{2\pi} \int_0^{2\pi} (\cos N(x - t - x_{j,m}) - \cos N(x - t - x_{j,m} - x_{j+1,1})) \, dt
\]

\[
= -\sum_{s=0}^{2N-1} \beta_{j,s} \phi_{j,s}(x - x_{j,m}) + \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}(x - x_{j,m}),
\]

so that

\[
\phi_{j+1,2m}(x) = \phi_{j+1,2m+1}(x) - \psi_{j,m}(x)
\]

\[
= -\sum_{s=0}^{2N-1} \beta_{j,s} \phi_{j,s}(x - x_{j,m}) + \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}(x - x_{j,m}).
\]
This yields the assertion for \( l = 0 \). In the general case \( l \geq 0 \) we write

\[
\phi_{j+1,2m}^l(x) - \phi_{j+1,2m+1}^l(x) = \psi_{j,m}^l(x)
\]

\[
= \frac{1}{(2b)^l} \int_{-b}^b \cdots \int_{-b}^b (\phi_{j+1,2m}(x - s_1 - \ldots - s_l) - \phi_{j+1,2m+1}(x - s_1 - \ldots - s_l) - \psi_{j,m}(x - s_1 - \ldots - s_l)) \, ds_1 \ldots ds_l
\]

\[
= - \sum_{s=0}^{2N-1} \beta_{j,s-m} \frac{1}{(2b)^l} \int_{-b}^b \cdots \int_{-b}^b \phi_{j,s}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
+ \frac{(-1)^m}{2N} \sum_{s=0}^{2N-1} (-1)^s \frac{1}{(2b)^l} \int_{-b}^b \cdots \int_{-b}^b \psi_{j,s}(x - s_1 - \ldots - s_l) \, ds_1 \ldots ds_l
\]

\[
= - \sum_{s=0}^{2N-1} \beta_{j,s-m} \phi_{j,s}^l + \frac{(-1)^m}{2N} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}^l.
\]

In the next Lemma we use for \( j,l \in \mathbb{N}_0 \) and \( k = 0, \ldots, 2N - 1 \) the abbreviations

\[
S_k^l = S_{j,k}^l := \phi_{j,k}^l + \psi_{j,k}^l
\]

and

\[
M_k^l = M_{j,k}^l := \phi_{j,k}^l - \psi_{j,k}^l.
\]

**Lemma 3.3** For \( l \in \mathbb{N}_0 \) and \( m = 0, \ldots, 2N - 1 \) we have

\[
\phi_{j+1,2m}^l - \phi_{j+1,2m-2}^l = \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} S_k^m - S_{m-1}^m
\]

\[
\quad + \frac{1}{2N} \sum_{k=0}^{2N-1} (-1)^{m-k} S_k^m.
\]

\[
\phi_{j+1,2m+1}^l - \phi_{j+1,2m-1}^l = \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} M_k^m
\]

\[
\quad + \frac{1}{2N} \sum_{k=0}^{2N-1} (-1)^{m-k} M_k^m.
\]
and therefore
\[
\phi^{j+1,2m} + \phi^{j+1,2m+1} - \phi^{j+1,2m-2} - \phi^{j+1,2m-1}
\]
\[
= \frac{2}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} \phi^{j+1,2m}_{j,k} - \phi^{j+1,m}_{j,m-1} + \phi^{j+1,m}_{j,m} - \phi^{j+1,2m-1}_{j,m-1}
\]
\[
+ \frac{1}{N} \sum_{k=0}^{2N-1} (-1)^{m-k} \phi^{j+1,2m}_{j,k}.
\]  
(37)

Proof  Using (17), (18) and (26) we obtain
\[
S^l_k = \phi^{j+1,2k+2} + \frac{1}{N} \sum_{s=0}^{2N-1} \Delta_{j,s} \phi^{j+1,2k+2}_{j,s}
\]
\[
= \phi^{j+1,2k+2} + \frac{1}{N} \sum_{s=0}^{2N-1} \Delta_{j,k-s} \phi^{j+1,2k+2}_{j+1,s}.
\]  
(38)

and
\[
M^l_k = \phi^{j+1,2k+1} + \frac{1}{N} \sum_{s=0}^{2N-1} \Delta_{j,s} \phi^{j+1,2k+1}_{j,s+1}
\]
\[
= \phi^{j+1,2k+1} + \frac{1}{N} \sum_{s=0}^{2N-1} \Delta_{j,k-s} \phi^{j+1,2k+1}_{j+1,s+1}.
\]  
(39)

