Wavelet and Allied Topics
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10. Trigonometric Wavelets

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1. Introduction

Trigonometric polynomials and the approximation of periodic functions by polynomials play an important role in harmonic analysis. Here we are interested in constructing time-localized bases for certain spaces of trigonometric polynomials. Adapting basic ideas of wavelet theory, translation invariant and orthonormal bases are employed, both of which are constructed from equidistant shifts of a single polynomial. One of our main goals is the investigation of basis transformations in a form that facilitates fast algorithms. We observe that the corresponding transformation matrices have a circulant structure and can be diagonalized by Fourier matrices. The resulting diagonal matrices contain the eigenvalues, which are computed explicitly. So, the algorithms can be easily realized using the Fast Fourier Transform (FFT).

Further, we consider the nesting of the sample spaces to obtain multiresolution analyses (MRA's). For the resulting orthogonal wavelet spaces, we proceed as above and find wavelet bases consisting of translates of a single polynomial. Again, translation invariant and orthonormal bases are constructed, which show the same time-frequency behaviour as the sample bases do. The basis transformations can be described analogously by circulant matrices.

Most important for practical reasons are the decomposition of signals in frequency bands, which correspond to the wavelet spaces, and their reconstruction. Focusing on basis transformations, the two-scale relations and decomposition formulas are also given in matrix notation suitable for the use of FFT methods. Then the transformations for signal data follow easily.

Let us start with the explanation of time localization and frequency localization for trigonometric polynomials. If we consider a polynomial $p_n \in \Pi_n$ then

$$p_n(x) = \sum_{k=-n}^{n} c_k(f)e^{ikx}$$

is a basis representation in terms of the naturally best frequency localized basis $\{e^{ikx}\}$. The coefficients $c_k(f)$ turn out to be the Fourier coefficients

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) e^{-ikt} dt.$$

On the other hand, if we want to write the polynomial p_n in a basis such that the coefficients give the most time localized information we end up with a representation of p_n as a Lagrange interpolation polynomial

$$p_n(x) = \frac{1}{n} \sum_{k=-n+1}^{n} p_n\left(\frac{k\pi}{n}\right) D_n^*\left(x - \frac{k\pi}{n}\right).$$

Here D_n^* is the modified Dirichlet kernel

$$\begin{aligned} D_n^*(x) &= \frac{1}{2} \sum_{k=-n+1}^{n-1} e^{ikx} + \frac{e^{inx} + e^{-inx}}{4} \\ &= \begin{cases} \dfrac{\sin nx}{2 \tan \frac{x}{2}}, & \text{if } x \notin 2\pi\mathbb{Z}, \\ n, & \text{if } x \in 2\pi\mathbb{Z}. \end{cases} \end{aligned}$$

Obviously, the coefficients $p_n\left(\frac{k\pi}{n}\right)$ yield the most time localized information of p_n.
These two bases namely

$$\{e^{ikx}\}_{k=-n}^{n} \qquad \text{and} \qquad \left\{D_n^*\left(x - \frac{k\pi}{n}\right)\right\}_{k=-n+1}^{n},$$

could be seen as extreme cases of time- and frequency localized bases in corresponding spaces. As one can assume there is no basis which is optimal in both senses. However, one could try to find a compromise although the different uncertainty principles behind are not the topic of this article (cf. [1], [2], [8], [12], [13]). Our aim is to construct a couple of bases consisting of equidistant translates of one underlying function with different properties.
The basic idea is the orthogonal decomposition

$$L_{2\pi}^2 = W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \dots .$$

Hence, every $f \in L_{2\pi}^2$ can be uniquely decomposed into the sum

$$f = \sum_{j=-1}^{\infty} f_j$$

with $f_j \in W_j, j = -1, 0, 1, 2, \dots$. How to choose now these spaces W_j ? The general idea is to compose W_k from functions living mainly in a certain frequency band. However, to ensure also time localization one needs now basis elements $\psi_{j,s}$ in W_j which are time–localized as much as possible. A particular interesting constellation is given by

$$\psi_{j,s}(x) = \Psi_j\left(x - \frac{s\pi}{2^j}\right), \qquad s = 0, \dots, 2^{j+1} - 1,$$

and

$$W_j = \text{span}\{\psi_{j,s} \;:\quad s = 0, \dots, 2^{j+1} - 1\}. \tag{1}$$

For different purposes it is also essential to put together some wavelet spaces to build new spaces. We write for $j = -1, 0, 1, \dots$

$$V_{j+1} = W_{-1} \oplus W_0 \oplus W_1 \oplus W_2 \oplus \dots \oplus W_j \; .$$

In applications one starts often with a projection of a given function into V_j . Therefore it is important to know bases for V_j . As usual we call a function Φ_j scaling function if the

$$\phi_{j,s}(x) = \Phi_j\left(x - \frac{s\pi}{2^j}\right), \quad s = 0, \dots, 2^{j+1} - 1,$$

constitute a basis for V_j , i.e.,

$$V_j = \text{span}\{\phi_{j,s} \;:\quad s = 0, \dots, 2^{j+1} - 1\}. \tag{2}$$

Now we want to discuss the possibilities of decomposition and reconstruction. First we start with a function $\Phi_{j+1} \in L^2_{2\pi}$ spanning V_{j+1} . The question whether we can decompose V_{j+1} into an orthogonal sum of V_j and W_j spanned by the translates of Φ_j and Ψ_j we can answer in the following affirmative way.

Theorem 1.1 Let $\Phi_{j+1} \in L^2_{2\pi}$ be given.

a) The translates $\phi_{j+1,s}$, $s = 0, \dots, 2^{j+2} - 1$ are linearly independent if and only if the Fourier coefficients

$$c_k = c_k(\Phi_{j+1})$$

satisfy

$$\sum_{m=-\infty}^{\infty} \mid c_{k+m \cdot 2^{j+2}} \mid^2 \; > \; 0 \quad \text{for all} \quad k = 0, \dots, 2^{j+2} - 1. \tag{3}$$

b) If (3) is satisfied then the space V_{j+1} of dimension 2^{j+2} spanned by the 2^{j+2} equidistant translates can be decomposed into an orthogonal sum

$$V_{j+1} = V_j \oplus W_j ,$$

where V_j and W_j are both of dimension 2^{j+1} and spanned by translates of some functions Φ_j and Ψ_j , respectively (cf. (1), (2)).

A proof can be found in [15] or [10]. Note that the decomposition is not uniquely determined.

Unfortunately, the converse of this decomposition theorem is not true. i.e., if one starts with two arbitrary spaces V_j and W_j of dimension 2^{j+1} and of the form (1) and (2) respectively, then $V_j \oplus W_j$ is not necessarily spanned by the 2^{j+2} equidistant translates of one function. We give here the following counter example from [15]. Let $j = 0$ and

$$\Phi_0(x) = \sum_{k=-3}^{0} e^{ikx}, \qquad \Psi_0(x) = \sum_{k=1}^{3} e^{ikx}.$$

Then

$$\begin{array}{lclcl} V_0 & = & \operatorname{span}\{\Phi_0,\ \Phi_0(\circ - \pi)\} & = & \operatorname{span}\{1 + e^{-2i\circ},\ e^{-i\circ} + e^{-3i\circ}\}, \\ W_0 & = & \operatorname{span}\{\Psi_0,\ \Psi_0(\circ - \pi)\} & = & \operatorname{span}\{e^{2i\circ},\ e^{i\circ} + e^{3i\circ}\}. \end{array}$$

Easily one can see $V_0 \perp W_0$. Writing

$$V_1 = V_0 \oplus W_0$$

the question is whether V_1 is spanned by a function Φ_1 and its 3 translates

$$V_1 = \operatorname{span}\left\{\Phi_1,\ \Phi_1\left(\circ - \frac{\pi}{2}\right),\ \Phi_1\ (\circ - \pi),\ \Phi_1\left(\circ - \frac{2\pi}{2}\right)\right\}. \tag{4}$$

Assume now that (4) is true. Then it follows for every $f \in V_1$ that the function $f\left(\circ - \frac{3\pi}{2}\right)$ is an element of V_1.
In particular, $f = e^{i\circ} + e^{3i\circ} \in W_0 \subset V_1$ but

$$f\left(\circ - \frac{3\pi}{2}\right) = -i(e^{i\circ} - e^{3i\circ}) \notin V_1 .$$

Hence, (4) is not satisfied, which means that reconstruction is not always possible. Therefore it is not the usual way to start with wavelet functions and to construct scaling functions out of them.

2. Modified Fourier Sums of Spline Functions

In this section we present counter examples of bases for particular polynomial spaces V_j and W_j . Other examples of trigonometric poynomial

wavelets are studied e.g. in [12], [13], [14]. More informations on the general construction principles of periodic wavelets can be found in [6], [8], [10], [15].
In the following we write $N = 2^j$ for $j \in \mathbb{N}_0$. For simplicity we choose as a starting point the following definition of polynomial spaces of scaling functions

$$V_j := \text{ span } \{1, \cos x, \ldots, \cos(N-1)x, \sin x, \ldots, \sin Nx\}. \tag{5}$$

Note that this definition is not of the form (2). However, we will show a representation of that kind in the sequel.
Note further that the examples presented here depend essentially on the fact that $\sin N\circ$ is included and $\cos N\circ$ is excluded. One could exchange the frequency $\sin N\circ$ by $\cos N\circ$ in the definition of V_j . Spaces of that kind are discussed in detail in [11], [13], [15].

Easily we obtain the following two properties which are satisfied for these spaces of polynomials

$$\begin{aligned} V_j &\subset V_{j+1}, \qquad j \in \mathbb{N}_0, \\ L^2_{2\pi} &= \operatorname{clos}_{L^2_{2\pi}} \left\{ \bigcup_{j=0}^{\infty} V_j \right\}. \end{aligned}$$

Analogously we define for $j \in \mathbb{N}_0$

$$W_j := \text{ span } \{\cos Nx, \ldots, \cos(2N-1)x, \sin(N+1)x, \ldots, \sin 2Nx\}. \tag{6}$$

The spaces V_j and W_j for $j \in \mathbb{N}_0$ have dimension $2N$ and it holds

$$\begin{aligned} V_{j+1} &= V_j \oplus W_j, \\ L^2_{2\pi} &= \operatorname{clos}_{L^2_{2\pi}} \left\{ V_0 \oplus \bigoplus_{j=0}^{\infty} W_j \right\}. \end{aligned}$$

Now we want to construct scaling functions and wavelets with the properties (2) and (1), respectively. This approach goes back to the paper of Chui and Mhaskar [3] and the generalizations presented here are taken from Ihsberner [5].

