BEST APPROXIMATION IN LIPSCHITZ SPACES*

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With the help of de la Vallée Poussin sums we describe best approximation in Lipschitz spaces. This yields equivalence theorems as well as estimates for special approximation processes.

Let $X$ be one of the spaces $C$ or $L^p$ $(1 \leq p < \infty)$ of $2\pi$-periodic complex-valued functions. If $f \in C$, we will write $\|f\|_C$ instead of $\|f\|_C$. For $0 \leq \beta \leq 1$ and $r = 0, 1, 2, ...$, we denote by $X^{r, \beta}$ the class of functions $f$ with the following property: There exists a $2\pi$-periodic $(r-1)$-times absolutely continuous function $g$ with $g^{(r)} \in X$ (in the case $r = 0$), $f = g$ in $X$ and

$$
\|g^{(r)}\|_{p, \beta} = \sup_{h \neq 0} |h|^{-\beta} \|g^{(r)}(\cdot + h) - g^{(r)}(\cdot)\|_p < \infty.
$$

The norm in $X^{r, \beta}$ is given by

$$
\|f\|_{p, r, \beta} = \sum_{k=0}^{r} \|g^{(k)}\|_p + \|g^{(r)}\|_{p, \beta}.
$$

Further we define the subspaces $X^{r, \beta}$, $0 \leq \beta < 1$, in the following way:

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\[ \mathcal{X}^{r,\beta} = \{ f \in X^{r,\beta} : \lim_{h \to 0} |h|^{-\beta} \| g(r)(\cdot+h) - g(r)(\cdot) \|_p = 0 \} \]

with norm \[ \| f \|_{p, r, \beta} \].

Let \( T_n \) be the set of trigonometric polynomials of order \( \leq n \).

The best approximation of \( f \in X^{r,\beta} \) is denoted by

\[ E_n(f, X^{r,\beta}) = \inf_{p_n \in T_n} \| f - p_n \|_{p, r, \beta}. \]


\[ a_{n,\ell} f = \frac{1}{2\ell+1} \sum_{k=-\ell}^{\ell} S_k f \quad (0 \leq \ell \leq n). \]

This yields the following Jackson-type theorem and further applications.

**Theorem 1.** Let \( r+\beta \leq m+\alpha \) and \( 0 \leq r \leq m \). For all \( n=0,1,2,\ldots \) and \( f \in X^{m,\alpha} \) there exists a constant \( C > 0 \) such that

\[ E_n(f, X^{r,\beta}) \leq C(n+1)^{r+\beta-m-\alpha} \| g(m) \|_{p,\alpha}. \]

Further we have for \( f \in X^{m,\alpha} \), \( 0 \leq \alpha < 1 \),

\[ E_n(f, X^{r,\beta}) = o(n^{r+\beta-m-\alpha}), \quad n \to \infty. \]

**Corollary 2.** For \( r+\beta \leq m+\alpha \), \( 0 \leq r \leq m \) and \( f \in X^{m,\alpha} \) it holds

\[ E_n(f, X^{r,\beta}) \leq C(n+1)^{r+\beta-m-\alpha} E_n(f, X^{m,\alpha}). \]

A simple application of the above results gives a Kandaliya-type estimate for special approximation processes.

**Corollary 3.** Let \( r+\beta \leq m+\alpha \), \( 0 \leq r \leq m \) and \( f \in X^{m,\alpha} \). If \( t_n \in T_n \) is such that

\[ \| f - t_n \|_{p, r, \beta} \leq C(n) E_n(f, X), \]

then

\[ \| f - t_n \|_{p, r, \beta} \leq cC(n)(n+1)^{r+\beta-m-\alpha} \| g(m) \|_{p,\alpha} \]

with a constant \( c > 0 \).

The Lipschitz spaces are, in general, not separable. That is why it is interesting to know for which \( f \in X^{r,\beta} \) the condition

(1) \[ E_n(f, X^{r,\beta}) = o(1), \quad n \to \infty, \]

is fulfilled.

**Theorem 4.** Let \( f \in X^{r,\beta} \) and \( 0 \leq \beta < 1 \). Then (1) is equivalent to \( f \in \mathcal{X}^{r,\beta} \).

In the following we need a Bernstein-type inequality for trigonometric polynomials.