Furthermore, for \( j \in \mathbb{N}_0 \) and \( k = 0, 1, \ldots, 2N-1 \) we compute with (15)
\[
\frac{1}{N} \sum_{s=0}^{2N-1} \Delta_{j,s} \Delta_{j,s-k} = N \delta_{k,0} - \frac{1}{2}(-1)^k.
\]  
(40)

With (40) and (38) we obtain for \( m = 0, 1, \ldots, 2N-1 \) that
\[
\frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} S^l_k
\]
\[
= \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} \phi^{j+1,2k+2}_{j,k} + \frac{1}{N} \sum_{k=0}^{2N-1} \left( \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k} \Delta_{j,k-s} \right) \phi^{j+1,2k}_{j,s}
\]
\[
= \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,(m-1)-k} \phi^{j+1,2k+2}_{j,k} + \frac{1}{N} \sum_{k=0}^{2N-1} \left( \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k} \Delta_{j,k+m-s} \right) \phi^{j+1,2k}_{j,s}
\]
\[
= S^l_{m-1} - \phi^{j+1,2m-2}_{j+1,2m-2} + \phi^{j+1,2m}_{j+1,2m} - \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^{s-m} \phi^{j+1,2s}_{j+1,2s},
\]
which implies
\[
\begin{align*}
\phi_{j+1,2m} - \phi_{j+1,2m-2} &= \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k} \phi_k^{j+1} - \phi_{m-1}^{j+1} + \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^{s-m} \phi_{j+1,2s}^{j+1,2s}. \\
&= \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k} \phi_k^{j+1} - \phi_{m-1}^{j+1} + \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^{s-m} \phi_{j+1,2s+1}^{j+1,2s+1}. 
\end{align*}
\] (41)

Analogously, with (39) and (40) we obtain
\[
\begin{align*}
\phi_{j+1,2m+1} - \phi_{j+1,2m-1} &= M^j_m - \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k} M_k^j + \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^{s-m} \phi_{j+1,2s+1}^{j+1,2s+1}.
\end{align*}
\] (42)

Again from (16) and
\[
\cos Nx_{j+1,2s} = (-1)^s, \quad \cos Nx_{j+1,2s+1} = 0 \quad \text{for } s = 0, \ldots, 2N - 1
\]
we obtain
\[
\cos N(x - t) = \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s D_{2N}^j (x - t - x_{j+1,2s})
\] (43)

and from
\[
\sin Nx_{j+1,2s} = 0, \quad \sin Nx_{j+1,2s+1} = (-1)^s \quad \text{for } s = 0, \ldots, 2N - 1
\]
we obtain
\[
\sin N(x - t) = \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s D_{2N}^j (x - t - x_{j+1,2s+1}).
\] (44)

Integrating (43) and (44) with respect to \( t \) on the interval \([0, \frac{\pi}{N}]\) and using Lemma 3.1 deduce
\[
\begin{align*}
\frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s \phi_{j+1,2s}(x) &= \frac{1}{N\pi} (\sin Nx + \cos Nx) \\
&= \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s S_s(x)
\end{align*}
\]

and
\[
\begin{align*}
\frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s \phi_{j+1,2s+1}(x) &= \frac{1}{N\pi} (\sin Nx - \cos Nx) \\
&= \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s M_s(x),
\end{align*}
\]
which implies for arbitrary \( l \in \mathbb{N}_0 \) the representations

\[
\frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s \phi_{j+1,2s}(x) = \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s S_s(x)
\]  

(45)

and

\[
\frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s \phi_{j+1,2s+1}(x) = \frac{1}{2N} \sum_{s=0}^{2N-1} (-1)^s M_s(x)
\]  

Putting (45) and (46) in (41) and (42) we obtain the assertion of the Lemma.

Now we are able to show the decomposition formulas.

