On a Sequence of Fast Decreasing Polynomial Operators

H. N. Mhaskar[1] and J. Prestin

Abstract. Let f be a piecewise analytic function on the unit interval (respectively, the unit circle of the complex plane). Starting from the Chebyshev (respectively, Fourier) coefficients of f, we construct a sequence of fast decreasing polynomials (respectively, trigonometric polynomials) which "detect" the points where f fails to be analytic, provided f is not infinitely differentiable at these points.

1. Introduction

A fast decreasing sequence of polynomials is a sequence of polynomials which are "large" near some points, and decrease exponentially fast at other points. Such polynomials have been constructed by Gaier, Ivanov, Saff, Totik, and others ([1], [6] and references therein) in connection with the approximation of piecewise analytic functions on a real interval. The error in approximating such functions decays exponentially at the points of analyticity, and (typically) polynomially at the singularities; i.e., at the points where the function fails to be analytic.

In [4], we used segment approximation to **detect** the singularities of a piecewise analytic function on a compact, real interval, provided the number of such points is known in advance. A central observation in our approach is the fact that a piecewise analytic function on an interval (in the sense of [4]) necessarily has jump discontinuities in some derivative at the singularities. Motivated by the work [1] of Gaier, we define in this paper a sequence of fast decreasing polynomials which enables us to detect the location of those singularities of a function where it is not infinitely many times differentiable. Our approach also does not require the prior knowledge of the number of such singularities.

Our construction uses the Chebyshev coefficients of the function. We would like to observe here that the singularity or analyticity of a function is a local property, whereas each of the Chebyshev coefficients represents global information about the function. In [5], we have constructed polynomial operators, based on the Jacobi coefficients, that detect the points where the derivatives of a function

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have jump discontinuities. However, these operators do not define fast decreasing polynomials. In the case of Chebyshev polynomials, we are able to use function-theoretic methods to construct operators which yield fast decreasing polynomials.

Our techniques enable us to detect the singularities of a piecewise analytic function defined on the unit circle as well. In fact, the starting point of our research is to construct fast decreasing trigonometric polynomials for this purpose.

We discuss the construction of fast decreasing trigonometric polynomials in the next section. In Section 3, we point out the connection with Chebyshev polynomials. In Section 4, we present some numerical examples.

2. Functions on the Unit Circle

Let U denote the complex unit circle, $f : U \to \mathbb{C}$ be integrable with respect to the arclength of U. The Fourier coefficients of f are then defined by

$$\hat{f}_k := \frac{1}{2\pi i}\int_U \frac{f(\omega)}{\omega^{k+1}}d\omega, \qquad k = 0, \pm 1, \pm 2, \dots . \tag{1}$$

Suppose that both the power series $\sum_{k=0}^{\infty} \hat{f}_k z^k$ and $\sum_{k=1}^{\infty} \hat{f}_{-k} z^k$ have radius of convergence equal to 1. Then they define analytic functions on the open unit disc, and have singularities on U in the sense defined in Hille's book [2, §10.2, p.4 and §10.4, p.15]. We say that f is piecewise analytic if the total number of singularities on U of both of these power series is finite. The singularities of f are then defined to be the singularities of these power series. If $r \geq 0$ is an integer, we will say that f has a singularity of order r at a point w_0 if f is $r-1$ times continuously differentiable at w_0 with respect to the arclength, and the rth angular derivative (i.e., derivative with respect to the arclength) has a jump discontinuity at w_0.

Next, we define certain operators related to those defined by Gaier in [1]. For $m = 0, 1, \dots$, and $z \in \mathbb{C}$, we write

$$s_m(f, z) := \sum_{k=-m}^{m} \hat{f}_k z^k.$$

Our operators are defined by applying a summability method with the s_m's. Towards this end, we write for integers $\ell \geq 1$, $m \geq 0$,

$$h_{\ell,m} := \begin{cases} 0, & \text{if } 0 \leq m \leq \ell - 1, \\ -\dfrac{1}{\ell}, & \text{if } \ell \leq m \leq 2\ell - 1, \\ \dfrac{1}{2\ell}, & \text{if } 2\ell \leq m \leq 4\ell - 1, \\ 0, & \text{if } m \geq 4\ell, \end{cases}$$

and $h_{0,m} := 0$. For integer $n \ge 0$, the summability factors are defined by

$$g_{n,m} := 2^{-n}\sum_{\ell=0}^{n}\binom{n}{\ell}\{h_{\ell,4m}+h_{\ell,4m+1}+h_{\ell,4m+2}+h_{\ell,4m+3}\}$$
$$= 2^{-n}\sum_{\ell=m+1}^{n}\binom{n}{\ell}\{h_{\ell,4m}+h_{\ell,4m+1}+h_{\ell,4m+2}+h_{\ell,4m+3}\}. \tag{2}$$