For shortness we write in the sequel

$$x_{j,k} := \frac{2\pi k}{2N}, \qquad k = 0, 1, \ldots, 2N-1.$$

A typical candidate for the scaling function seems to be the modified Dirichlet kernel and its translates which is motivated by the localization properties

$$D_N^*\left(\frac{2\pi k}{2N}\right) = \begin{cases} 0, & \text{if } k \not\equiv 0 \bmod 2N, \\ N, & \text{if } k \equiv 0 \bmod 2N. \end{cases} \tag{7}$$

However, D_N^* is not an element of V_j as defined here. To overcome this difficulty we proceed here by integrating the modified Dirichlet kernel. Therefore we define scaling functions for $s = 0, \ldots, 2N-1$ by

$$\phi_{j,s}(x) \quad := \quad \Phi_j(x - x_{j,s})$$

with

$$\Phi_j(x) \quad := \quad \frac{1}{\pi} \int_0^{x_{j,1}} D_N^*(x-t)\, dt. \tag{8}$$

By the help of the convolution of periodic functions we can write

$$\Phi_j(x) = \frac{1}{\pi}(h_j * D_N^*)(x),$$

where the 2π-periodic function h_j is given by

$$h_j(x) \quad := \quad \begin{cases} 1, & \text{if } x \in [0, x_{j,1}), \\ 0, & \text{otherwise in } (-\pi, \pi). \end{cases}$$

Note that h_j is also a (Haar-)scaling function for a spline multiresolution.

Analogously we define wavelets for $s = 0, \ldots, 2N-1$ by

$$\psi_{j,s}(x) := \Psi_j(x - x_{j,s}),$$

where

$$\begin{aligned} \Psi_j(x) \quad := \quad & \frac{1}{\pi} \int_0^{x_{j+1,1}} (D_{2N}^*(x-t) - D_N^*(x-t))\, dt \\ & - \frac{1}{\pi} \int_{x_{j+1,1}}^{x_{j,1}} (D_{2N}^*(x-t) - D_N^*(x-t))\, dt. \end{aligned} \tag{9}$$

Again we rewrite this definition by introducing the corresponding 2π-periodic (Haar-)spline wavelet

$$g_j(x) \quad := \quad \begin{cases} 1, & \text{if } x \in [0, x_{j+1,1}), \\ -1, & \text{if } x \in [x_{j+1,1}, x_{j,1}), \\ 0, & \text{otherwise in } (-\pi, \pi). \end{cases}$$

We obtain

$$\Psi_j(x) = \frac{1}{\pi}(g_j * (D_{2N}^* - D_N^*))(x).$$

These scaling functions and wavelets can also be seen as modified Fourier sums of the Haar functions h_j and g_j, respectively.

Writing the modified Fourier sum for $f \in L^2_{2\pi}$ and $m \in \mathbb{N}$ as

$$S^*_m(f,x) := \frac{a_0}{2} + \sum_{k=1}^{m-1} (a_k \cos kx + b_k \sin kx) + \frac{a_m}{2} \cos mx + \frac{b_m}{2} \sin mx,$$

where

$$a_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \cos kt \; dt, \qquad b_k := \frac{1}{\pi} \int_{-\pi}^{\pi} f(t) \sin kt \; dt,$$

we derive immediately

$$\begin{aligned} \Phi_j(x) &= S^*_N(h_j, x), \\ \Psi_j(x) &= S^*_{2N}(g_j, x) - S^*_N(g_j, x). \end{aligned}$$

Consequently we obtain the following imbedding.

Lemma 2.1 For $s = 0, \ldots, 2N-1$ the scaling functions $\phi_{j,s}$ are elements of V_j and the wavelets $\psi_{j,s}$ are elements of W_j .

Proof Representing the scaling functions $\phi_{j,s}$ in terms of linear combinations of the sine and cosine functions we have

$$\begin{aligned} \phi_{j,s}(x) &= \frac{1}{2N} + \sum_{k=1}^{N-1} \frac{1}{\pi k} \big((\sin x_{j,k(s+1)} - \sin x_{j,ks}) \cos kx \\ &\quad + (\cos x_{j,ks} - \cos x_{j,k(s+1)}) \sin kx \big) + \frac{(-1)^s}{N\pi} \sin Nx \end{aligned}$$

which means $\phi_{j,s} \in V_j$. For the wavelets $\psi_{j,s}$ we write

$$\begin{aligned} \psi_{j,s}(x) = {} & \frac{(-1)^s}{N\pi} \cos Nx + \frac{1}{N\pi} \sin 2Nx \\ & + \sum_{k=N+1}^{2N-1} \frac{1}{\pi k} \big((2 \sin x_{j+1,k(2s+1)} - \sin x_{j,k(s+1)} - \sin x_{j,ks}) \cos kx \\ & + (\cos x_{j,ks} - 2 \cos x_{j+1,k(2s+1)} + \cos x_{j,k(s+1)}) \sin kx \big) \end{aligned}$$

and hence $\psi_{j,s} \in W_j$. ■

It remains to prove the linear independence of the translates so that the scaling functions and wavelets really constitute a basis. Using (3) we will

show that later.
Let us mention here the lack of localization as a particular drawback of the scaling functions and wavelets defined above. As it is known for Dirichlet kernels there are large oscillations also away from the point of amplitude. This can be illustrated by different measures. The most famous is the $L^1_{2\pi}$-norm which includes an additional term of logarithmic order. Another one is an uncertainty product for periodic functions given by Breitenberger (cf. [2], [8], [12]). A third one is the ability of detection of singularities with such wavelets (cf. [7]). We do not go here into the details of these time-frequency localization properties.
Our aim here is now to introduce better localized scaling functions and wavelets for the spaces V_j and W_j defined in (5) and (6), respectively. Therefore we generalize Haar functions to splines of higher order. They are more smooth which yields scaling functions and wavelets which are more localized in the sense that they have less oscillations. In other words one could also say they have a smaller $L^1_{2\pi}$-norm, or they have a smaller uncertainty-product or they are better suited for the detection of singularities of higher order.

As usual we define these splines by convolution. Therefore for $0 < b < \pi$ let

$$\widetilde{h} := \frac{1}{2b}\,\chi_{[-b,b]}, \quad \text{in } (-\pi,\pi) \tag{10}$$

and 2π-periodically extended. For $l \in \mathbb{N}$ and $bl < \pi$ let

$$\widetilde{H}_l \quad := \quad \begin{cases} \widetilde{h}, & \text{if } l = 1, \\ \underbrace{\widetilde{h} * \ldots * \widetilde{h}}_{l-1 \ \text{convolutions}}, & \text{if } l > 1. \end{cases} \tag{11}$$

Let us note that $\widetilde{H}_l$ is zero in $(-\pi,\pi) \setminus (-lb, lb)$ and for $l > 1$ the spline $\widetilde{H}_l$ is $l-2$ times continuously differentiable. Furthermore, in the intervals $[-l\,b + 2b\,(k-1), -l\,b + 2b\,k]$ for $k = 0, \ldots, l$ the spline is an algebraic polynomial of degree $(l-1)$. To simplify our further considerations we assume that $b \sim \frac{1}{N}$ where N is the highest order of sampling we start with and b is not a rational multiple of π.

After these preliminaries the following definitions are very straightforward.

Definition 2.1 For $l \in \mathbb{N}$ and $s = 0, \ldots, 2N-1$ we write

$$\begin{aligned} \phi^l_{j,s}(x) = \Phi^l_j(x - x_{j,s}) \quad &:= \quad S^*_N(\widetilde{H}_l * h_j, x - x_{j,s}) \\ &= \quad (\widetilde{H}_l * \phi_{j,s})(x) \end{aligned} \tag{12}$$

and

$$\begin{aligned}\psi_{j,s}^{l}(x) = \Psi_{j}^{l}(x - x_{j,s}) &:= S_{2N}^{*}(\widetilde{H}_l * g_j, x - x_{j,s}) - S_{N}^{*}(\widetilde{H}_l * g_j, x - x_{j,s}) \\ &= (\widetilde{H}_l * \psi_{j,s})(x). \end{aligned} \tag{13}$$

For $l = 0$ we write analogously

$$\phi_{j,s}^{0}(x) = \Phi_{j}^{0}(x - x_{j,s}) \quad := \quad \phi_{j,s}(x)$$

and

$$\psi_{j,s}^{0}(x) = \Psi_{j}^{0}(x - x_{j,s}) \quad := \quad \psi_{j,s}(x).$$

Again we have easily the following imbedding.

Theorem 2.1 For $s = 0, \ldots, 2N-1$ the scaling functions $\phi_{j,s}^{l}$ are elements of V_j and the wavelets $\psi_{j,s}^{l}$ are elements of W_j.