**Proof of Theorem 3.2** Let \( m \) be an integer with \( 0 \leq m \leq 2N - 1 \). Note that

\[
\sum_{s=0}^{4N-1} \phi_{j+1,s+2m} = \sum_{s=0}^{2N-1} \phi_{j,s} = 1.
\]

Therefore

\[
\phi_{j+1,2m} = \phi_{j+1,2m} - \frac{1}{4N} \sum_{s=0}^{4N-1} \phi_{j+1,s+2m} + \frac{1}{4N} \sum_{s=0}^{2N-1} \phi_{j,s}
\]

\[
= \frac{1}{4N} \sum_{s=0}^{2N-1} \phi_{j,s} + \phi_{j+1,2m} - \frac{1}{4N} \sum_{s=0}^{2N-1} (\phi_{j+1,2s+2m} + \phi_{j+1,2s+2m+1}).
\]

With \( \lambda_{j,s} \) defined in (25) and \( j \in \mathbb{N}_0 \) we use Abel summation

\[
\sum_{s=0}^{2N-1} \alpha_s = 2N \alpha_0 - \sum_{s=0}^{2N-1} \lambda_{j,s}(\alpha_s - \alpha_{s-1}).
\]

which implies for \( \alpha_s := \phi_{j+1,2s+2m} + \phi_{j+1,2s+2m+1} \) the representation

\[
\phi_{j+1,2m} = \frac{1}{4N} \sum_{s=0}^{2N-1} \phi_{j,s} + \phi_{j+1,2m} - \frac{1}{2} (\phi_{j+1,2m} + \phi_{j+1,2m+1})
\]

\[
+ \frac{1}{4N} \sum_{s=0}^{2N-1} \lambda_{j,s}(\phi_{j+1,2s+2m} + \phi_{j+1,2s+1+2s}
\]

\[
- \phi_{j+1,2m+2s} - \phi_{j+1,2m+1+2s}).
\]
\[
\frac{1}{4N} \sum_{s=0}^{2N-1} \phi_j, s + \frac{1}{2} (\phi_{j+1,2m} + \phi_{j+1,2m+1}) \\
+ \frac{1}{4N} \sum_{s=0}^{2N-1} \lambda_{j, s} m \phi_{j+1,2s+1} - \phi_{j+1,2s+1}.
\]

Hence,
\[
\phi_{j+1,2m+1} = \phi_{j+1,2m+1} - \phi_{j+1,2m} + \phi_{j+1,2m} \\
= \frac{1}{4N} \sum_{s=0}^{2N-1} \phi_{j, s} + \frac{1}{4N} \sum_{s=0}^{2N-1} \lambda_{j, s} m (\phi_{j+1,2s+1} - \phi_{j+1,2s+1}) - \frac{1}{2} (\phi_{j+1,2m} - \phi_{j+1,2m+1}).
\]

Now using Lemma 3.2 and equation (37) we obtain from (47) and (48) the decomposition formulas.

4. Algorithms

We start with the algorithm for decomposition. For \( j \in \mathbb{N}_0 \) let functions \( f_j \in V_j \) and \( g_j \in W_j \) be given as
\[
f_j(x) = \sum_{k=0}^{2N-1} C_{j,k} \phi_{j,k}(x)
\]
and
\[
g_j(x) = \sum_{k=0}^{2N-1} D_{j,k} \psi_{j,k}(x).
\]

By the basis property the sequences of coefficients \((C_{j,k})_{k=0}^{2N-1}\) and \((D_{j,k})_{k=0}^{2N-1}\) determine the functions \( f_j \) and \( g_j \) uniquely. With the help of these sequences of coefficients we can describe the algorithm for the decomposition of \( f_{j+1} \in V_{j+1} \) in the orthogonal sum of \( f_j \in V_j \) and \( g_j \in W_j \). Writing for \( s = -4N + 2, \ldots, 4N - 1 \)
\[
A^j_s := c_{j, -\lfloor \frac{s}{2}\rfloor} - \frac{1}{2} (-1)^{s} \beta_{j, -\lfloor \frac{s}{2}\rfloor}
\]
and
\[
B^j_s := d_{j, -\lfloor \frac{s}{2}\rfloor} + (-1)^{s} \left( \frac{(-1)^{-\lfloor \frac{s}{2}\rfloor}}{4N} + \frac{1}{2} \gamma_{0, -\lfloor \frac{s}{2}\rfloor} \right)
\]
Trigonometric Wavelets

we can reformulate (27) and (28) for \( s = 0, \ldots, 4N - 1 \) as

\[
\phi_{j+1,s}^s(x) = \sum_{k=0}^{2N-1} (A_{s-2k}^j \phi_{j,k}^s(x) + B_{s-2k}^j \psi_{j,k}^s(x)).
\]

