A Theorem of Gopengauz Type with Added Interpolatory Conditions

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Abstract

The theorem of Gopengauz guarantees the existence of a polynomial which well approximates a function $f \in C^q[-1, 1]$, while at the same time its kth derivative ($k \le q$) well approximates the kth derivative of the function, and moreover the polynomial and its derivatives respectively interpolate the function and its derivatives at ± 1. With more generality, we shall prescribe that the polynomial interpolate the function at up to $q + 1$ points near 1 and up to $q + 1$ points near -1. The points may coalesce, in which case one also interpolates at the coalescent point a number of derivatives one less than the multiplicity of coalescence. Aside from intrinsic theoretical interest, our results are clearly applicable in describing more precisely the error incurred in certain linear processes of simultaneous approximation, such as interpolation with added nodes near ± 1. The original theorem of Gopengauz will be shown to follow as a special case.

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Introduction:

Let q be a natural number. Then for $n \ge 2q+1$ we consider sets of points $S_n := \{s_{0,n} \ldots, s_{q,n}, t_{0,n}, \ldots, t_{q,n}\}$, in which $-1 \le s_{0,n} \le \ldots \le s_{q,n} < -1 + \frac{1}{n^2}$ and $1 \ge t_{0,n} \ge \ldots \ge t_{q,n} > 1 - \frac{1}{n^2}$. Now that we have indicated clearly the dependence of the elements of S_n upon n, the second subscript, n, is clearly redundant and will be suppressed. Associated with each set S_n is also the polynomial Q_n defined by

$$Q_n(x) := (x - s_0) \ldots (x - s_q)(x - t_q) \ldots (x - t_0)$$

which has zeroes which depend on n but is always of degree $2q + 1$.

We recall that the notation $\omega(f;\cdot)$ denotes the modulus of continuity of the function f, defined by

$$\omega(f;\delta) := sup_{|x-y|\leq\delta}|f(x)-f(y)|,$$

and we are ready to state our results.

Theorem 1: *Let $f \in C^q[-1,1]$, with $q \geq 0$. Then for each $n \geq 2q+1$, there exists a polynomial P_n of degree at most n such that for $k=0,\ldots,q$ and for $-1 \leq x \leq 1$*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \leq K \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-k} \omega\left(f^{(q)}; \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \quad (1)$$

with K independent of n and f and also satisfying

$$|f(x) - P_n(x)| \leq$$

$$\frac{K}{n^q} \min_{i,j} \left(\sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\, \omega\left(f^{(q)}; \min_{\ell\neq i,\, m\neq j} \frac{\sqrt{|(x-t_\ell)(x-s_m)|}}{n}\right)\right). \quad (2)$$

A related result is:

Theorem 2: *Let $f \in C^q[-1,1]$ and let P_n satisfy (1) for some constant K. If P_n also interpolates f on S_n, then P_n also satisfies (2) with a constant CK, in which C is an absolute constant.*

From Theorem 1 and Theorem 2 and their proofs we have *a fortiori* the following:

Corollary: *Let S_{n,j_1,j_2} be a set containing points $s_0,\ldots,s_{j_1}$ in $[-1,-1+\frac{1}{n^2}]$ and points $t_0,\ldots,t_{j_2}$ in $[1-\frac{1}{n^2},1]$, the count in both cases including any multiplicity. If $j_1, j_2 \leq q$ and if $f \in C^q[-1,1]$, then there is a polynomial P_n which interpolates f on S_{n,j_1,j_2} and satisfies (1) with a constant K independent of n and f. Furthermore, if P_n is any polynomial interpolating f on S_{n,j_1,j_2} and satisfying (1) with any constant K (whether independent of the other quantities or not), then for $0 \leq x \leq 1$*

$$|f(x) - P_n(x)| \leq$$

$$\frac{CK}{n^{j_2}} \min_{0\leq i\leq j_2} \left(\sqrt{\frac{|(x-t_0)\ldots(x-t_{j_2})|}{|x-t_i|}} \left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)^{q-j_2} \omega\left(f^{(q)}; \min_{\ell\neq i} \frac{\sqrt{|x-t_\ell|}}{n}\right)\right).$$

A similar inequality involving the points $s_0, \ldots, s_{j_1}$ is valid for $-1 \le x \le 0$.