Our operators are now defined for integer $n \ge 1$ by

$$\tau_n(f) := \sum_{m=0}^{n-1} g_{n,m}s_m(f) =: \sum_{k=-n+1}^{n-1}\tilde{g}_{n,|k|}\hat{f}_k z^k.$$

We observe that for each integer $n \ge 1$, τ_n is a linear operator defined on the class of all functions on U integrable with respect to the arclength, and for each such function f, $\tau_n(f)$ is a Laurent polynomial of degree at most $n-1$ (cf. Fig. 2).

For $f : U \to \mathbb{C}$, and $x \ge 0$, $\|f\|$ denotes $\sup_{z\in U}|f(z)|$, and we write

$$E_x(f) := \inf_{T\in\mathbb{H}_x}\|f-T\|,$$

where $\mathbb{H}_x$ denotes the class of all functions of the form $\sum_{|k|\le x} a_k(\cdot)^k$. Restricted to U, elements of $\mathbb{H}_x$ are just trigonometric polynomials of order at most x in the argument variable.

For integer $r \ge 0$, and $z \in U$, we define

$$\Gamma_r(z) := \sum_{k=-\infty, k\ne 0}^{\infty}\frac{z^k}{(ik)^{r+1}}.$$

The function Γ_r is analytic on U, except at $z = 1$, where its rth angular derivative (i.e., derivative with respect to the arclength) has a jump discontinuity. Hence, if $f : U \to \mathbb{C}$ has a singularity of order r at $w_0 \in U$, then for a judiciously chosen complex constant $a(f)$, the rth angular derivative of $f - a(f)\Gamma_r(\cdot/w_0)$ is continuous at w_0.

In the sequel, we adopt the following notations and conventions. Letters $c, c_1, c_2, \ldots$, will denote positive constants depending on explicitly indicated quantities only. Their values may be different at different occurrences, even within the same formula. The main theorem of this section is the following.

Theorem 2.1. *Let $f : U \to \mathbb{C}$ be continuous. For each integer $n \ge 1$,*

$$\|\tau_n(f)\| \le c_1 e^{-cn}\|f\| + 6E_{n/12}(f). \tag{3}$$

If $w_0 \in U$, and f is analytic in the disc $\{z : |z - w_0| < d\}$ for some $0 < d := d(f, w_0) < 1$, then

$$|\tau_n(f, w_0)| \le \frac{c(f)}{d^4}\left(1 - \frac{d^2}{900}\right)^{n/2}. \tag{4}$$

Suppose that f has only finitely many singularities, and that the only discontinuities in all angular derivatives of f at these singularities are jump discontinuities. If $r \geq 0$ is an integer, there is an arc $J := J_r(f)$ of the circle, such that if $w_0 \in U$ is a singularity of f of order r, then

$$n^r |\tau_n(f, w)| \geq c(f, r), \qquad (ww_0^{-1})^n \in J. \tag{5}$$

In order to prove Theorem 2.1, we need some preparation. First, we define the de la Vallée Poussin operators by $v_0(f, z) := \hat{f}_0$ and

$$v_\ell(f, z) := \frac{1}{\ell} \sum_{m=\ell}^{2\ell-1} s_m(f, z), \qquad \ell = 1, 2, \ldots .$$

We also need the Euler means of the difference of these operators. Thus, we define for $n = 0, 1, \ldots$

$$\tau_n^*(f, z) := 2^{-n} \sum_{\ell=0}^{n} \binom{n}{\ell} \Big(v_{2\ell}(f, z) - v_\ell(f, z) \Big).$$