Then

\[
f_{j+1}(x) = f_j(x) + g_j(x)
\]
can be written as

\[
\begin{align*}
2N-1 \sum_{k=0} C_{j,k} \phi_{j,k}^s(x) + \sum_{k=0}^{2N-1} D_{j,k} \psi_{j,k}^s(x) &= \sum_{s=0}^{4N-1} C_{j+1,s} \left( \sum_{k=0}^{2N-1} A_{s-2k}^j \phi_{j,k}^s(x) + B_{s-2k}^j \psi_{j,k}^s(x) \right) \\
&= \sum_{k=0}^{2N-1} \left( \sum_{s=0}^{4N-1} C_{j+1,s} A_{s-2k}^j \phi_{j,k}^s(x) \right) + \sum_{k=0}^{2N-1} \left( \sum_{s=0}^{4N-1} C_{j+1,s} B_{s-2k}^j \psi_{j,k}^s(x) \right).
\end{align*}
\]

The linear independence of the basis functions \( \phi_{j,k}^s \) and \( \psi_{j,k}^s \) implies that the coefficients of \( f_j \) and \( g_j \) are uniquely determined and we obtain for \( k = 0, \ldots, 2N - 1 \) that

\[
C_{j,k} = \sum_{s=0}^{4N-1} C_{j+1,s} A_{s-2k}^s,
\]

\[
D_{j,k} = \sum_{s=0}^{4N-1} C_{j+1,s} B_{s-2k}^s.
\]

Now we want to describe the opposite direction. With the help of the sequence of coefficients \((C_{j,k})_{k=0}^{2N-1}\) and \((D_{j,k})_{k=0}^{2N-1}\) we reconstruct a function \( f_{j+1} \in V_{j+1} \) out of \( f_j \in V_j \) and \( g_j \in W_j \). For \( k = -4N + 2, \ldots, 4N - 1 \) we define using (19) and (20)

\[
P_{k}^j := \frac{1}{2N} R_j(x_{j+1,k})
\]

and

\[
Q_{k}^j := \frac{1}{2N} \tilde{R}_j(x_{j+1,k}).
\]
Then the two-scale relations (17) and (18) for $s = 0, \ldots, 2N - 1$ can be written as

$$
\phi_{j,s}(x) = \sum_{k=0}^{4N-1} P_{k-2s} \phi_{j+1,k}(x),
$$

$$
\psi_{j,s}(x) = \sum_{k=0}^{4N-1} Q_{k-2s} \phi_{j+1,k}(x).
$$

That means we can reformulate

$$
f_{j+1}(x) = f_j(x) + g_j(x)
$$

as

$$
\sum_{k=0}^{4N-1} C_{j+1,k} \phi_{j+1,k}(x)
= \sum_{s=0}^{2N-1} C_{j,s} \left( \sum_{k=0}^{4N-1} P_{k-2s} \phi_{j+1,k}(x) \right) + \sum_{s=0}^{2N-1} D_{j,s} \left( \sum_{k=0}^{4N-1} Q_{k-2s} \phi_{j+1,k}(x) \right)
= \sum_{k=0}^{4N-1} \left( \sum_{s=0}^{2N-1} C_{j,s} P_{k-2j} + D_{j,s} Q_{k-2j} \right) \phi_{j+1,k}(x).
$$

By comparison of coefficients for $k = 0, \ldots, 4N - 1$ we finish with

$$
C_{j+1,k} = \sum_{s=0}^{2N-1} C_{j,s} P_{k-2j} + D_{j,s} Q_{k-2s}.
$$

Finally we want to describe these algorithms in matrix notation which enables us also to introduce the Fast Fourier Transform methods in a simple manner. For $l \in \mathbb{N}_0$ we define the vectors of translates of the level $j$

$$
\Phi_{j,l} := (\phi_{j,s})_{s=0}^{2^{2N-1}} \quad \text{and} \quad \Psi_{j,l} := (\psi_{j,s})_{s=0}^{2^{2N-1}}.
$$