An important special case occurs when in the set S_n we have $s_j = -1$ and $t_j = 1$ for $j = 0, \ldots, q$. Then as a consequence of Theorem 1 and Theorem 2 we get (2), and then because of the fact that $f^{(k)}(\pm 1) - P_n^{(k)}(\pm 1) = 0$ for $k = 0, \ldots, q$ we again get (2) for each of the derivatives (with q replaced, of course, with $q - k$). Therefore, Theorem 1 and Theorem 2 together imply the original theorem of Gopengauz [3], which states

Theorem: *Let $f \in C^q[-1, 1]$. Then for each $n \ge 4q + 5$ (*we need only $n \ge 2q + 1$ here*), there exists a polynomial P_n of degree at most n such that for $k = 0, \ldots, q$ and for $-1 \le x \le 1$*

$$|f^{(k)}(x) - P_n^{(k)}(x)| \le K \left(\frac{\sqrt{1-x^2}}{n} \right)^{q-k} \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n} \right) \tag{3}$$

with K independent of n and f.

We turn now to the proofs of our two theorems. Because Theorem 2 will be used in the proof of Theorem 1, we will prove Theorem 2 first and then Theorem 1.

Proof of Theorem 2:

The proof of Theorem 2 divides itself naturally into two cases: that $q = 0$ is one of the cases, and that $q > 0$ is the other.

The Case $q = 0$: We begin with (1) for some given constant K and with

$$|f(t_0) - P_n(t_0)| = 0.$$

We must show the stronger inequality

$$|f(x) - P_n(x)| \le C' K\, \omega\left(f;\ \frac{\sqrt{|(x - t_0)(x - s_0)|}}{n} \right). \tag{4}$$

If $x = t_0$ or if $x = s_0$, then (4) follows no matter what the constant is. More than this, in Balázs and Kilgore ([1]) it is shown that if $t_0 = 1$ and $s_0 = -1$ we get

$$|f(x) - P_n(x)| \le (7K + 7)\, \omega\left(f;\ \frac{\sqrt{1-x^2}}{n} \right). \tag{5}$$

Applying (5) on the interval $[s_0, t_0]$, of length not less than $2(1-\frac{1}{n^2})$, we have there immediately the estimate

$$|f(x) - P_n(x)| \le (7K+7)\left(\frac{n^2}{n^2-1}\right)\ \omega\left(f;\ \frac{\sqrt{|(x-t_0)(x-s_0)|}}{n}\right).$$

Therefore, it remains to show (4) for $-1 \le x < s_0$ and for $t_0 < x \le 1$, in both of which cases we have $\sqrt{1-x^2} < \frac{1}{n}$. Since these two cases are similar, we will assume with no loss of generality that $t_0 < x \le 1$. Taking into account that $f(t_0) = P_n(t_0)$, we may estimate

$$|f(x) - P_n(x)| \le |f(x) - f(t_0)| + |P_n(x) - P_n(t_0)|, \tag{6}$$

and

$$|f(x) - f(t_0)| \le \omega(f;\ |x - t_0|) \le \omega\left(f;\ \frac{\sqrt{|(x-t_0)(x-s_0)|}}{n}\right). \tag{7}$$

Now we notice that for arbitrary $t \in [-1, 1]$

$$|P_n(t) - P_n(t_0)| \le |P_n(t) - f(t)| + |f(t) - f(t_0)|,$$

whence also for arbitrary $t \in [-1, 1]$

$$|P_n(t) - P_n(t_0)| \le K\,\omega\left(f;\ \frac{\sqrt{1-t^2}}{n} + \frac{1}{n^2}\right) + \omega(f;\ |t-t_0|). \tag{8}$$