Next, let

$$F(z) := f(z^4).$$

For $z = \rho e^{i\theta}$, we will write $z^{1/4} := \rho^{1/4} \exp(i\theta/4)$, $\theta \in (-\pi, \pi]$. If $z_0 \in U$, $|z - z_0^{1/4}| < \delta < 1$, then

$$|z^4 - z_0| < |z - z_0^{1/4}|(1 + |z|)(1 + |z|^2) < 15\delta.$$

Hence, if f is analytic in the disc of radius $d < 1$, centered at some $z_0 \in U$, then F is analytic in the disc of radius $d/15$ around $z_0^{1/4}$ (and hence, also in the discs of radius $d/15$ around $-z_0^{1/4}$ and $\pm i z_0^{1/4}$). Similarly, f has a singularity of order r at $z_0 \in U$ if and only if F has singularities of order r at $\pm z_0^{1/4}$ and $\pm i z_0^{1/4}$.

Further, we observe that for integer $m \geq 0$,

$$s_{4m}(F, z) = s_{4m+1}(F, z) = s_{4m+2}(F, z) = s_{4m+3}(F, z) = s_m(f, z^4).$$

Hence, for integer $\ell \geq 1$,

$$\begin{aligned} v_{2\ell}(F, z) - v_\ell(F, z) &= \sum_{m=0}^{4\ell-1} h_{\ell,m} s_m(F, z) \\ &= \sum_{m=0}^{\ell-1} \{h_{\ell,4m} + h_{\ell,4m+1} + h_{\ell,4m+2} + h_{\ell,4m+3}\} s_m(f, z^4) \\ &=: \sum_{k=-\ell+1}^{\ell-1} \tilde{h}_{\ell,|k|} \hat{f}_k z^{4k}. \end{aligned}$$

Consequently, we obtain that

$$\begin{aligned}
&\tau_n^*(F,z)\\
&= 2^{-n}\sum_{\ell=0}^{n}\binom{n}{\ell}\Big(v_{2\ell}(F,z)-v_\ell(F,z)\Big)\\
&= 2^{-n}\sum_{m=0}^{n-1}\bigg\{\sum_{\ell=m+1}^{n}\binom{n}{\ell}\Big[h_{\ell,4m}+h_{\ell,4m+1}+h_{\ell,4m+2}+h_{\ell,4m+3}\Big]\bigg\}s_m(f,z^4)\\
&= \tau_n(f,z^4).
\end{aligned}$$

Thus, Theorem 2.1 will be proved if we prove the following.

Theorem 2.2. *Let $F : U \to \mathbb{C}$ be continuous, and*

$$F(z) = F(-z) = F(iz) = F(-iz), \qquad z \in \mathbb{C}.$$

For each integer $n \geq 0$,

$$\|\tau_n^*(F)\| \leq 0.15e^{-n/18}\|F\| + 6E_{n/3}(F). \tag{6}$$

If $w_0 \in U$, and F is analytic in the disc $\{z : |z - w_0| < d\}$ for some $0 < d := d(F, w_0) < 1$, then

$$|\tau_n^*(F,w_0)| \leq \frac{c(F)}{d^4}\left(1-\frac{d^2}{4}\right)^{n/2}. \tag{7}$$

Suppose that F has only finitely many singularities, and that the only discontinuities in all angular derivatives of F at these singularities are jump discontinuities. If $r \geq 0$ is an integer, there is an arc $J := J_r(F)$ of the circle, such that if $w_0 \in U$ is a singularity of F of order r, then

$$n^r|\tau_n^*(F,w)| \geq c(F,r), \qquad (ww_0^{-1})^n \in J. \tag{8}$$

Proof. It is well known that

$$\|v_\ell(F)\| \leq 3\|F\|, \quad \|v_\ell(F) - v_{2\ell}(F)\| \leq 6E_\ell(F), \quad \ell = 0, 1, \ldots .$$

Hence, (6) follows from an estimate for binomial sums (cf. [3]) and Stirling's inequality

$$2^{-n}\sum_{\ell=0}^{[n/3]-1}\binom{n}{\ell} \leq 2^{-n}\binom{n}{[n/3]} \leq \left(\frac{3}{2^{5/3}}\right)^n \frac{3e^{1/(12n)}}{2\sqrt{\pi n}} \leq 0.15e^{(\ln 3 - 5\ln 2)n},$$

where the last inequality is true for $n \geq 32$. For $n < 32$ the first sum can be estimated directly by computation.