Proof By Lemma 2.1 the functions $\phi_{j,s}^{l}$ are elements of V_j. Then a convolution with the even function $\widetilde{H}_l$ remains in V_j. The same holds true for the wavelets. ■

To show basis properties of these scaling functions and wavelets we compute the complex Fourier coefficients. Therefore, for $f, g \in L_{2\pi}^{1}$, $k \in \mathbb{Z}$ and $y \in \mathbb{R}$ we make extensive use of

$$\begin{aligned} c_k(f * g) &= 2\pi\, c_k(f)\, c_k(g), \\ c_k(f(\cdot - y)) &= c_k(f)\, e^{-iky}. \end{aligned}$$

Moreover, the Fourier coefficients of the characteristic function $\chi_{[-b,b]}$ for $k \in \mathbb{Z}$ can be calculated by

$$c_k(\chi_{[-b,b]}) = \frac{1}{2\pi}\int_{-b}^{b} e^{-ikt}\, dt = \frac{2b}{2\pi}\mathrm{sinc}\ kb,$$

where the sinus cardinalis is defined by

$$\mathrm{sinc}\ x = \begin{cases} \frac{\sin x}{x}, & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$$

With $x_{j,1} = \pi/N$ and the observation $\chi_{(-b/2,b/2)}(\cdot - b/2) = \chi_{(0,b)}$ we obtain

$$\begin{aligned}
c_k(\Phi_j^0) &= \frac{1}{\pi}\, c_k\left(\chi_{[-\frac{\pi}{2N},\frac{\pi}{2N}]}\left(\cdot-\frac{\pi}{2N}\right) * D_N^*\right) \\
&= 2\, c_k\left(\chi_{[-\frac{\pi}{2N},\frac{\pi}{2N}]}\right)\, e^{\frac{-i\,k\pi}{2N}}\, c_k(D_N^*) \\
&= \frac{1}{N}\,\mathrm{sinc}\,\frac{k\pi}{2N}\, e^{\frac{-i\,k\pi}{2N}}\, c_k(D_N^*) \\
&= \begin{cases} \frac{1}{2N}, & \text{if } k=0, \\ \frac{i}{2k\pi}\left(e^{\frac{-i\,k\pi}{N}}-1\right), & \text{if } 0<|k|<N, \\ \frac{-i}{2N\pi}, & \text{if } |k|=N, \\ 0, & \text{if } |k|>N. \end{cases}
\end{aligned}$$

With

$$c_k(\widetilde{H}_l) = \frac{1}{2\pi}\mathrm{sinc}^l\, kb$$

we obtain

$$\begin{aligned}
c_k(\Phi_j^l) &= c_k(\Phi_j^0 * \widetilde{H}_l) \\
&= 2\pi\, c_k(\Phi_j^0)\, c_k(\widetilde{H}_l) \\
&= c_k(\Phi_j^0)\,\mathrm{sinc}^l\, kb.
\end{aligned}$$

Hence the scaling functions Φ_j^l possess for $l \in \mathbb{N}_0$ the following representation

$$\Phi_j^l(x) = \sum_{k=-N}^{N} c_k(\Phi_j^l) e^{ikx},$$

where

$$\begin{aligned}
c_k(\Phi_j^l) &= c_k(\Phi_j^0)\,\mathrm{sinc}^l\, kb \\
&= \begin{cases} \frac{1}{2N}, & \text{if } k=0, \\ \frac{i}{2k\pi}\left(e^{\frac{-\pi ik}{N}}-1\right)\,\mathrm{sinc}^l\, kb, & \text{if } |k|=1,\dots,N-1, \\ \frac{-i}{2N\pi}\,\mathrm{sinc}^l\, Nb, & \text{if } |k|=N, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}$$

Analogously, for $l \in \mathbb{N}_0$ the wavelet functions Ψ_j^l satisfy

$$\Psi_j^l(x) = \sum_{|k|=N}^{2N} c_k(\Psi_j^l)\, e^{ikx},$$

where

$$\begin{aligned} c_k(\Psi_j^l) &= c_k(\Psi_j^0)\ \mathrm{sinc}^l\, kb \\ &= \begin{cases} \frac{1}{2N\pi}\ \mathrm{sinc}^l\, Nb, & \text{if } |k| = N, \\ \frac{i}{k\pi}\ (1-\cos x_{j+1,k})\ e^{\frac{-\pi i k}{2N}}\ \mathrm{sinc}^l\, kb, & \text{if } N < |k| < 2N, \\ \frac{-i}{2N\pi}\ \mathrm{sinc}^l\, 2Nb, & \text{if } |k| = 2N, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

Now we are enabled to show the basis properties of these functions.

Theorem 2.2 For $l = 0, 1, 2, \ldots$ the system

$$B_j(\Phi_j^l) := \{\Phi_j^l(x - x_{j,s})\}_{s=0}^{2N-1} = \{\phi_{j,s}^l(x)\}_{s=0}^{2N-1}$$

is a basis for V_j and

$$B_j(\Psi_j^l) := \{\Psi_j^l(x - x_{j,s})\}_{s=0}^{2N-1} = \{\psi_{j,s}^l(x)\}_{s=0}^{2N-1}$$

is a basis for W_j .

Proof We use Theorem 1.1 which states that $B_j(\Phi_j^l)$ is a basis of

$$\mathrm{span}\{\Phi_j^l(x - x_{j,s}) :\ s = 0, \ldots, 2N-1\},$$

iff

$$\sum_{m=-\infty}^{\infty} |c_{s+m2N}(\Phi_j^l)|^2 > 0 \ \text{ for } s = 0, \ldots, 2N-1.$$

The nonvanishing Fourier coefficients of Φ_j^l are exactly

$$c_{-N}(\Phi_j^l), \quad \ldots, \quad c_{-1}(\Phi_j^l), \quad c_0(\Phi_j^l), \quad c_1(\Phi_j^l), \quad \ldots, \quad c_N(\Phi_j^l).$$

Hence, the sum for $s = 0, \ldots, 2N-1$ can be written as

$$\sum_{m=-\infty}^{\infty} |c_{s+m2N}(\Phi_j^l)|^2 = |c_s(\Phi_j^l)|^2 + |c_{s-2N}(\Phi_j^l)|^2 > 0.$$

From Lemma 2.1 and Theorem 2.1 we have the imbedding of these linear independent functions into V_j . The space V_j has dimension $2N$. Thus,

$$V_j \quad = \quad \mathrm{span}\{\Phi_j^l(x - x_{j,s}) :\ s = 0, \ldots, 2N-1\}$$

and $B_j(\Phi_j^l)$ is a basis of V_j .

Analogously we prove the second part. For arbitrary $l \in \mathbb{N}_0$ the non-vanishing Fourier coefficients are

$$c_{-2N}(\Psi_j^l), \quad \ldots, \quad c_{-N}(\Psi_j^l), \quad c_N(\Psi_j^l), \quad \ldots, \quad c_{2N}(\Psi_j^l)$$

and we obtain

$$\sum_{m=-\infty}^{\infty} |c_{s+m2N}(\Phi_j^l)|^2 = \sum_{m=-1}^{1} |c_{s+m2N}(\Psi_j^l)|^2 > 0 \ \text{ for } s = 0, \ldots, 2N-1.$$

This implies that $B_j(\Psi_j^l)$ is a basis of span $\{\Psi_j^l(x - x_{j,s}) : \ s = 0, \ldots, 2N - 1\}$. Because of the dimension of W_j is $2N$ and Theorem 2.1 we obtain that $B_j(\Psi_j^l)$ is a basis for W_j. ■

Note that by definition of V_j und W_k the orthogonality

$$\phi_{j,s}^l \perp \psi_{k,r}^m$$

for $k \geq j$ and arbitrary $r, s, l, m \in \mathbb{N}_0$ is easily seen.

However, these bases $B_j(\Phi_j^l)$ and $B_j(\Psi_j^l)$ are not orthogonal bases in the sense that the translates itself are orthogonal to each other. But there is a simple orthonormalization principle available. Particularly, one is interested in a procedure which ends up again in a translation invariant basis. Also for this purpose the sums of the form (3) play a very important role. In [10, 15] the following result is proved.

Lemma 2.2 If Φ_j^* and Ψ_j^* are 2π-periodic functions with Fourier coefficients

$$\begin{aligned} c_k(\Phi_j^{l*}) &= c_k(\Phi_j^{0*}) \frac{\operatorname{sinc}^l kb}{|\operatorname{sinc}^l kb|} \\ &= \begin{cases} \frac{1}{\sqrt{2N}}, & \text{if } k = 0, \\ \frac{e^{\frac{-\pi i k}{2N}} \sin^l kb}{\sqrt{2N} \ |\sin^l kb|}, & \text{if } |k| = 1, \ldots, N-1, \\ \frac{-i \sin^l Nb}{2\sqrt{N} \ |\sin^l Nb|}, & \text{if } |k| = N, \\ 0, & \text{otherwise,} \end{cases} \end{aligned}$$

and

$$\begin{aligned} c_k(\Psi_j^{l*}) &= c_k(\Psi_j^{0*})\frac{\operatorname{sinc}^l kb}{|\operatorname{sinc}^l kb|} \\ &= \begin{cases} \frac{\sin^l Nb}{2\sqrt{N}\ |\sin^l Nb|}, & \text{if } |k| = N, \\ \frac{i\ e^{\frac{-\pi i k}{2N}} \sin^l kb}{\sqrt{2N}\ |\sin^l kb|}, & \text{if } |k| = N+1, \dots, 2N-1, \\ \frac{-i\ \sin^l 2Nb}{2\sqrt{N}\ |\sin^l 2Nb|}, & \text{if } |k| = 2N, \\ 0, & \text{otherwise}, \end{cases} \end{aligned}$$

respectively, then

$$B_j(\Phi_j^{l*}) := \{\Phi_j^{l*}(x - x_{j,s})\}_{s=0}^{2N-1} = \{\phi_{j,s}^{l*}(x)\}_{s=0}^{2N-1}$$

is an orthonormal basis for V_j and

$$B_j(\Psi_j^{l*}) := \{\Psi_j^{l*}(x - x_{j,s})\}_{s=0}^{2N-1} = \{\psi_{j,s}^{l*}(x)\}_{s=0}^{2N-1}$$

is an orthonormal basis for W_j .

Proof In [10, 15] the following result is proved. If

$$\sum_{m=-\infty}^{\infty} |c_{s+m2N}(\Phi_j^l)|^2 > 0 \quad \text{for } s = 0, \dots, 2N-1,$$

i.e., if the translates are linearly independent, then the orthonormalization procedure consists in putting

$$c_k(\Phi_j^{l*}) := \frac{c_k(\Phi_j^l)}{\sqrt{2N}\ \sqrt{\sum_{m=-\infty}^{\infty} |c_{k_j+m2N}(\Phi_j^l)|^2}} \tag{14}$$

$$(\, k \in \mathbb{Z}\,,\ k_j = k \mod 2N\,),$$

to construct Φ_j^{l*} . Out of (14) the assertions can be easily deduced. ■

3. Decomposition and Reconstruction

A cornerstone for the development of reconstruction and decomposition algorithms for the trigonometric multiresolution analysis described in Section 2 is the knowledge of the corresponding reconstruction and decomposition sequences for the spaces V_j and W_j . This section is devoted to the computation of these two-scale sequences for reconstruction and decomposition.

As $V_j \subset V_{j+1}$, there must be coefficients $p_{j,n,s}$ such that

$$\phi_{j,n} = \sum_{s=0}^{4N-1} p_{j,n,s}\phi_{j+1,s}.$$

The Theorem 3.1 establishes the precise values of these $p_{j,n,s}$.

The other pair of important sequences for our approach are the so-called decomposition sequences. As $V_{j+1} = V_j \oplus W_j$ for $j \in \mathbb{N}_0$, any $\phi_{j+1,s} \in V_{j+1}$ can be written as a linear combination of the basis functions of V_j and W_j , i.e. $\phi_{j,k}$ and $\psi_{j,k}$.