Furthermore, let $U_{n+1}$ be the permutation matrix which sorts the elements of the vector $\Phi_{j+1,l}$ starting with the elements with even index followed by the elements with odd index, i.e.,

$$
U_{j+1} \Phi_{j+1,l} = \begin{pmatrix}
(\phi_{j+1,2s})_{s=0}^{2^{N-1}} \\
(\phi_{j+1,2s+1})_{s=0}^{2^{N-1}}
\end{pmatrix}.
$$
With the notations (51), (52), and with the matrix

\[
C_{j+1} := \begin{pmatrix}
(P_n^{2^s-2k})_{k=0,s=0}^{2N-1} & (P_j^{2^s+1-2k})_{k=0,s=0}^{2N-1} \\
(Q_j^{2^s-2k})_{k=0,s=0}^{2N-1} & (Q_j^{2^s+1-2k})_{k=0,s=0}^{2N-1}
\end{pmatrix}
\]  

(53)

we rewrite Theorem 3.1 as

\[
\begin{pmatrix}
\Phi_{j,l} \\
\Psi_{j,l}
\end{pmatrix} = C_{j+1} U_{j+1} \Phi_{j+1,l}.
\]

Considering the decomposition relation from Theorem 3.2 we obtain from (49), (50) with the matrix

\[
D_{j+1} := \begin{pmatrix}
(A_j^{2^s-2k})_{k=0,s=0}^{2N-1} & (B_j^{2^s+1-2k})_{k=0,s=0}^{2N-1} \\
(A_j^{2^s+1-2k})_{k=0,s=0}^{2N-1} & (B_j^{2^s+1-2k})_{k=0,s=0}^{2N-1}
\end{pmatrix}
\]

the representation

\[
U_{j+1} \Phi_{j+1,l} = D_{j+1} \begin{pmatrix}
\Phi_{j,l} \\
\Psi_{j,l}
\end{pmatrix}.
\]

Writing

\[
P_j := (P_j^{2^s})_{k=0,s=0}^{2N-1} \quad \text{and} \quad A_j := (A_j^{2^s})_{k=0,s=0}^{2N-1}
\]

we summarize our representations in the following result.

**Theorem 4.1** The coefficient matrices \( C_{j+1} \) and \( D_{j+1} \) for reconstruction and decomposition are of the following block structure

\[
C_{j+1} = \begin{pmatrix}
P_j & P_j^T \\
P_j & -P_j^T
\end{pmatrix}
\]

and

\[
D_{j+1} = \begin{pmatrix}
A_j & A_j \\
A_j^T & -A_j^T
\end{pmatrix}.
\]

**Proof** By (53) the equality for \( C_{j+1} \) is equivalent with

\[
P_j^{2^s+1-2k} = P_j^{2^s-2k},
\]

\[
Q_j^{2^s+1-2k} = -Q_j^{2^s-2k},
\]

\[
Q_j^{2^s-2k} = P_j^{2^s-2k}.
\]
for \( s, k = 0, \ldots, 2N - 1 \). These scalar equations can be proved by (51) and (19)

\[
P_{s+k}^{j} = \frac{1}{2N} \left( D_N^j(x_{j+1,2s+1-k}) + D_N^j(x_{j+1,2s+1-2k}) \right) = \frac{1}{2N} \left( D_N^j(x_{j+1,2k-s-1}) + D_N^j(x_{j+1,2k-2s}) \right) = P_{2k-2s}^j
\]

and analogously by (52) and (20) for \( r := 2s - 2k, \ (-4N+1 < r < 4N-1) \) with

\[
Q_{2r}^j = \frac{1}{2N} \left( D_N^j(x_{j,r}) - D_N^j(x_{j,r} - x_{j+1,1}) - D_N^j(x_{j,r} - x_{j+1,1}) \right)
= \begin{cases}
\frac{1}{2} + \frac{1}{2N} D_N^j(x_{j+1,1}), & \text{if } r \equiv 0 \mod 2N; \\
\frac{1}{2N} D_N^j(x_{j,r} - x_{j+1,1}), & \text{if } l \not\equiv 0 \mod 2N;
\end{cases}
\]

\[
= \frac{1}{2N} \left( D_N^j(x_{j,r}) + D_N^j(x_{j,r} - x_{j+1,1}) \right) = P_{2r}^j.
\]

The second part can be seen immediately from the fact that \( D_{j+1} \) is the inverse matrix of \( C_{j+1} \). Indeed, using the identity matrix \( I_{j+1} \) of order \( 4N \) we have

\[
I_{j+1} = C_{j+1} D_{j+1} = \begin{pmatrix} P_j & P_j^T \\ P_j & -P_j^T \end{pmatrix} \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}.
\]