To prove the next assertion of the theorem, we assume without loss of generality that $w_0 = 1$, and that F is analytic in the disc $\{z : |z-1| < d\}$ for some d, $0 < d < 0.7$. Following Gaier [1], we consider the arc γ of the circle $|z+1| = \sqrt{4-d^2}$ intercepted by U and inside U. Let w and W be the end points of γ in the upper and lower half plane respectively (cf. Figure 1). Both w and W

are on the arc from $e^{-i\pi/4}$ to $e^{i\pi/4}$. Let C_0 be the Jordan arc from $e^{-i\pi/4}$ to $e^{i\pi/4}$, following γ from W to w, and U otherwise. Let

$$C_1 := C_0 \cup (iC_0) \cup (-C_0) \cup (-iC_0),$$

and C_2 be the reflection of C_1 in the unit circle. We observe that the lens-shaped region between γ and its reflection in U lies in $\{z \;:\; |z-1| < d\}$, and hence, F is analytic in this region.

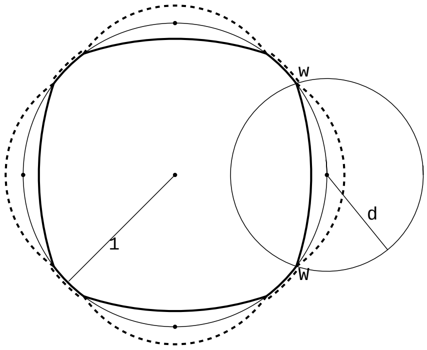

Fig. 1. The unit circle with curves C_1 (thick line) and C_2 (dotted line).

Following Gaier [1], we obtain from (1) and Cauchy's formula that for integer $m \geq 0$,

$$s_m(F,1) = F(1) + \frac{1}{2\pi i}\oint_{C_1} \frac{F(\zeta)}{(\zeta-1)}\zeta^m d\zeta - \frac{1}{2\pi i}\oint_{C_2} \frac{F(\zeta)}{(\zeta-1)}\zeta^{-m-1} d\zeta,$$

and hence, after a little computation, that for integer $\ell \geq 1$,

$$\begin{aligned} v_{2\ell}(F,1) - v_\ell(F,1) = & \frac{1}{4\pi i}\oint_{C_1} \frac{F(\zeta)}{(\zeta-1)^2}\frac{\zeta^{4\ell} - 3\zeta^{2\ell} + 2\zeta^\ell}{\ell} d\zeta \\ & - \frac{1}{4\pi i}\oint_{C_2} \frac{F(\zeta)}{(\zeta-1)^2}\frac{\zeta^{-4\ell} - 3\zeta^{-2\ell} + 2\zeta^{-\ell}}{\ell} d\zeta, \end{aligned}$$

and for integer $n \geq 1$,

$$\tau_n^*(F,1) = \frac{1}{4\pi i}\oint_{C_1} \frac{F(\zeta)}{(\zeta-1)^2} H_n(\zeta) d\zeta - \frac{1}{4\pi i}\oint_{C_2} \frac{F(\zeta)}{(\zeta-1)^2} H_n(\zeta^{-1}) d\zeta, \qquad (9)$$

where

$$H_n(z) := 2^{-n}\sum_{\ell=1}^{n} \binom{n}{\ell} \frac{z^{4\ell} - 3z^{2\ell} + 2z^\ell}{\ell}.$$

Writing $h_n(t) = t^{-1}((1+t)^n - 1)$, we observe that

$$H_n(z) = 2^{-n}\left\{\int_{z^2}^{z^4} h_n(t)dt - 2\int_z^{z^2} h_n(t)dt\right\},$$

where the integrations are over any paths joining the indicated end points. For $\zeta \in C_1$, we may choose these paths to lie entirely inside C_1 and outside the disc $\{z \,:\, |z| \le (\sqrt{4-d^2}-1)^{1/4}\}$. With this choice,

$$|h_n(t)| \le \frac{(4-d^2)^{n/2}+1}{(\sqrt{3}-1)^{1/4}}$$

for all t on all these paths. Therefore, for $\zeta \in C_1$,

$$|H_n(\zeta)| \le \frac{2\pi}{(\sqrt{3}-1)^{1/4}}\left\{\left(1-\frac{d^2}{4}\right)^{n/2}+2^{-n}\right\} \le 10.72\left(1-\frac{d^2}{4}\right)^{n/2}.$$

Along with (9), this implies

$$|\tau_n^*(F,1)| \le \frac{10.72}{(2-\sqrt{4-d^2})^2}\left(1-\frac{d^2}{4}\right)^{n/2} c(F),$$

which leads to (7).