Note that we simplify our investigations by using the same b in (10) for any space V_j and W_j . The general case can be obtained again by simple basis transformations of the same kind as described in the sequel.

An important part of the following observations is based on interpolation type representations of trigonometric polynomials (cf. [16], Chap. 10). The essential point is that

$$\frac{1}{m} D_m^* \left(x - \frac{2k+1}{2m} \pi \right) \quad \text{for} \quad k = 0, \ldots, 2m-1$$

can be used as fundamental polynomials of Lagrange interpolation (see (7)). More precisely, we obtain for trigonometric polynomials of the form

$$\tilde{T}(x) = \frac{1}{2} a_0 + \sum_{k=1}^{m-1} (a_k \cos kx + b_k \sin kx) + b_m \sin mx$$

that

$$\tilde{T}(x) = \frac{1}{m} \sum_{k=0}^{2m-1} T\left(\frac{2k+1}{2m} \pi \right) D_m^* \left(x - \frac{2k+1}{2m} \pi \right).$$

Analogous statements could be made for the fundamental poynomials

$$\frac{1}{m} D_m^* \left(x - \frac{2k}{2m} \pi \right) \quad \text{for} \quad k = 0, \ldots, 2m-1.$$

Hence, for arbitrary polynomials

$$T(x) = \frac{1}{2} a_0 + \sum_{k=1}^{m} (a_k \cos kx + b_k \sin kx)$$

we have the two different representations

$$\begin{aligned} T(x) &= a_m \cos mx + \frac{1}{m} \sum_{k=0}^{2m-1} T\left(\frac{2k+1}{2m} \pi \right) D_m^* \left(x - \frac{2k+1}{2m} \pi \right) && (15) \\ &= b_m \sin mx + \frac{1}{m} \sum_{k=0}^{2m-1} T\left(\frac{2k}{2m} \pi \right) D_m^* \left(x - \frac{2k}{2m} \pi \right). && (16) \end{aligned}$$

Theorem 3.1 (Two-scale relations) For $l \in \mathbb{N}_0$ and $k = 0, \ldots, 2N-1$ it holds

$$\phi_{j,k}^l(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_{j+1,s}) \phi_{j+1,2k+s}^l(x) \tag{17}$$

and

$$\psi^l_{j,k}(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} \widetilde{R}_j\,(x_{j+1,s}) \phi^l_{j+1,2k+s}(x), \tag{18}$$

where the coefficients are point evaluations of the trigonometric polynomials

$$R_j(x) \quad := \quad D^*_N(x) + D^*_N(x - x_{j+1,1}) \tag{19}$$

and

$$\widetilde{R}_j\,(x) \quad := \quad D^*_{2N}(x) - D^*_{2N}(x - x_{j+1,1}) - D^*_N(x) + D^*_N(x - x_{j+1,1}). \tag{20}$$

Proof With (19), (20) and substitution we obtain for $l = 0$

$$\phi^0_{j,k}(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2N}} R_j(x - x_{j,k} - t)\, dt \tag{21}$$

and

$$\psi^0_{j,k}(x) = \frac{1}{\pi} \int_0^{\frac{\pi}{2N}} \widetilde{R}_j\,(x - x_{j,k} - t)\, dt. \tag{22}$$

Note that R_j and $\widetilde{R}_j$ are trigonometric polynomials of degree $2N$ without $2N$ th sine frequency. Applying (16), namely

$$R_j(x - x_{j,k} - t) = \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_{j+1,s}) D^*_{2N}(x - x_{j+1,2k+s} - t) \tag{23}$$

and

$$\widetilde{R}_j\,(x - x_{j,k} - t) = \frac{1}{2N} \sum_{s=0}^{4N-1} \widetilde{R}_j\,(x_{j+1,s}) D^*_{2N}(x - x_{j+1,2k+s} - t) \tag{24}$$

we deduce from (21) and (23) that

$$\phi^0_{j,k}(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} R_j(x_{j+1,s}) \phi^0_{j+1,2k+s}(x).$$

Analogously, from (22) and (24) it follows

$$\psi^0_{j,k}(x) = \frac{1}{2N} \sum_{s=0}^{4N-1} \widetilde{R}_j\,(x_{j+1,s}) \psi^0_{j+1,2k+s}(x).$$

Using l-times convolution yields for arbitrary $l \in \mathbb{N}_0$ (cf. (12) and (13)) that

$$\begin{aligned}
&\phi_{j,k}^{l}(x) \\
&= \frac{1}{(2b)^l}\int_{-b}^{b}\dots\int_{-b}^{b}\phi_{j,k}^{0}(x-s_1-\dots-s_l)\,ds_1\dots ds_l \\
&= \frac{1}{(2b)^l}\int_{-b}^{b}\dots\int_{-b}^{b}\frac{1}{2N}\sum_{s=0}^{4N-1}(R_j(x_{j+1,s})\times \\
&\qquad \phi_{j+1,2k+s}^{0}(x-s_1-\dots-s_l))\,ds_1\dots ds_l \\
&= \frac{1}{2N}\sum_{s=0}^{4N-1}R_j(x_{j+1,s})\frac{1}{(2b)^l}\times \\
&\qquad \int_{-b}^{b}\dots\int_{-b}^{b}\phi_{j+1,2k+s}^{0}(x-s_1-\dots-s_l)\,ds_1\dots ds_l \\
&= \frac{1}{2N}\sum_{s=0}^{4N-1}R_j(x_{j+1,s})\phi_{j+1,2k+s}^{l}(x)
\end{aligned}$$

and

$$\begin{aligned}
\psi_{j,k}^{l}(x) &= \frac{1}{(2b)^l}\int_{-b}^{b}\dots\int_{-b}^{b}\psi_{j,k}^{0}(x-s_1-\dots-s_l)\,ds_1\dots ds_l \\
&= \frac{1}{(2b)^l}\int_{-b}^{b}\dots\int_{-b}^{b}\frac{1}{2N}\times \\
&\qquad \sum_{s=0}^{4N-1}\widetilde{R}_j\,(x_{j+1,s})\psi_{j+1,2k+s}^{0}(x-s_1-\dots-s_l)\,ds_1\dots ds_l \\
&= \frac{1}{2N}\sum_{s=0}^{4N-1}\widetilde{R}_j\,(x_{j+1,s})\frac{1}{(2b)^l}\times \\
&\qquad \int_{-b}^{b}\dots\int_{-b}^{b}\psi_{j+1,2k+s}^{0}(x-s_1-\dots-s_l)\,ds_1\dots ds_l
\end{aligned}$$

$$= \frac{1}{2N} \sum_{s=0}^{4N-1} \widetilde{R}_j\ (x_{j+1,s}) \psi^l_{j+1,2k+s}(x). \qquad \blacksquare$$

The two-scale relations allow a reconstruction of signals. To obtain formulas for the inverse operation we introduce some abbreviations. Let us define for all $j \in \mathbb{N}_0$ a $2N$-periodic bi-infinite sequence $\lambda_{j,s}$ by

$$\lambda_{j,s} := s, \qquad s = 1, \ldots, 2^{j+1}. \tag{25}$$

Furthermore let

$$\begin{aligned} \beta_{j,k} &:= \frac{1}{N} \sum_{s=1}^{N-1} \tan \frac{x_{j+1,s}}{2} \sin x_{j,sk}, \\ \Delta_{j,k} &:= D_N^* \left(\frac{2k+1}{2N} \pi \right), \\ c_{j,k} &:= \frac{1}{2} \delta_{k,0} + \frac{(-1)^k}{4N}, \\ d_{j,k} &:= \frac{1}{2N} \left\{ \frac{1}{N} \sum_{s=0}^{2N-1} \left(\lambda_{j,k-s} - \frac{\lambda_{j,k} + \lambda_{j,k+1}}{2} \right) \Delta_{j,s} \right\}. \end{aligned} \tag{26}$$

Theorem 3.2 (Decomposition formula) For all $l \in \mathbb{N}_0$ and $m < 2N$ we have

$$\begin{aligned} \phi^l_{j+1,2m}(x) = {} & \sum_{k=0}^{2N-1} (c_{j,k} - \frac{1}{2}\beta_{j,k})\, \phi^l_{j,k+m}(x) + \frac{1}{2}\psi^l_{j,m}(x) \\ & + \frac{(-1)^m}{4N} \sum_{k=0}^{2N-1} (-1)^k\, \psi^l_{j,k}(x) + \sum_{k=0}^{2N-1} d_{j,k}\, \psi^l_{j,k+m}(x) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \phi^l_{j+1,2m+1}(x) = {} & \sum_{k=0}^{2N-1} (c_{j,k} + \frac{1}{2}\beta_{j,k})\, \phi^l_{j,k+m}(x) - \frac{1}{2}\psi^l_{j,m}(x) \\ & - \frac{(-1)^m}{4N} \sum_{k=0}^{2N-1} (-1)^k\, \psi^l_{j,k}(x) + \sum_{k=0}^{2N-1} d_{j,k}\, \psi^l_{j,k+m}(x). \end{aligned} \tag{28}$$

To show this result we need some preliminary observations.