Now one needs to obtain an estimate for $|P_n(x) - P_n(t_0)|$ of the form

$$|P_n(x) - P_n(t_0)| \le \text{const}\ \omega\left(f;\ \frac{\sqrt{|(x-t_0)(x-s_0)|}}{n}\right), \tag{9}$$

in which $t_0 < x \le 1$. If one wishes to estimate the constant, an economical but tedious method for obtaining this estimate is laid out in Balázs and Kilgore [1]. One begins by letting $t = \cos\theta$, $t_0 = \cos\theta_0$, $x = \cos\theta_x$, and $P_n(x) - P_n(t_0) = T_n(\theta)$, an even trigonometric polynomial. Then, using an identity of Balázs and Kilgore (cf. [1] or [2]) for the derivative of a trigonometric polynomial we can write

$$\begin{aligned} T_n'(\theta) &= \frac{1}{8n}\sum_{j=1}^{4n} T_n(\theta+\theta_j)\left(\frac{\sin\frac{n\theta_j}{2}}{n\sin\frac{\theta_j}{2}}\right)^2 \frac{(-1)^{j+1}}{(\sin\frac{1}{2}\theta_j)^2} \\ &\le \frac{1}{8n}\sum_{j=1}^{4n} |T_n(\theta+\theta_j)|\left(\frac{\sin\frac{n\theta_j}{2}}{n\sin\frac{\theta_j}{2}}\right)^2 \frac{1}{(\sin\frac{1}{2}\theta_j)^2} \end{aligned} \tag{10}$$

in which

$$\theta_j := \frac{2j-1}{4n}\pi.$$

Inserting the estimate (8) into (10) and using the fact that $\omega(\lambda\delta) \le (\lambda + 1)\omega(\delta)$, one obtains an estimate for $|T_n'(\theta)|$ valid on the interval $[\theta_x, \theta_0]$, and then one obtains (9) by integration from θ_x to θ_0. An alternative is to use Brudnyi's inequality, in which the constant has also been estimated in Balázs and Kilgore [2]. We omit further details here.

This completes the argument for the case that $q = 0$.

The Case $q > 0$: It remains to show for $q \ge 1$ that if the polynomials P_n satisfy (1) with constant K and also interpolate f on the set S_n, then for $i, j = 0, \ldots, q$ and for $\ell \ne i, m \ne j$ we can obtain

$$|f(x) - P_n(x)| \le$$
$$C''Kn^{-q}\left(\sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\right)\omega\left(f^{(q)};\ \frac{\sqrt{|(x-t_\ell)(x-s_m)|}}{n}\right). \tag{11}$$

We must begin by considering the relationship between $\sqrt{1-x^2}+\frac{1}{n}$ and a typical factor of $|Q_n|$, of the form $|(x-t_i)(x-s_j)|$, for arbitrary i and j, on various subsets of the interval $[-1,1]$. On the subinterval $[-1+\frac{1}{n^2}, 1-\frac{1}{n^2}]$ we have

$$\sqrt{1-x^2}+\frac{1}{n} \ \le\ |(x-t_i)(x-s_j)|^{\frac{1}{2}}\ \frac{\sqrt{1-x^2}+\frac{1}{n}}{\sqrt{(1-\frac{1}{n^2})^2-x^2}}\ .$$

In turn,

$$\frac{\sqrt{1-x^2}+\frac{1}{n}}{\sqrt{(1-\frac{1}{n^2})^2-x^2}} = \left(\frac{1-x^2+\frac{2}{n}\sqrt{1-x^2}+\frac{1}{n^2}}{1-x^2-\frac{2}{n^2}+\frac{1}{n^4}}\right)^{\frac{1}{2}} \tag{12}$$

If we assume that $\sqrt{1-x^2} \ge \frac{c}{n}$, where c is sufficiently large (for example $c \ge 2$), then (12) gives a decreasing function of $\sqrt{1-x^2}$, and therefore we get for these x

$$\frac{\sqrt{1-x^2}+\frac{1}{n}}{\sqrt{(1-\frac{1}{n^2})^2-x^2}} = \left(\frac{\frac{c^2}{n^2}+\frac{2c}{n^2}+\frac{1}{n^2}}{\frac{c^2}{n^2}-\frac{2}{n^2}+\frac{1}{n^4}}\right)^{\frac{1}{2}} \le \frac{c+1}{\sqrt{c^2-2}}\,. \tag{13}$$

Now if $\sqrt{1-x^2} \le \frac{c}{n}$ we must consider two subcases.

(i) If $s_j \le x \le t_i$ and $\sqrt{1-x^2} \le \frac{c}{n}$, then $|(x-t_i)(x-s_j)| \le \frac{c}{n}$.