**Theorem 5.** If \( r+\beta \leq \ell+m+\alpha \), then for any \( p_n \in T_n \)

\[ \| p_n(\cdot) \|_{p, m, \alpha} \leq \| p_n(\cdot) \|_{p, r, \beta} \cdot \begin{cases} 1, & \text{if} \; n = 0, \\ \max(4, 1 + \frac{m}{r+\beta}), & \text{if} \; n = 1, \\ 4n^{\ell+m+\alpha-r-\beta}, & \text{if} \; n > 1. \end{cases} \]
Now we can formulate the main equivalence theorem. (For arbitrary Banach spaces an analogous result is due to P. L. Butzer and K. Scherer [2].) We apply the usual k-th modulus of continuity \( \omega_k(f, x, h) \) (see e.g. [1]).

**Theorem 6.** Let \( 0 \leq r + \beta < r' + \beta' < m + \alpha < m + \gamma, \: 0 \leq \alpha, \beta, \beta', \gamma \leq 1, \: f \in X^{r', \beta}, \) and

\[
E_n(f, x^{r', \beta}) = \| f - p_n^r p_{r', \beta} \|.
\]

Then the following assertions are equivalent:

1. \( f \in X^{m, \alpha}, \: \| f \| \lesssim n^{-\alpha} \) for all \( n \),
2. \( f \in X^{m-1,0}, \: \omega_2(g(m-1), X, h) = o(h), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \),
3. \( f \in X^{m,0}, \: \omega_2(g(m), X, h) = o(h), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \),
4. \( \omega_k(f, x, h) = o(h), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \),
5. \( \omega_k(g(r), X, h) = o(h^{m+\alpha}), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \),
6. \( \omega_k(g(r'), X, h) = o(h^{m+\alpha}), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \).

Assume that one of the additional conditions is fulfilled:

a) There exist constants \( A \) and \( B \) with \( 0 < A < B < 1 \), \( A n^{-\alpha} \lesssim B n \) for all \( n \).

b) \( 1 < p < \infty \) and there exists a constant \( 0 < B < 1 \) with \( \ell \lesssim B n \) for all \( n \).

c) \( m + \alpha - r - \beta < 1 \) and there exists a constant \( 0 < A < 1 \) with \( \ell \geq A n \) for all \( n \).

Then the following assertions are equivalent to 1)-10):

11. \( \| f - p_{n, \alpha} f \|_{p^r, r', \beta} \lesssim n^{\alpha} \) for all \( n \).
12. \( \| (g - p_{n, \alpha} f)(r) \|_{p^r, r', \beta} \lesssim n^{\alpha} \) for all \( n \).

If a) or b) is fulfilled, then the following assertions are equivalent to 1)-12):

13. \( \| q_{n, \alpha} f \|_{p^s, s', \gamma} \lesssim n^{s+\gamma-m-\alpha} \) for all \( n \).
14. \( \| (q_{n, \alpha} f)(s) \|_{p^s, s', \gamma} \lesssim n^{s+\gamma-m-\alpha} \) for all \( n \).

Theorem 6 is also true in a "small-\( o \)" version.

**Theorem 7.** Under the assumptions of Theorem 6 the following assertions are equivalent:

1' \( f \in X^{m, \alpha}, \: \| f \| \lesssim n^{-\alpha} \) for all \( n \),
2' \( f \in X^{m-1,0}, \: \omega_2(g(m-1), X, h) = o(h), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \),
3' \( f \in X^{m,0}, \: \omega_2(g(m), X, h) = o(h), \: h \to 0^+, \: \| f \| \lesssim n^{\alpha} \) for all \( n \),

J' \( J = 2, \ldots, 14 \).

Here we get J') from J), if we replace "large-\( o \)" by "small-\( o \)".

To prove the equivalence theorems, we use the following estimates.
**Lemma 7.** For $0 \leq \xi \leq n$ we have

$$
\| \sigma_n f \|_{X, \xi, \beta} \leq \| \sigma_n f \|_{X, X} \cdot 
$$

$$
\| f - \sigma_n f \|_{p, r, \beta} \leq (1 + \| \sigma_n f \|_{X, X}) \, E_n(f, X^r, \beta)
$$

and

$$
\| f - \sigma_n f \|_{p, r, \beta} \leq 36 \sum_{j=0}^{n+\xi} \frac{E_{j+\xi}(f, X^r, \beta)}{\xi+j+1}.
$$

The last estimate is a consequence of a result due to W. Dahmen [3].

**References**


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