Lemma 3.1 For $N \in \mathbb{N}$ and $l \in \mathbb{N}_0$ we have

$$\left(\frac{1}{Nb} \right)^l \sin^l Nb \sin Nx = \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \phi^l_{j,s}(x) \tag{29}$$

and

$$\left(\frac{1}{Nb}\right)^l \sin^l Nb \cos Nx = \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \psi_{j,s}^l(x). \tag{30}$$

Proof We start with

$$\begin{aligned}
&\left(\frac{2}{N}\right)^l \sin^l Nb \sin Nx \\
&= \int_{-b}^{b} \cdots \int_{-b}^{b} \sin N(x - s_1 - \ldots - s_l)\, ds_1 \ldots ds_l \\
&= \frac{N}{2} \int_{-b}^{b} \cdots \int_{-b}^{b} \int_0^{\frac{\pi}{N}} \cos N(x - t - s_1 - \ldots - s_l)\, dt\, ds_1 \ldots ds_l .
\end{aligned}$$

From (16) we deduce

$$\cos Nx = \frac{1}{N} \sum_{s=0}^{2N-1} \cos(Nx_{j,s}) D_N^*(x - x_{j,s}).$$

Thus

$$\begin{aligned}
&\left(\frac{1}{Nb}\right)^l \sin^l Nb \sin Nx \\
&= \frac{1}{2} \sum_{s=0}^{2N-1} \cos(Nx_{j,s}) \frac{1}{(2b)^l} \\
&\qquad \int_{-b}^{b} \cdots \int_{-b}^{b} \int_0^{\frac{\pi}{N}} D_N^*(x - x_{j,s} - t - s_1 - \ldots - s_l)\, dt\, ds_1 \ldots ds_l \\
&= \frac{\pi}{2} \sum_{s=0}^{2N-1} (-1)^s \phi_{j,s}^l(x),
\end{aligned}$$

which proves (29). To show (30) we note that

$$\cos Nx = \frac{N}{2} \left\{ \int_0^{\frac{\pi}{2N}} \cos N(x-t)\, dt - \int_{\frac{\pi}{2N}}^{\frac{\pi}{N}} \cos N(x-t)\, dt \right\}. \tag{31}$$

Rewriting the righthand side by using (16) we obtain

$$\begin{aligned}\cos N(x-t) &= 2\cos N(x-t) - \cos N(x-t)\\ &= \frac{2}{2N}\sum_{k=0}^{4N-1}\cos(Nx_{j+1,k})D_{2N}^*(x-t-x_{j+1,k})\\ &\quad -\frac{1}{N}\sum_{k=0}^{2N-1}\cos(Nx_{j,k})D_N^*(x-t-x_{j,k})\\ &= \frac{1}{N}\sum_{k=0}^{2N-1}(-1)^k\left(D_{2N}^*(x-t-x_{j,k}) - D_N^*(x-t-x_{j,k})\right). \quad (32)\end{aligned}$$

Inserting (32) in (31) and applying the definition of $\psi_{j,s}$ we obtain

$$\cos Nx = \frac{\pi}{2}\sum_{s=0}^{2N-1}(-1)^s\psi_{j,s}(x).$$

Hence,

$$\begin{aligned}&\left(\frac{1}{Nb}\right)^l \sin^l Nb\,\cos Nx\\ &= \frac{1}{(2b)^l}\int_{-b}^{b}\dots\int_{-b}^{b}\cos N(x-s_1-\dots-s_l)\,ds_1\dots ds_l\\ &= \frac{\pi}{2}\sum_{s=0}^{2N-1}(-1)^s\frac{1}{(2b)^l}\int_{-b}^{b}\dots\int_{-b}^{b}\psi_{j,s}(x-s_1-\dots-s_l)\,ds_1\dots ds_l\\ &= \frac{\pi}{2}\sum_{s=0}^{2N-1}(-1)^s\psi_{j,s}^l(x). \qquad \blacksquare\end{aligned}$$

Lemma 3.2 For $l\in\mathbb{N}_0$ und $m=0,1,\dots,2N-1$ it holds

$$\begin{aligned}&\phi_{j+1,2m}^l - \phi_{j+1,2m+1}^l\\ &= -\sum_{s=0}^{2N-1}\beta_{j,s-m}\,\phi_{j,s}^l + \psi_{j,m}^l + \frac{(-1)^m}{2N}\sum_{s=0}^{2N-1}(-1)^s\,\psi_{j,s}^l.\end{aligned}$$

Proof In a first step we prove the Lemma for $l=0$. Applying the definitions (8) and (9) we obtain

$$\begin{aligned}
&\phi_{j+1,2m}(x) - \phi_{j+1,2m+1}(x) - \psi_{j,m}(x) \\
&\quad = \quad \Phi_{j+1}(x - x_{j,m}) - \Phi_{j+1}(x - x_{j,m} - x_{j+1,1}) - \Psi_j(x - x_{j,m}) \\
&\quad = \quad \frac{1}{\pi} \int_0^{\frac{\pi}{2N}} \left(D_N^*\left(x - t - x_{j,m}\right) - D_N^*\left(x - t - x_{j,m} - x_{j+1,1}\right)\right) dt \\
&\quad = \quad \frac{1}{\pi} \sum_{k=1}^{N-1} \int_0^{\frac{\pi}{2N}} (\cos k(x - t - x_{j,m}) \\
&\qquad\qquad - \cos k\left(x - t - x_{j,m} - x_{j+1,1}\right) dt \\
&\qquad\quad + \frac{1}{2\pi} \int_0^{\frac{\pi}{2N}} (\cos N(x - t - x_{j,m})) \\
&\qquad\qquad - \cos N\left(x - t - x_{j,m} - x_{j+1,1}\right) dt.
\end{aligned}$$

Exploring the identity

$$\int_0^{\frac{\pi}{N}} \sin k(x-t)\, dt = \frac{2}{k} \sin \frac{k\pi}{2N} \sin k\left(x - \frac{\pi}{2N}\right), \qquad k \in \mathbb{N},$$

we simplify for $k = 1, \ldots, N$ the integral

$$\begin{aligned}
&\int_0^{\frac{\pi}{2N}} \left(\cos k(x-t) - \cos k\left(x - t - x_{j+1,1}\right)\right) dt \\
&\qquad = \quad -2 \sin \frac{k\pi}{4N} \int_0^{\frac{\pi}{2N}} \sin k\left(x - t - \frac{\pi}{4N}\right) dt \qquad (33) \\
&\qquad = \quad -\frac{4 \sin^2\left(\frac{k\pi}{4N}\right)}{k} \sin k\left(x - \frac{\pi}{2N}\right) \\
&\qquad = \quad -\frac{2 \sin^2\left(\frac{k\pi}{4N}\right)}{\sin\left(\frac{k\pi}{2N}\right)} \int_0^{\frac{\pi}{N}} \sin k(x-t)\, dt \\
&\qquad = \quad -\tan\left(\frac{k\pi}{4N}\right) \int_0^{\frac{\pi}{N}} \sin k(x-t)\, dt. \qquad (34)
\end{aligned}$$

Moreover, for $k = N$ it follows from (33) and (30) that

$$\begin{aligned}
&\int_0^{\frac{\pi}{2N}} \left(\cos N(x-t) - \cos N\left(x-t-x_{j+1,1}\right)\right) dt \\
&\qquad = -2\sin\frac{N\pi}{4N}\int_0^{\frac{\pi}{2N}} \sin N\left(x-t-\frac{\pi}{4N}\right) dt \\
&\qquad = -\frac{\sqrt{2}}{N}\left(\cos\left(Nx-\frac{3\pi}{4}\right)-\cos\left(Nx-\frac{\pi}{4}\right)\right) \\
&\qquad = \frac{2}{N}\cos Nx \\
&\qquad = \frac{\pi}{N}\sum_{s=0}^{2N-1}(-1)^s\psi_{j,s}(x). \qquad (35)
\end{aligned}$$

On the other hand, for $k = 1, \ldots, N-1$ we apply (34) and (16) to obtain

$$\begin{aligned}
&\int_0^{\frac{\pi}{2N}} \left(\cos k(x-t) - \cos k\left(x-t-x_{j+1,1}\right)\right) dt \\
&\qquad = -\frac{1}{N}\tan\left(\frac{k\pi}{4N}\right)\sum_{s=0}^{2N-1}\sin kx_{j,s}\int_0^{\frac{\pi}{N}} D_N^*(x-t-x_{j,s})\,dt \\
&\qquad = -\frac{\pi}{N}\tan\left(\frac{k\pi}{4N}\right)\sum_{s=0}^{2N-1}(\sin x_{j,sk})\phi_{j,s}(x). \qquad (36)
\end{aligned}$$

With (36), (35) and the definition of the $\beta_{j,s}$ we argue

$$\begin{aligned}
&\frac{1}{\pi}\sum_{k=1}^{N-1}\int_0^{\frac{\pi}{2N}} \left(\cos k(x-t-x_{j,m}) - \cos k\left(x-t-x_{j,m}-x_{j+1,1}\right)\right) dt \\
&\qquad +\frac{1}{2\pi}\int_0^{\frac{\pi}{2N}} \left(\cos N(x-t-x_{j,m}) - \cos N\left(x-t-x_{j,m}-x_{j+1,1}\right)\right) dt \\
&= -\sum_{s=0}^{2N-1}\beta_{j,s}\phi_{j,s}(x-x_{j,m}) + \frac{1}{2N}\sum_{s=0}^{2N-1}(-1)^s\psi_{j,s}(x-x_{j,m}),
\end{aligned}$$

so that

$$\begin{aligned}
&\phi_{j+1,2m}(x) - \phi_{j+1,2m+1}(x) - \psi_{j,m}(x) \\
&\qquad = -\sum_{s=0}^{2N-1}\beta_{j,s}\phi_{j,s}(x-x_{j,m}) + \frac{1}{2N}\sum_{s=0}^{2N-1}(-1)^s\psi_{j,s}(x-x_{j,m}).
\end{aligned}$$

This yields the assertion for $l=0$. In the general case $l \geq 0$ we write

$$\begin{aligned}
&\phi^l_{j+1,2m}(x) - \phi^l_{j+1,2m+1}(x) - \psi^l_{j,m}(x) \\
&\quad = \frac{1}{(2b)^l} \int_{-b}^{b} \dots \int_{-b}^{b} (\phi_{j+1,2m}(x - s_1 - \dots - s_l) \\
&\qquad -\phi_{j+1,2m+1}(x - s_1 - \dots - s_l) - \psi_{j,m}(x - s_1 - \dots - s_l))\, ds_1 \dots ds_l \\
&\quad = -\sum_{s=0}^{2N-1} \beta_{j,s-m} \frac{1}{(2b)^l} \int_{-b}^{b} \dots \int_{-b}^{b} \phi_{j,s}(x - s_1 - \dots - s_l)\, ds_1 \dots ds_l \\
&\qquad + \frac{(-1)^m}{2N} \sum_{s=0}^{2N-1} (-1)^s \frac{1}{(2b)^l} \int_{-b}^{b} \dots \int_{-b}^{b} \psi_{j,s}(x - s_1 - \dots - s_l)\, ds_1 \dots ds_l \\
&\quad = -\sum_{s=0}^{2N-1} \beta_{j,s-m}\, \phi^l_{j,s} + \frac{(-1)^m}{2N} \sum_{s=0}^{2N-1} (-1)^s\, \psi^l_{j,s}. \qquad \blacksquare
\end{aligned}$$