(ii) If $t_i \le x \le 1$ or if $-1 \le x \le s_j$, then without loss of generality let $t_i \le x \le 1$. Then $|x - t_i| \le \frac{1}{n^2}$, and $|x - s_j| \le 2$. Thus in this case $|(x-t_i)(x-s_j)|^{\frac{1}{2}} \le \frac{\sqrt{2}}{n}$.

We are ready now to obtain estimates for the constant C'' in (11). Choosing for example $c = 2$, we recall that $\omega(\lambda\delta) \le ([\lambda]+1)\omega(\delta)$ and apply the estimate (13) to (1) to obtain for $\sqrt{1-x^2} \ge \frac{2}{n}$ that

$$|f(x) - P_n(x)| \le$$

$$\left(\frac{3}{\sqrt{2}}\right)^q K n^{-q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\ \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \le$$

$$3 \cdot \left(\frac{3}{\sqrt{2}}\right)^q K n^{-q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\ \omega\left(f^{(q)};\ \frac{\sqrt{|(x-t_\ell)(x-s_m)|}}{n}\right). \quad (14)$$

On the other hand, if $\sqrt{1-x^2} \le \frac{2}{n}$, then it follows from our previous analysis that for $\ell \ne i$, $m \ne j$ we can estimate

$$\left|\frac{Q_n(x)}{(x-t_i)(x-s_j)(x-t_\ell)(x-s_m)}\right| \le 2^{q-1},$$

and hence using Rolle's theorem we obtain

$$\begin{aligned}\frac{|f(x)-P_n(x)|\sqrt{|(x-t_i)(x-s_j)|}}{\sqrt{|(x-t_\ell)(x-s_m)|}\sqrt{|Q_n(x)|}} &\le \left(\frac{\sqrt{2}}{n}\right)^{q-1} \frac{|(f(x)-P_n(x))(x-t_i)(x-s_j)|}{|Q_n(x)|} \\ &\le \frac{(\sqrt{2})^{q-1}}{q!\, n^{q-1}} |f^{(q)}(z) - P_n^{(q)}(z)|, \quad (15)\end{aligned}$$

for some intermediate point z lying between x and 1. Since $\sqrt{1-z^2} \le \sqrt{1-x^2} \le \frac{2}{n}$, we may now further estimate

$$\begin{aligned}|f^{(q)}(z) - P_n^{(q)}(z)| \cdot \left(\frac{2}{n}\right)^{q-1} &\le K\left(\frac{2}{n}\right)^{q-1}\left(\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)\frac{\omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right)}{\frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}} \\ &\le 3\, K\left(\frac{2}{n}\right)^q \frac{\omega\left(f^{(q)};\ \frac{\sqrt{|(x-t_\ell)(x-s_m)|}}{n}\right)}{\sqrt{|(x-t_\ell)(x-s_m)|}},\end{aligned}$$

in which we have also used the fact that $\delta^{-1}\omega(\delta) \le 2\eta^{-1}\omega(\eta)$ for $0 < \delta < \eta$. Therefore, for $\sqrt{1-x^2} \le \frac{2}{n}$ we obtain

$$|f(x) - P_n(x)| \le$$

$$\frac{3 \cdot 2^q\, K}{q!\, n^q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\ \omega\left(f^{(q)};\ \frac{\sqrt{|(x-t_\ell)(x-s_m)|}}{n}\right). \tag{16}$$

The estimate (14) is therefore seen to be dominant on the entire interval $[-1,1]$, with the much stronger inequality (16) holding near the ends.

The estimate for C'' in (14) can be improved for large values of q by allowing the parameter c to vary with q. Indeed, we can obtain a constant C'' which is independent of q as well as of the other quantities involved and is thus absolute. To do this, we can choose $c = (q!)^{\frac{1}{q}}$. Then similarly as in (15) and (16) we can estimate

$$\begin{aligned} |f(x) - P_n(x)| &\le \frac{1}{n^q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\, |f^{(q)}(z) - P_n^{(q)}(z)| \\ &\le \frac{K}{n^q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\ \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right), \end{aligned} \tag{17}$$

valid for $\sqrt{1-x^2} \le \frac{(q!)^{\frac{1}{q}}}{n}$.