Next, let w_0 be a singularity of F. Without loss of generality, we may again assume that $w_0 = 1$. Let R be the maximum of the orders of all the singularities of F. Our assumption on F implies that

$$F(z) = \omega \sum_{\nu=0}^{3} \Gamma_r(zi^\nu) + \Phi_1(z) + \Phi_2(z), \qquad z \in U,$$

where $\omega \in \mathbb{C}$, Φ_1 is a (finite) linear combination of terms of the form $\Gamma_k(zw_j^{-1})$, $w_j \ne 1$, $0 \le k \le R$, and Φ_2 has R continuous angular derivatives on U. In particular, Φ_1 is analytic at 1, and both Φ_1 and Φ_2 satisfy

$$\Phi_j(z) = \Phi_j(iz) = \Phi_j(-z) = \Phi_j(-iz), \quad j = 1, 2.$$

From (6) and Favard's theorem, $\tau_n^*(\Phi_2, w) = o(n^{-R})$ for $w \in U$. From (7), $\tau_n^*(\Phi_1, w) = o(e^{-c(F)n})$ for w on some arc J_1 around 1. We observe that

$$\tau_n^*(\Gamma_r(i^\nu\cdot), w) = \tau_n^*(\Gamma_r, i^\nu w)$$

for $\nu = 0, 1, 2, 3$ and $w \in U$. Hence, the estimate (8) will be proved if we prove that

$$n^r|\tau_n^*(\Gamma_r, i^\nu w)| = \mathcal{O}(n^{-1}), \qquad \nu = 1, 2, 3, \tag{10}$$

and

$$n^r|\tau_n^*(\Gamma_r, w)| \ge c_r \tag{11}$$

for sufficiently large integer values of n, and w^n on some arc of U which intersects J_1.

We write $w =: e^{i\theta}$, $\tau_{n,r} := \tau_n^*(\Gamma_r)$, $v_{\ell,r} := v_\ell(\Gamma_r)$, and

$$h(t) := \frac{1}{\pi}\begin{cases} 0, & \text{if } t \in [0, \pi/2), \\ t^{-r-1}(2t-\pi), & \text{if } t \in [\pi/2, \pi), \\ t^{-r-1}(2\pi - t), & \text{if } t \in [\pi, 2\pi). \end{cases}$$

A simple summation by parts argument leads to the formula (for integer $\ell \ge 1$)

$$v_{2\ell,r}(w) - v_{\ell,r}(w) = 4\pi \left(\frac{2\pi}{4\ell}\right)^r \frac{1}{4\ell} \sum_{k=0}^{4\ell-1} h\left(\frac{2\pi k}{4\ell}\right) \cos(k\theta - (r+1)\pi/2).$$

Next, we recall that for any integer $p \ge 0$ and $\ell \ge 1$,

$$\ell^{-p} = \frac{\ell!}{(\ell+p)!}\left(1 + \mathcal{O}(\ell^{-1})\right) \le c(p)\frac{\ell!}{(\ell+p)!}. \tag{12}$$

Hence, an application of Lemma 3.2 of [5] gives

$$\begin{aligned} v_{2\ell,r}(w) - v_{\ell,r}(w) &= 2\left(\frac{2\pi}{4\ell}\right)^r \left\{\int_0^{2\pi} h(t)\cos(\ell\alpha t - \beta)dt + \mathcal{O}(\ell^{-1})\right\} \\ &= 2(\frac{\pi}{2})^r \frac{\ell!}{(\ell+r)!}\left\{\int_0^{2\pi} h(t)\cos(\ell\alpha t - \beta)dt + \mathcal{O}(\ell^{-1})\right\}, \end{aligned} \tag{13}$$

where $\alpha := 2\theta/\pi$, $\beta := (r+1)\pi/2$.