In the next Lemma we use for $j, l \in \mathbb{N}_0$ und $k = 0, \dots, 2N-1$ the abbreviations

$$S^l_k = S^l_{j,k} := \phi^l_{j,k} + \psi^l_{j,k}$$

and

$$M^l_k = M^l_{j,k} := \phi^l_{j,k} - \psi^l_{j,k}.$$

Lemma 3.3 For $l \in \mathbb{N}_0$ and $m = 0, \dots, 2N-1$ we have

$$\begin{aligned}
\phi^l_{j+1,2m} - \phi^l_{j+1,2m-2} &= \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} S^l_k - S^l_{m-1} \\
&\qquad + \frac{1}{2N} \sum_{k=0}^{2N-1} (-1)^{m-k} S^l_k, \\
\phi^l_{j+1,2m+1} - \phi^l_{j+1,2m-1} &= M^l_m - \frac{1}{N} \sum_{k=0}^{2N-1} \Delta_{j,k-m} M^l_k \\
&\qquad + \frac{1}{2N} \sum_{k=0}^{2N-1} (-1)^{m-k} M^l_k
\end{aligned}$$

and therefore

$$\begin{aligned}
&\phi^l_{j+1,2m} + \phi^l_{j+1,2m+1} - \phi^l_{j+1,2m-2} - \phi^l_{j+1,2m-1} \\
&\quad = \frac{2}{N}\sum_{k=0}^{2N-1} \Delta_{j,k-m}\psi^l_{j,k} - \psi^l_{j,m} - \psi^l_{j,m-1} + \phi^l_{j,m} - \phi^l_{j,m-1} \\
&\qquad + \frac{1}{N}\sum_{k=0}^{2N-1} (-1)^{m-k}\phi^l_{j,k}. \qquad (37)
\end{aligned}$$

Proof Using (17), (18) and (26) we obtain

$$\begin{aligned}
S^l_k &= \phi^l_{j+1,2k} + \frac{1}{N}\sum_{s=0}^{2N-1} \Delta_{j,s-1}\phi^l_{j+1,2k+2s} \\
&= \phi^l_{j+1,2k} + \frac{1}{N}\sum_{s=0}^{2N-1} \Delta_{j,k-s}\phi^l_{j+1,2s} \qquad (38)
\end{aligned}$$

and

$$\begin{aligned}
M^l_k &= \phi^l_{j+1,2k+1} + \frac{1}{N}\sum_{s=0}^{2N-1} \Delta_{j,s}\phi^l_{j+1,2k+2s+1} \\
&= \phi^l_{j+1,2k+1} + \frac{1}{N}\sum_{s=0}^{2N-1} \Delta_{j,k-s-1}\phi^l_{j+1,2s+1}. \qquad (39)
\end{aligned}$$

Furthermore, for $j \in \mathbb{N}_0$ and $k = 0, 1, \ldots, 2N-1$ we compute with (15)

$$\frac{1}{N}\sum_{s=0}^{2N-1} \Delta_{j,s}\Delta_{j,s-k} = N\delta_{k,0} - \frac{1}{2}(-1)^k. \qquad (40)$$

With (40) and (38) we obtain for $m = 0, 1, \ldots, 2N-1$ that

$$\begin{aligned}
&\frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,k-m}S^l_k \\
&= \frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,k-m}\phi^l_{j+1,2k} + \frac{1}{N}\sum_{s=0}^{2N-1}\left(\frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,k-m}\Delta_{j,k-s}\right)\phi^l_{j+1,2s} \\
&= \frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,(m-1)-k}\phi^l_{j+1,2k} + \frac{1}{N}\sum_{s=0}^{2N-1}\left(\frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,k}\Delta_{j,k+m-s}\right)\phi^l_{j+1,2s} \\
&= S^l_{m-1} - \phi^l_{j+1,2m-2} + \phi^l_{j+1,2m} - \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^{s-m}\phi^l_{j+1,2s},
\end{aligned}$$

which implies

$$\begin{aligned}
&\phi^l_{j+1,2m} - \phi^l_{j+1,2m-2} \\
&\quad = \frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,k-m} S^l_k - S^l_{m-1} + \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^{s-m}\phi^l_{j+1,2s}.
\end{aligned} \tag{41}$$

Analogously, with (39) and (40) we obtain

$$\begin{aligned}
&\phi^l_{j+1,2m+1} - \phi^l_{j+1,2m-1} \\
&\quad = M^l_m - \frac{1}{N}\sum_{k=0}^{2N-1} \Delta_{j,k-m} M^l_k + \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^{s-m}\phi^l_{j+1,2s+1}.
\end{aligned} \tag{42}$$

Again from (16) and

$$\cos Nx_{j+1,2s} = (-1)^s, \qquad \cos Nx_{j+1,2s+1} = 0 \quad \text{for} \quad s = 0, \ldots, 2N-1$$

we obtain

$$\cos N(x-t) = \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^s D^*_{2N}(x - t - x_{j+1,2s}) \tag{43}$$

and from

$$\sin Nx_{j+1,2s} = 0, \qquad \sin Nx_{j+1,2s+1} = (-1)^s \quad \text{for} \quad s = 0, \ldots, 2N-1$$

we obtain

$$\sin N(x-t) = \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^s D^*_{2N}(x - t - x_{j+1,2s+1}). \tag{44}$$

Integrating (43) and (44) with respect to t on the interval $[0, \frac{\pi}{2N}]$ and using Lemma 3.1 deduce

$$\begin{aligned}
\frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^s \phi_{j+1,2s}(x) &= \frac{1}{N\pi}(\sin Nx + \cos Nx) \\
&= \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^s S_s(x)
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^s \phi_{j+1,2s+1}(x) &= \frac{1}{N\pi}(\sin Nx - \cos Nx) \\
&= \frac{1}{2N}\sum_{s=0}^{2N-1} (-1)^s M_s(x),
\end{aligned}$$

which implies for arbitrary $l \in \mathbb{N}_0$ the representations

$$\frac{1}{2N}\sum_{s=0}^{2N-1}(-1)^s\phi^l_{j+1,2s}(x) = \frac{1}{2N}\sum_{s=0}^{2N-1}(-1)^s S^l_s(x) \tag{45}$$

and

$$\frac{1}{2N}\sum_{s=0}^{2N-1}(-1)^s\phi^l_{j+1,2s+1}(x) = \frac{1}{2N}\sum_{s=0}^{2N-1}(-1)^s M^l_s(x). \tag{46}$$

Putting (45) and (46) in (41) and (42) we obtain the assertion of the Lemma. ∎

Now we are able to show the decomposition formulas.

Proof of Theorem 3.2 Let m be an integer with $0 \le m \le 2N-1$. Note that

$$\sum_{s=0}^{4N-1}\phi^l_{j+1,s+2m} = \sum_{s=0}^{2N-1}\phi^l_{j,s} = 1.$$

Therefore

$$\begin{aligned}\phi^l_{j+1,2m} &= \phi^l_{j+1,2m} - \frac{1}{4N}\sum_{s=0}^{4N-1}\phi^l_{j+1,s+2m} + \frac{1}{4N}\sum_{s=0}^{2N-1}\phi^l_{j,s}\\ &= \frac{1}{4N}\sum_{s=0}^{2N-1}\phi^l_{j,s} + \phi^l_{j+1,2m}\\ &\quad -\frac{1}{4N}\sum_{s=0}^{2N-1}(\phi^l_{j+1,2s+2m} + \phi^l_{j+1,2s+2m+1}).\end{aligned}$$

With $\lambda_{j,s}$ defined in (25) and $j \in \mathbb{N}_0$ we use Abel summation

$$\sum_{s=0}^{2N-1} a_s = 2Na_0 - \sum_{s=0}^{2N-1}\lambda_{j,s}(a_s - a_{s-1}),$$

which implies for $a_s := \phi^l_{j+1,2s+2m} + \phi^l_{j+1,2s+2m+1}$ the representation

$$\begin{aligned}\phi^l_{j+1,2m} &= \frac{1}{4N}\sum_{s=0}^{2N-1}\phi^l_{j,s} + \phi^l_{j+1,2m} - \frac{1}{2}(\phi^l_{j+1,2m} + \phi^l_{j+1,2m+1})\\ &\quad + \frac{1}{4N}\sum_{s=0}^{2N-1}\lambda_{j,s}(\phi^l_{j+1,2m+2s} + \phi^l_{j+1,2m+1+2s}\\ &\quad -\phi^l_{j+1,2m-2+2s} - \phi^l_{j+1,2m-1+2s})\end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4N}\sum_{s=0}^{2N-1}\phi^l_{j,s} + \frac{1}{2}(\phi^l_{j+1,2m} - \phi^l_{j+1,2m+1}) \qquad (47)\\
&+\frac{1}{4N}\sum_{s=0}^{2N-1}\lambda_{j,s-m}(\phi^l_{j+1,2s} + \phi^l_{j+1,2s+1}\\
&-\phi^l_{j+1,2s-2} - \phi^l_{j+1,2s-1}).
\end{aligned}$$

Hence,

$$\begin{aligned}
\phi^l_{j+1,2m+1} &= \phi^l_{j+1,2m+1} - \phi^l_{j+1,2m} + \phi^l_{j+1,2m}\\
&= \frac{1}{4N}\sum_{s=0}^{2N-1}\phi^l_{j,s} + \frac{1}{4N}\sum_{s=0}^{2N-1}\lambda_{j,s-m}(\phi^l_{j+1,2s} + \phi^l_{j+1,2s+1}\\
&\quad -\phi^l_{j+1,2s-2} - \phi^l_{j+1,2s-1}) - \frac{1}{2}(\phi^l_{j+1,2m} - \phi^l_{j+1,2m+1}) \qquad (48)
\end{aligned}$$

Now using Lemma 3.2 and equation (37) we obtain from (47) and (48) the decomposition formulas. ■