Now, from Stirling's formula we know that

$$e^{-1} q \le (q!)^{\frac{1}{q}}.$$

Hence, if we now let $\sqrt{1-x^2} \ge \frac{q}{e\,n}$, we will get an estimate which is also valid for $\sqrt{1-x^2} \ge \frac{(q!)^{\frac{1}{q}}}{n}$. Thus, choosing now $c = \frac{q}{e}$ in (13) we obtain (for $q > 4$)

$$\left(\frac{\sqrt{1-x^2} + \frac{1}{n}}{\sqrt{(1-\frac{1}{n^2})^2 - x^2}}\right)^q \le \left(1 + \frac{2q + 3e}{q^2 - 2e^2}\right)^{\frac{q}{2}} \le e^e,$$

and from this we obtain

$$|f(x) - P_n(x)| \le \frac{e^e\, K}{n^q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\ \omega\left(f^{(q)};\ \frac{\sqrt{1-x^2}}{n} + \frac{1}{n^2}\right) \tag{18}$$

for $\sqrt{1-x^2} \ge \frac{q}{e}$, and, combining (18) with (17) we see that the estimate (17) in fact holds on the entire interval $[-1,1]$.

Now, just as was done in deriving (14) we can obtain from (18) the estimate

$$|f(x) - P_n(x)| \leq$$
$$\frac{3\, e^e\, K}{n^q} \sqrt{\frac{|Q_n(x)|}{|(x-t_i)(x-s_j)|}}\ \omega\left(f^{(q)}; \frac{\sqrt{|(x-t_\ell)(x-s_m)|}}{n}\right) \tag{19}$$

for $\sqrt{1-x^2} \geq \frac{2}{n}$. If on the other hand $\sqrt{1-x^2} \leq \frac{2}{n}$ we have this estimate already, as a consequence of (16). Therefore, (19) holds on the entire interval $[-1,1]$, and our proof is completed in the case $q > 0$, with $C'' \leq \min\{\frac{3^{q+1}}{(\sqrt{2})^q}, 3e^e\}$. For the constant C we must now take the maximum of C' and C'', and the proof of Theorem 2 is concluded.

Proof of Theorem 1:

We will construct a polynomial P_n which satisfies both (1) and (2), with a constant independent of both n and f. According to a result of Trigub [4], a polynomial p_n exists which satisfies (1) with a constant K independent of n and f. Beginning with this polynomial p_n, we will construct polynomials $p_{n,0}, \ldots, p_{n,q}$. For each $j = 0, \ldots, q$ the polynomial $p_{n,j}$ will be seen still to satisfy (1) (with a new constant K_q depending on q and K) and to satisfy

$$(f - p_{n,j})(t_k) = (f - p_{n,j})(t_k) = 0 \ \text{ for } \ k = 0, \ldots, j. \tag{20}$$

Then we will define

$$P_n := p_{n,q}. \tag{21}$$

It is clear from Theorem 2 that the polynomial P_n will then satisfy both (1) and (2).

We define first a polynomial $p_{n,0,0}$ by

$$p_{n,0,0}(x) = (f(s_0) - p_n(s_0))\left(\frac{1-x}{2}\right) + (f(t_0) - p_n(t_0))\left(\frac{1+x}{2}\right).$$

Then both (1) and (20) follow for $p_n + p_{n,0,0}$ in place of p_n. Thus we can let $p_{n,0} := p_n + p_{n,0,0}$ and, defining P_n by (21), we have (1) with a new constant $K_0 \leq 2K$. If $q = 0$ we are now finished.

We now construct the polynomial $p_{n,0}$ in the eventuality that $q > 0$. Let us assume inductively that we have constructed a polynomial $p_{n,j-1}$ for which (20) holds and for which (1) holds with a constant of K_{j-1}. We now show how to construct the polynomial $p_{n,j}$. To this end, we let m denote the greatest even integer such that $mq < n$, and we let $T_m(x) := \cos(m \arccos x)$, the Chebyshev polynomial of degree m. The properties of T_m which we need here are that $T_m(\pm 1) = 1$; $\|T_m\| \leq 1$ (by $\|\cdot\|$ we mean here the supremum norm on $[-1,1]$); and $T'_m(1) = m^2$; $T'_m(-1) = -m^2$. Using these properties, we define for each $j = 0, \ldots, q$