Let $\nu \in \{1, 2, 3\}$. Using this formula with $i^\nu w$ in place of w, we get

$$v_{2\ell,r}(i^\nu w) - v_{\ell,r}(i^\nu w) = 2(\frac{\pi}{2})^r \frac{\ell!}{(\ell+r)!}\left\{\int_0^{2\pi} h(t)\cos(\ell\alpha_\nu t - \beta)dt + \mathcal{O}(\ell^{-1})\right\},$$

where $\alpha_\nu = \alpha + \nu$. For $|\theta| < \pi/8$, we have $|\alpha_\nu| \ge c$. Hence, using integration by parts, we see that

$$\int_0^{2\pi} h(t)\cos(\ell\alpha_\nu t - \beta)dt = \mathcal{O}(\ell^{-1}). \tag{14}$$

Now, using (12), we obtain

$$\sum_{\ell=1}^{n} \binom{n}{\ell} \ell^{-p} \le c(p)\frac{n!}{(n+p)!} \sum_{\ell=1}^{n} \binom{n+p}{\ell+p} \le c_1(p) n^{-p} 2^n. \tag{15}$$

Together with (13) and (14), this estimate (with $p = 1$) leads to (10).

Next, we deduce from (13) and (15) that

$$\begin{aligned} &\tau_{n,r}(w) \\ &= 2(\frac{\pi}{2})^r \frac{n!}{(n+r)!}\mathrm{Re}\left\{e^{-i\beta}\int_0^{2\pi} h(t)e^{-ir\alpha t}2^{-n}\sum_{l=r+1}^{n+r}\binom{n+r}{\ell}e^{i\alpha t\ell}dt \right. \\ &\qquad\qquad \left. + \mathcal{O}(n^{-1})\right\} \\ &= 2\pi^r \frac{n!}{(n+r)!}\mathrm{Re}\left\{e^{-i\beta}\int_0^{2\pi} h(t)e^{-ir\alpha t}((1+e^{i\alpha t})/2)^{n+r}dt \right. \\ &\qquad\qquad \left. + \mathcal{O}(n^{-1})\right\}. \end{aligned}$$

For any $A > 0$ and $|\alpha| \le A/n$, this yields

$$\begin{aligned}\tau_{n,r}(w) &= 2\pi^r \frac{n!}{(n+r)!}\mathrm{Re}\Big\{e^{-i\beta}\int_0^{2\pi} h(t)e^{in\alpha t}dt + \mathcal{O}(n^{-1})\Big\} \\ &= 2\pi^r \frac{n!}{(n+r)!}\Big\{\int_0^{2\pi} h(t)\cos(n\alpha t-\beta)dt + \mathcal{O}(n^{-1})\Big\}, \qquad (16)\end{aligned}$$

where the unspecified constants may depend also on A. Now, the entire function

$$z \mapsto \int_0^{2\pi} h(t)\cos(zt-\beta)dt$$

is not identically equal to zero, and hence, every neighborhood of 0 contains a real interval I such that

$$\Big|\int_0^{2\pi} h(t)\cos(zt-\beta)dt\Big| \ge c_r$$

for all $z \in I$. Along with (16), this leads to (11). □

3. Functions on $[-1,1]$

Let $f : [-1,1] \to \mathbb{R}$. We will say that f is piecewise analytic if there exist points $-1 =: y_0 < y_1 < \cdots < y_s := 1$ such that f is analytic in each of the open intervals (y_{j-1}, y_j) for $j = 1, \dots, s$. The points y_j will be called the singularities of f. This notion is more general than that used in [4], where the restriction of f to each of the *closed* intervals $[y_{j-1}, y_j]$ was assumed to be analytic. For the analysis in this section, we do have to assume the existence of all the one-sided derivatives of f at the points y_j, including at ± 1. Our objective is to detect the location of the points y_j that are in the open interval $(-1,1)$, given the sequence of Chebyshev coefficients of f (see (17) below for the definition). Unlike [4], we do not require that the number of these "interior singularities" be known in advance. We say that a point $y \in (-1,1)$ is a singularity of f of order r if f has r continuous derivatives in a deleted neighborhood of y, and the rth derivative of f has a jump discontinuity at y.