4. Algorithms

We start with the algorithm for decomposition. For $l \in \mathbb{N}_0$ let functions $f_j \in V_j$ and $g_j \in W_j$ be given as

$$f_j(x) = \sum_{k=0}^{2N-1} C_{j,k}\phi^l_{j,k}(x)$$

and

$$g_j(x) = \sum_{k=0}^{2N-1} D_{j,k}\psi^l_{j,k}(x).$$

By the basis property the sequences of coefficients $(C_{j,k})_{k=0}^{2N-1}$ and $(D_{j,k})_{k=0}^{2N-1}$ determine the functions f_j and g_j uniquely. With the help of these sequences of coefficients we can describe the algorithm for the decomposition of $f_{j+1} \in V_{j+1}$ in the orthogonal sum of $f_j \in V_j$ and $g_j \in W_j$. Writing for $s = -4N+2, \dots, 4N-1$

$$A^j_s := c_{j,-\lfloor\frac{s}{2}\rfloor} - \frac{1}{2}(-1)^s\beta_{j,-\lfloor\frac{s}{2}\rfloor} \qquad (49)$$

and

$$B^j_s := d_{j,-\lfloor\frac{s}{2}\rfloor} + (-1)^s\left(\frac{(-1)^{-\lfloor\frac{s}{2}\rfloor}}{4N} + \frac{1}{2}\delta_{0,-\lfloor\frac{s}{2}\rfloor}\right) \qquad (50)$$

we can reformulate (27) and (28) for $s = 0, \dots, 4N-1$ as

$$\phi^l_{j+1,s}(x) = \sum_{k=0}^{2N-1} (A^j_{s-2k}\phi^l_{j,k}(x) + B^j_{s-2k}\psi^l_{j,k}(x)).$$

Then

$$f_{j+1}(x) = f_j(x) + g_j(x)$$

can be written as

$$\begin{aligned}
&\sum_{k=0}^{2N-1} C_{j,k}\phi^l_{j,k}(x) + \sum_{k=0}^{2N-1} D_{j,k}\psi^l_{j,k}(x) \\
&\quad = \sum_{s=0}^{4N-1} C_{j+1,s}\left(\sum_{k=0}^{2N-1} A^j_{s-2k}\phi^l_{j,k}(x) + B^j_{s-2k}\psi^l_{j,k}(x)\right) \\
&\quad = \sum_{k=0}^{2N-1}\left(\sum_{s=0}^{4N-1} C_{j+1,s}A^j_{s-2k}\right)\phi^l_{j,k}(x) \\
&\qquad\quad + \sum_{k=0}^{2N-1}\left(\sum_{s=0}^{4N-1} C_{j+1,s}B^j_{s-2k}\right)\psi^l_{j,k}(x).
\end{aligned}$$

The linear independence of the basis functions $\phi^l_{j,k}$ and $\psi^l_{j,k}$ implies that the coefficients of f_j and g_j are uniquely determined and we obtain for $k = 0, \dots, 2N-1$ that

$$\begin{aligned}
C_{j,k} &= \sum_{s=0}^{4N-1} C_{j+1,s}A^j_{s-2k}, \\
D_{j,k} &= \sum_{s=0}^{4N-1} C_{j+1,s}B^j_{s-2k}.
\end{aligned}$$

Now we want to describe the opposite direction. With the help of the sequence of coefficients $(C_{j,k})_{k=0}^{2N-1}$ and $(D_{j,k})_{k=0}^{2N-1}$ we reconstruct a function $f_{j+1} \in V_{j+1}$ out of $f_j \in V_j$ and $g_j \in W_j$. For $k = -4N+2, \dots, 4N-1$ we define using (19) and (20)

$$P^j_k := \frac{1}{2N} R_j(x_{j+1,k}) \tag{51}$$

and

$$Q^j_k := \frac{1}{2N} \tilde{R}_j\,(x_{j+1,k}). \tag{52}$$

Then the two-scale relations (17) and (18) for $s = 0, \ldots, 2N-1$ can be written as

$$\begin{aligned}
\phi_{j,s}^{l}(x) &= \sum_{k=0}^{4N-1} P_{k-2s}^{j} \phi_{j+1,k}^{l}(x), \\
\psi_{j,s}^{l}(x) &= \sum_{k=0}^{4N-1} Q_{k-2s}^{j} \phi_{j+1,k}^{l}(x).
\end{aligned}$$

That means we can reformulate

$$f_{j+1}(x) = f_j(x) + g_j(x)$$

as

$$\begin{aligned}
&\sum_{k=0}^{4N-1} C_{j+1,k} \phi_{j+1,k}^{l}(x) \\
&= \sum_{s=0}^{2N-1} C_{j,s} \left(\sum_{k=0}^{4N-1} P_{k-2s}^{j} \phi_{j+1,k}^{l}(x) \right) + \\
&\qquad \sum_{s=0}^{2N-1} D_{j,s} \left(\sum_{k=0}^{4N-1} Q_{k-2s}^{j} \phi_{j+1,k}^{l}(x) \right) \\
&= \sum_{k=0}^{4N-1} \left(\sum_{s=0}^{2N-1} C_{j,s} P_{k-2j}^{j} + D_{j,s} Q_{k-2j}^{j} \right) \phi_{j+1,k}^{l}(x).
\end{aligned}$$

By comparison of coefficients for $k = 0, \ldots, 4N-1$ we finish with

$$C_{j+1,k} = \sum_{s=0}^{2N-1} C_{j,s} P_{k-2j}^{j} + D_{j,s} Q_{k-2s}^{j}.$$

Finally we want to describe these algorithms in matrix notation which enables us also to introduce the Fast Fourier Transform methods in a simple manner. For $l \in \mathbb{N}_0$ we define the vectors of translates of the level j

$$\Phi_{j,l} := \left(\phi_{j,s}^{l}\right)_{s=0}^{2N-1} \quad \text{and} \quad \Psi_{j,l} \; := \; \left(\psi_{j,s}^{l}\right)_{s=0}^{2N-1}.$$

Furthermore, let $\boldsymbol{U}_{n+1}$ be the permutation matrix which sorts the elements of the vector $\Phi_{j+1,l}$ starting with the elements with even index followed by the elements with odd index, i.e.,

$$\boldsymbol{U}_{j+1} \Phi_{j+1,l} = \left(\frac{(\phi_{j+1,2s}^{l})_{s=0}^{2N-1}}{(\phi_{j+1,2s+1}^{l})_{s=0}^{2N-1}} \right).$$

With the notations (51), (52), and with the matrix

$$\boldsymbol{C}_{j+1} := \left(\begin{array}{c|c} (P^n_{2s-2k})^{2N-1}_{k=0,s=0} & (P^j_{2s+1-2k})^{2N-1}_{k=0,s=0} \\ \hline (Q^j_{2s-2k})^{2N-1}_{k=0,s=0} & (Q^j_{2s+1-2k})^{2N-1}_{k=0,s=0} \end{array}\right) \tag{53}$$

we rewrite Theorem 3.1 as

$$\left(\begin{array}{c} \Phi_{j,l} \\ \hline \Psi_{j,l} \end{array}\right) = \boldsymbol{C}_{j+1}\boldsymbol{U}_{j+1}\Phi_{j+1,l}.$$

Considering the decomposition relation from Theorem 3.2 we obtain from (49), (50) with the matrix

$$\boldsymbol{D}_{j+1} := \left(\begin{array}{c|c} (A^j_{2k-2s})^{2N-1}_{k=0,s=0} & (B^j_{2k-2s})^{2N-1}_{k=0,s=0} \\ \hline (A^j_{2k+1-2s})^{2N-1}_{k=0,s=0} & (B^j_{2k+1-2s})^{2N-1}_{k=0,s=0} \end{array}\right)$$

the representation

$$\boldsymbol{U}_{j+1}\Phi_{j+1,l} \quad = \quad \boldsymbol{D}_{j+1}\left(\begin{array}{c} \Phi_{j,l} \\ \hline \Psi_{j,l} \end{array}\right).$$

Writing

$$\boldsymbol{P}_j \ := \ (P^j_{2s-2k})^{2N-1}_{k=0,s=0} \quad \text{and} \quad \boldsymbol{A}_j \ := \ (A^j_{2k-2s})^{2N-1}_{k=0,s=0}$$

we summarize our representations in the following result.

Theorem 4.1 The coefficient matrices $\boldsymbol{C}_{j+1}$ and $\boldsymbol{D}_{j+1}$ for reconstruction and decomposition are of the following block structure

$$\boldsymbol{C}_{j+1} = \left(\begin{array}{c|c} \boldsymbol{P}_j & \boldsymbol{P}_j^T \\ \hline \boldsymbol{P}_j & -\boldsymbol{P}_j^T \end{array}\right)$$

and

$$\boldsymbol{D}_{j+1} = \left(\begin{array}{c|c} \boldsymbol{A}_j & \boldsymbol{A}_j \\ \hline \boldsymbol{A}_j^T & -\boldsymbol{A}_j^T \end{array}\right).$$

Proof By (53) the equality for $\boldsymbol{C}_{j+1}$ is equivalent with

$$\begin{aligned} P^j_{2s+1-2k} &= P^j_{2k-2s}, \\ Q^j_{2s+1-2k} &= -Q^j_{2k-2s}, \\ Q^n_{2j-2k} &= P^j_{2s-2k} \end{aligned}$$

for $s,k=0,\dots,2N-1$. These scalar equations can be proved by (51) and (19)

$$\begin{aligned}
P^j_{2s+1-2k} &= \frac{1}{2N}(D^*_N(x_{j+1,2s+1-2k})+D^*_N(x_{j+1,2s+1-2k-1}))\\
&= \frac{1}{2N}(D^*_N(x_{j+1,2k-2s-1})+D^*_N(x_{j+1,2k-2s}))\\
&= P^j_{2k-2s}
\end{aligned}$$

and analogously by (52) and (20) for $r:=2s-2k$, $(-4N+1<r<4N-1)$ with

$$\begin{aligned}
&Q^j_{2r}\\
&= \frac{1}{2N}(D^*_{2N}(x_{j,r})-D^*_{2N}(x_{j,r}-x_{j+1,1})-D^*_N(x_{j,r})+D^*_N(x_{j,r}-x_{j+1,1}))\\
&= \begin{cases} \frac{1}{2}+\frac{1}{2N}D^*_N(x_{j+1,1}), & \text{if } r\equiv 0 \bmod 2N,\\ \frac{1}{2N}D^*_N(x_{j,r}-x_{j+1,1}), & \text{if } l\not\equiv 0 \bmod 2N,\end{cases}\\
&= \frac{1}{2N}(D^*_N(x_{j,r})+D^*_N(x_{j,r}-x_{j+1,1}))\\
&= P^j_{2r}.
\end{aligned}$$

The second part can be seen immediately from the fact that $\boldsymbol{D}_{j+1}$ is the inverse matrix of $\boldsymbol{C}_{j+1}$. Indeed, using the identity matrix $\boldsymbol{I}_{j+1}$ of order $4N$ we have

$$\begin{aligned}
\boldsymbol{I}_{j+1} &= \boldsymbol{C}_{j+1}\boldsymbol{D}_{j+1}\\
&= \left(\begin{array}{c|c} \boldsymbol{P}_j & \boldsymbol{P}_j^T\\ \hline \boldsymbol{P}_j & -\boldsymbol{P}_j^T\end{array}\right)\left(\begin{array}{c|c} \boldsymbol{D}_{11} & \boldsymbol{D}_{12}\\ \hline \boldsymbol{D}_{21} & \boldsymbol{D}_{22}\end{array}\right).
\end{aligned} \tag{54}$$