Solving the linear system out of (54) we obtain

\[
A_j = D_{11} = \frac{1}{2} P_j^{-1},
\]

\[
D_{21} = D_{11}^T, \\
D_{12} = D_{11}, \\
D_{22} = -D_{11}^T,
\]

which proves the Theorem. \( \blacksquare \)

It is easy to see that the matrices \( P_j \) and \( A_j \) are circulant matrices, i.e.

\[
P_j = \text{circ} \ (P_{2s})_{s=0}^{2N-1}, \quad A_j = \text{circ} \ (A_{4N-2s})_{s=0}^{2N-1}.
\]
Remember that for a vector \( a = (a_0, a_1, \ldots, a_{N-1})^T \) the circulant matrix constructed from \( a \) is given by

\[
\text{circ } a := \{ a_{(k-s)\mod N} \}_{s=0, k=0}^{N-1}
\]

\[
= \{ a_{s,k} \}_{s=0, k=0}^{N-1} := \begin{pmatrix}
    a_0 & a_1 & \cdots & a_{N-1} \\
    a_{N-1} & a_0 & \cdots & a_{N-2} \\
    \vdots & \vdots & \ddots & \vdots \\
    a_1 & a_2 & \cdots & a_0
\end{pmatrix}.
\]

For more details cf. Davis [4]. Particularly important is the factorization of a circulant matrix by Fourier matrices

\[
F_N = \frac{1}{\sqrt{N}} \left( w_N^{sk} \right)_{k=0, s=0}^{N-1} \quad \text{with} \quad w_N = e^{-\frac{2\pi i}{N}}
\]

so that one obtains for an \( N \times N \) circulant matrix \( Y \) the diagonal matrix

\[
F_N Y F_N = \text{diag}(Y),
\]

which contains on the main diagonal the eigenvalues \( d_r(Y) \) \( (r = 0, \ldots, N-1) \) of \( Y \). Finally, for applying fast algorithms we only have to determine these eigenvalues.

**Lemma 4.1** The matrices \( P_j \) and \( A_j \) are uniquely determined by the vector of their eigenvalues

\[
d_r(P_j) = \begin{cases} 
    1, & \text{if } r = 0, \\
    \frac{1}{2} + \frac{1}{2} e^{\frac{2\pi i r}{N}}, & \text{if } 1 \leq r \leq N-1, \\
    \frac{1}{2}, & \text{if } r = N, \\
    \frac{1}{2} - \frac{1}{2} e^{\frac{2\pi i r}{N}}, & \text{if } N+1 \leq r \leq 2N-1,
\end{cases}
\]

and

\[
d_r(A_j) = \begin{cases} 
    \frac{1}{2}, & \text{if } r = 0, \\
    \frac{1}{1 + e^{2\pi i r/N}}, & \text{if } 1 \leq r \leq N-1, \\
    1, & \text{if } r = N, \\
    \frac{1}{1 - e^{2\pi i r/N}}, & \text{if } N+1 \leq r \leq 2N-1.
\end{cases}
\]

**Proof** One easily computes the eigenvalues \( d_r(P_j) \) \( (r = 0, \ldots, 2N-1) \)
by (cf. [4])

\[
d_r(P_j) = \frac{1}{2N} \sum_{s=0}^{2N-1} P^j_s e^{i\frac{2\pi s}{2N}}
\]

\[
= \frac{1}{2N} \sum_{s=0}^{2N-1} \left( N\delta_{0,s} + D^j_s (x_{j,s} - x_{j+1,1}) \right) e^{i\frac{2\pi s}{2N}}
\]

\[
= \frac{1}{2} + \frac{1}{2N} \sum_{s=0}^{2N-1} \left( \frac{1}{2} + \sum_{m=1}^{N-1} \cos \left( \frac{m\pi s}{N} - \frac{m\pi}{2N} \right) \right)
+ \frac{1}{2} \cos \left( \frac{N\pi s}{N} - \frac{N\pi}{2N} \right) \left( \cos \frac{r\pi s}{N} + i \sin \frac{r\pi s}{N} \right).
\]

The addition formulas together with

\[
\sum_{s=0}^{2N-1} \cos \left( x - \frac{sk\pi}{N} \right) = \begin{cases} 2N \cos x, & \text{if } k \equiv 0 \mod 2N, \\ 0, & \text{otherwise}, \end{cases}
\]

and

\[
\sum_{s=0}^{2N-1} \sin \left( x - \frac{sk\pi}{N} \right) = \begin{cases} 2N \sin x, & \text{if } k \equiv 0 \mod 2N, \\ 0, & \text{otherwise}, \end{cases}
\]

imply the assertion for \( d_r(P_j) \).