$$T_{m,j}(x) := T_m(\frac{2x - s_j - t_j}{s_j - t_j}),$$

noting that $T_{m,j}(s_j) = T_{m,j}(t_j) = 1$, while

$$T'_{m,j}(t_j) = -T'_{m,j}(s_j) = \frac{2m^2}{t_j - s_j}$$

and that there are constants α and β depending upon q but otherwise independent of m and n such that if $-1 \leq x \leq -1 + \frac{1}{n^2}$ and $1 - \frac{1}{n^2} \leq y \leq 1$ we have

$$\alpha \leq \min\{T_{m,j}(x), T_{m,j}(y)\} \leq \max\{T_{m,j}(x), T_{m,j}(y)\} \leq \beta. \tag{22}$$

Now, let us assume that we have defined a polynomial $p_{n,j-1}$ such that (20) holds for $p_{n,j-1}$, and (1) holds with a constant K_{q-1}. We now construct for each $k = 0, \ldots, j$ a polynomial $p_{n,j,k}$ as follows:

$$p_{n,j,j}(x) \quad := \quad \{(f(s_j) - p_{n,j-1}(s_j))\left(\frac{t_j - x}{t_j - s_j}\right)\} \cdot \prod_{\ell=1}^{j-1} \frac{1 - T_{m,\ell}(x)}{1 - T_{m,\ell}(s_j)} \tag{23}$$

$$+\{(f(t_j) - p_{n,j-1}(t_j))\left(\frac{s_j - x}{s_j - t_j}\right)\} \cdot \prod_{\ell=1}^{j-1} \frac{1 - T_{m,\ell}(x)}{1 - T_{m,\ell}(t_j)}. \tag{24}$$

The polynomial $p_{n,j,j}$ enjoys the property that

$$p_{n,j,j}(s_k) = p_{n,j,j}(t_k) = 0 \;\text{ for }\; k = 0, \ldots, j-1,$$

while $p_{n,j,j}$ also interpolates $f - p_{n,j-1}$ exactly at s_j and t_j. Since the order of the points $s_0, \ldots, s_j$ and $t_0, \ldots, t_j$ was actually not relevant in this construction, we can by relabelling the points in a suitable manner similarly construct polynomials $p_{n,j,k}$ respectively interpolating $f - p_{n,j-1}$ on the points s_k and t_k for each $k = 0, \ldots, j-1$. Having done this, we can define

$$p_{n,j} := p_{n,j-1} + \sum_{k=0}^{j} p_{n,j,k}, \tag{25}$$

and clearly (20) is satisfied at the index j. To see that (1) is also satisfied, it suffices to show that

$$\|p_{n,j,k}\| \;\le\; M_j\, K_j\, n^{-q}\, \omega(f^{(q)}; \frac{1}{n^2}), \tag{26}$$

in which M_j and K_j are constants depending upon j, with $K_0 \le 2K$. And to see this we may without loss of generality choose $k = j$. Rewriting (23) as

$$p_{n,j,j}(x) \;=$$

$$\{\frac{(f(s_j) - p_{n,j-1}(s_j))}{(s_j - s_0)\dots(s_j - s_{j-1})}\left(\frac{t_j - x}{t_j - s_j}\right)\}\cdot\prod_{\ell=1}^{j-1}\frac{(s_j - s_\ell)(1 - T_{m,\ell}(x)}{1 - T_{m,\ell}(s_j)}$$

$$+\{\frac{(f(t_j) - p_{n,j-1}(t_j))}{(t_j - t_0)\dots(t_j - t_{j-1})}\left(\frac{s_j - x}{s_j - t_j}\right)\}\cdot\prod_{\ell=1}^{j-1}\frac{(t_j - t_\ell)(1 - T_{m,\ell}(x))}{1 - T_{m,\ell}(t_j)},$$

we easily get (26) by using Rolle's theorem, taking also (22) into consideration. From (26) we now clearly get (1) for the polynomial $p_{n,j}$ constructed in (25) with a new constant K_j, and we can repeat the induction until we reach $j = q$.

As previously stated, we now let $P_n := p_{n,q}$. At this point, it remains only to show that (2) follows from the fact that P_n interpolates f on S_n. This follows from Theorem 2, and then the proof of Theorem 1 is completed, with a new value of K equal to CK_q, where C is the constant obtained in Theorem 2.

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