Solving the linear ystem out of (54) we obtain

$$\begin{aligned}
\boldsymbol{A}_j &= \boldsymbol{D}_{11}=\frac{1}{2}\boldsymbol{P}_j^{-1}, \qquad (55)\\
\boldsymbol{D}_{21} &= \boldsymbol{D}_{11}^T,\\
\boldsymbol{D}_{12} &= \boldsymbol{D}_{11},\\
\boldsymbol{D}_{22} &= -\boldsymbol{D}_{11}^T,
\end{aligned}$$

which proves the Theorem. ∎

It is easy to see that the matrices $\boldsymbol{P}_j$ and $\boldsymbol{A}_j$ are circulant matrices, i.e.

$$\boldsymbol{P}_j = \text{circ }(P_{2s})_{s=0}^{2N-1}, \qquad \boldsymbol{A}_j = \text{circ }(A_{4N-2s})_{s=0}^{2N-1}. \tag{56}$$

Remember that for a vector $a = (a_0, a_1, \dots, a_{N-1})^T$ the circulant matrix constructed from a is given by

$$\begin{aligned} \operatorname{circ} a &:= \left(a_{(k-s) \bmod N}\right)_{s=0,k=0}^{N-1} \\ &= \left(a_{s,k}\right)_{s=0,k=0}^{N-1} := \begin{pmatrix} a_0 & a_1 & \cdots & a_{N-1} \\ a_{N-1} & a_0 & \cdots & a_{N-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \cdots & a_0 \end{pmatrix}. \end{aligned}$$

For more details cf. Davis [4]. Particularly important is the factorization of a circulant matrix by Fourier matrices

$$\boldsymbol{F}_N = \frac{1}{\sqrt{N}} \left(w_N^{sk}\right)_{k=0,s=0}^{N-1} \quad \text{with} \quad w_N = e^{\frac{-2\pi i}{N}}$$

so that one obtains for an $N \times N$ circulant matrix $\boldsymbol{Y}$ the diagonal matrix

$$\boldsymbol{F}_N \ \boldsymbol{Y} \ \overline{\boldsymbol{F}}_N = \operatorname{diag} d(\boldsymbol{Y}),$$

which contains on the main diagonal the eigenvalues $d_r(\boldsymbol{Y})$ $(r = 0, \dots, N-1)$ of $\boldsymbol{Y}$. Finally, for applying fast algorithms we only have to determine these eigenvalues.

Lemma 4.1 The matrices $\boldsymbol{P}_j$ und $\boldsymbol{A}_j$ are uniquely determined by the vector of their eigenvalues

$$d_r(\boldsymbol{P}_j) = \begin{cases} 1, & \text{if } r = 0, \\ \frac{1}{2} + \frac{1}{2} e^{\frac{i\pi r}{2N}}, & \text{if } 1 \le r \le N-1, \\ \frac{1}{2}, & \text{if } r = N, \\ \frac{1}{2} - \frac{1}{2} e^{\frac{i\pi r}{2N}}, & \text{if } N+1 \le r \le 2N-1, \end{cases}$$

and

$$d_r(\boldsymbol{A}_j) = \begin{cases} \frac{1}{2}, & \text{if } r = 0, \\ \frac{1}{1+e^{\frac{i\pi r}{2N}}}, & \text{if } 1 \le r \le N-1, \\ 1, & \text{if } r = N, \\ \frac{1}{1-e^{\frac{i\pi r}{2N}}}, & \text{if } N+1 \le r \le 2N-1. \end{cases}$$

Proof One easily computes the eigenvalues $d_r(\boldsymbol{P}_j)$ $(r = 0, \dots, 2N-1)$

by (cf. [4])

$$
\begin{aligned}
d_r(\boldsymbol{P}_j) &= \frac{1}{2N}\sum_{s=0}^{2N-1} P_{2s}^{j} e^{\frac{is\pi r}{N}} \\
&= \frac{1}{2N}\sum_{s=0}^{2N-1} (N\delta_{0,s} + D_N^*(x_{j,s} - x_{j+1,1}))e^{\frac{is\pi r}{N}} \\
&= \frac{1}{2} + \frac{1}{2N}\sum_{s=0}^{2N-1}\left(\frac{1}{2} + \sum_{m=1}^{N-1}\cos\left(\frac{m\pi s}{N} - \frac{m\pi}{2N}\right)\right. \\
&\qquad \left. + \frac{1}{2}\cos\left(\frac{N\pi s}{N} - \frac{N\pi}{2N}\right)\right)\left(\cos\frac{r\pi s}{N} + i\sin\frac{r\pi s}{N}\right).
\end{aligned}
$$

The addition formulas together with

$$
\sum_{s=0}^{2N-1}\cos\left(x - \frac{sk\pi}{N}\right) = \begin{cases} 2N\cos x, & \text{if } k \equiv 0 \bmod 2N, \\ 0, & \text{otherwise}, \end{cases}
$$

and

$$
\sum_{s=0}^{2N-1}\sin\left(x - \frac{sk\pi}{N}\right) = \begin{cases} 2N\sin x, & \text{if } k \equiv 0 \bmod 2N, \\ 0, & \text{otherwise}, \end{cases}
$$

imply the assertion for $d_r(\boldsymbol{P}_j)$.

Taking into account (55) we obtain the eigenvalues of the matrix $\boldsymbol{A}_j$. Namely with $\overline{\boldsymbol{F}}_{2N} = \boldsymbol{F}_{2N}^{-1}$ we have

$$
\begin{aligned}
\overline{\boldsymbol{F}}_{2N}\,\mathrm{diag}\,d(\boldsymbol{A}_j)\,\boldsymbol{F}_{2N} &= \frac{1}{2}\left(\overline{\boldsymbol{F}}_{2N}\,\mathrm{diag}\,d(\boldsymbol{P}_j)\,\boldsymbol{F}_{2N}\right)^{-1} \\
&= \frac{1}{2}\overline{\boldsymbol{F}}_{2N}\,(\mathrm{diag}\,d(\boldsymbol{P}_j))^{-1}\,\boldsymbol{F}_{2N},
\end{aligned}
$$

which finally yields

$$
d_r(\boldsymbol{A}_j) = \frac{1}{2\,d_r(\boldsymbol{P}_j)} \qquad \text{for } r = 0, \ldots, 2N-1. \qquad \blacksquare
$$

Let us finish this section with the investigation of the basis transformations between different bases

$$
\left(\phi_{j,s}^{l}(x)\right)_{s=0}^{2N-1} \quad \text{and} \quad \left(\phi_{j,s}^{l-m}(x)\right)_{s=0}^{2N-1}
$$

of the same space V_j. That means we seek for the matrix $\boldsymbol{T}_{j,m} = (t_{s,k}^{j,m})_{s=0,k=0}^{2N-1}$ with

$$
\Phi_{j,l} = \boldsymbol{T}_{j,m}\Phi_{j,l-m}.
$$

Again the circulant structure of the matrix $\boldsymbol{T}_{j,m}$ is the main ingredient of our considerations.

Lemma 4.2 The matrix $\boldsymbol{T}_{j,m} = \overline{\boldsymbol{F}}_{2N} \, diag \, d(\boldsymbol{T}_{j,m}) \, \boldsymbol{F}_{2N}$ has the entries

$$t^{j,m}_{s,k} = \frac{1}{2N} \sum_{r=0}^{2N-1} \operatorname{sinc}^m rb \;\; e^{\frac{ir(k-s)\pi}{N}}$$

with eigenvalues

$$d_r(\boldsymbol{T}_{j,m}) = \operatorname{sinc}^m rb.$$

Proof Considering for $s = -N, \ldots, N$ the complex Fourier coefficients $c_s(\Phi^l_j)$ and $c_s(\Phi^{l-m}_j)$ we obtain with (11), (12) and

$$\begin{aligned} \Phi^l_j(x - x_{j,q}) &= (\Phi^{l-m}_j * \widetilde{H}_m)(x - x_{j,q}) \\ &= \sum_{k=0}^{2N-1} t^j_{q,k} \Phi^{l-m}_j (x - x_{j,k}) \;\; \text{for } q = 0, \ldots, 2N-1, \end{aligned}$$

that for $s = -N, \ldots, N$

$$\begin{aligned} c_s\left(\Phi^l_j\left(\cdot - \frac{\pi q}{N}\right)\right) &= 2\pi \; c_s(\Phi^{l-m}_j) \; c_s(\widetilde{H}_m) \; e^{-\frac{i\pi s q}{N}} \\ &= \sum_{k=0}^{2N-1} t^{j,m}_{q,k} \; c_s(\Phi^{l-m}_j) \; e^{-\frac{i\pi s k}{N}}. \end{aligned}$$

Hence

$$\sum_{k=0}^{2N-1} t^{j,m}_{q,k} \; e^{-\frac{i\pi s(k-q)}{N}} = 2\pi \; c_s(\widetilde{H}_m) \;=\; \operatorname{sinc}^m sb$$

which means that we have computed the eigenvalues of $\boldsymbol{T}_{j,m}$ so that

$$d_r(\boldsymbol{T}_{j,m}) = \sum_{k=0}^{2N-1} t^{j,m}_{0,k} \; e^{\frac{i\pi r k}{N}} \;=\; \operatorname{sinc}^m rb.$$

By discrete Fourier transform the matrix $\boldsymbol{T}_{j,m}$ has the entries

$$t^{j,m}_{q,k} = \frac{1}{2N} \sum_{r=0}^{2N-1} d_r(\boldsymbol{T}_{j,m}) \; e^{\frac{ir(k-q)\pi}{N}}. \qquad \blacksquare$$

Let us finally mention that the inverse relation reads as

$$\Phi_{j,l-m} = \boldsymbol{T}^{-1}_{j,m} \Phi_{j,l},$$

with $\boldsymbol{T}_{j,m}^{-1} = \overline{\boldsymbol{F}}_{2N} \operatorname{diag} (d_r(\boldsymbol{T}_{j,m})^{-1})_{r=0}^{2N-1} \ \boldsymbol{F}_{2N}$.

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