Taking into account (55) we obtain the eigenvalues of the matrix \( A_j \). Namely with \( F_{2N} = F_{2N}^{-1} \) we have

\[
F_{2N} \text{ diag } d(A_j) \ F_{2N} = \frac{1}{2} (F_{2N} \text{ diag } d(P_j) \ F_{2N}^{-1})^{-1}
\]

\[
= \frac{1}{2} F_{2N} (\text{diag } d(P_j))^{-1} F_{2N},
\]

which finally yields

\[
d_r(A_j) = \frac{1}{2 d_r(P_j)} \quad \text{for } r = 0, \ldots, 2N - 1.
\]

Let us finish this section with the investigation of the basis transformations between different bases

\[
(\phi^j_{s,t}(x))_{s=0}^{2N-1} \quad \text{and} \quad (\phi^{j-m}_{s,t}(x))_{s=0}^{2N-1}
\]

of the same space \( V_j \). That means we seek for the matrix \( T_{j,m} = (T^j_{s,k})_{s=0}^{2N-1}, k=0 \) with

\[
\Phi_{j,l} = T_{j,m} \Phi_{j,l-m}.
\]
Again the circulant structure of the matrix $T_{j,m}$ is the main ingredient of our considerations.

**Lemma 4.2** The matrix $T_{j,m} = F_{2N} \text{diag}(T_{j,m}) F_{2N}$ has the entries

$$t_{j,m}^{i,k} = \frac{1}{2N} \sum_{r=0}^{2N-1} \text{sinc}^m r b \ e^{\frac{i \pi (r-i) x}{N}}$$

with eigenvalues

$$d_r(T_{j,m}) = \text{sinc}^m r b.$$

**Proof.** Considering for $s = -N, \ldots, N$ the complex Fourier coefficients $c_s(\Phi_j^l)$ and $c_s(\Phi_j^{l-m})$ we obtain with (11), (12) and

$$\Phi_j^l(x - x_{j,q}) = (\Phi_j^{l-m} \ast \tilde{H}_m)(x - x_{j,q})$$

$$= \sum_{k=0}^{2N-1} t_{q,k}^{j,m} \Phi_j^{l-m}(x - x_{j,k}) \text{ for } q = 0, \ldots, 2N - 1,$$

that for $s = -N, \ldots, N$

$$c_s(\Phi_j^l \cdot - \frac{\pi q}{N}) = 2\pi c_s(\Phi_j^{l-m}) c_s(\tilde{H}_m) e^{-\frac{i\pi r s}{N}}$$

$$= \sum_{k=0}^{2N-1} t_{q,k}^{j,m} c_s(\Phi_j^{l-m}) e^{-\frac{i\pi r s}{N}}.$$

Hence

$$\sum_{k=0}^{2N-1} t_{q,k}^{j,m} e^{-\frac{i\pi r s}{N}} = 2\pi c_s(\tilde{H}_m) = \text{sinc}^m s b$$

which means that we have computed the eigenvalues of $T_{j,m}$ so that

$$d_r(T_{j,m}) = \sum_{k=0}^{2N-1} t_{q,k}^{j,m} e^{\frac{i\pi r s}{N}} = \text{sinc}^m r b.$$

By discrete Fourier transform the matrix $T_{j,m}$ has the entries

$$t_{q,k}^{j,m} = \frac{1}{2N} \sum_{r=0}^{2N-1} d_r(T_{j,m}) e^{\frac{i \pi (r-i) x}{N}}.$$

Let us finally mention that the inverse relation reads as

$$\Phi_j^{l-m} = T_{j,m}^{-1} \Phi_j^l.$$
with \( T_{jm}^{-1} = {\cal F}^{-1}_{2N} \text{diag}(d_{\ell}(T_{jm})^{-1})_{\ell=0}^{2N-1} {\cal F}_{2N} \).

References