We recall that the Chebyshev polynomials are defined by the formula

$$T_n(\cos\theta) := \cos(n\theta), \qquad n = 0, 1, 2, \dots,\ \theta \in [0,\pi].$$

They satisfy the following orthogonality relations

$$\int_{-1}^{1} T_n(t)T_m(t)\frac{dt}{\sqrt{1-t^2}} = \begin{cases} \pi, & \text{if } n = m = 0, \\ \pi/2, & \text{if } n = m \ne 0, \\ 0, & \text{if } n \ne m. \end{cases}$$

If f is a continuous function on $[-1,1]$, we may define the Chebyshev coefficients for integer $k \ge 0$ by

$$f_k^T := \frac{1}{\sqrt{\pi}} \begin{cases} \displaystyle\int_{-1}^{1} f(t)(1-t^2)^{-1/2}dt, & \text{if } k = 0, \\ \sqrt{2}\displaystyle\int_{-1}^{1} f(t)T_k(t)(1-t^2)^{-1/2}dt, & \text{if } k \neq 0. \end{cases} \tag{17}$$

The partial sums of the Chebyshev expansion of f are then defined for integer $m \ge 0$ and $x \in [-1,1]$ by

$$S_m(f,x) := \frac{1}{\sqrt{\pi}} f_0^T + \sqrt{\frac{2}{\pi}} \sum_{k=1}^{m} f_k^T T_k(x).$$

With the summability factors $g_{n,m}$ as in (2), we may now define the operators $\mathcal{T}_n$ by

$$\mathcal{T}_n(f,x) := \sum_{m=0}^{n-1} g_{n,m} S_m(f,x), \qquad n = 1,2,\ldots,\ x \in [-1,1].$$

Finally, we write

$$\|f\|_I := \max_{x \in [-1,1]} |f(x)|,$$

and for $y \ge 0$,

$$\epsilon_y(f) := \min_{P \in \Pi_y} \|f - P\|_I,$$

where Π_y denotes the class of all algebraic polynomials of degree at most y.

Theorem 2.1 now leads to the following analogous result.

Theorem 3.1. *Let $f : [-1,1] \to \mathbb{R}$ be continuous. For each integer $n \ge 1$,*

$$\|\mathcal{T}_n(f)\|_I \le c_1 e^{-cn} \|f\|_I + 6\epsilon_{n/12}(f). \tag{18}$$

If $x_0 \in [-1,1]$, and f is analytic in the disc $\{z \in \mathbb{C} \ :\ |z - x_0| < d\}$ for some $0 < d := d(f,x_0) < 1$, then

$$|\mathcal{T}_n(f,w_0)| \le \frac{c(f)}{d^4}(1 - c_1 d^2)^{n/2}. \tag{19}$$

Suppose that f fails to be analytic only at finitely many points, and that the only discontiuities in all derivatives of f at these points are jump discontinuities. We also assume that all the one-sided derivatives of f at ± 1 are defined. If $r \ge 0$ is an integer, there is a subinterval $J \subseteq [0,\pi]$ such that if $y =: \cos\phi \in (-1,1)$ is a singularity of f of order r, then

$$n^r |\mathcal{T}_n(f,\cos\theta)| \ge c(f,r), \qquad n|\theta - \phi| \in J. \tag{20}$$

Proof. We define $f^\circ : U \to \mathbb{R}$ by

$$f^\circ(e^{i\theta}) := f(\cos\theta), \qquad \theta \in [-\pi,\pi],$$

and consider the Joukowski transformation

$$z = \frac{1}{2}\left(w + \frac{1}{w}\right), \qquad |w| > 1,$$

that maps the exterior of $[-1,1]$ conformally onto the exterior of the unit disc in the w-plane. It is easy to verify from the definitions that

$$S_m(f,\cos\theta) = s_m(f^\circ, e^{i\theta}), \qquad \theta \in [-\pi,\pi],$$

and hence, that

$$\mathcal{T}_n(f,\cos\theta) = \tau_n(f^\circ, e^{i\theta}), \qquad \theta \in [-\pi,\pi].$$

Together with (3), this implies (18).

We observe that

$$z - \cos\theta = \frac{1}{2w}(w - e^{i\theta})(w - e^{-i\theta}),$$

and that the right-hand side is a rational function of w with pole at the origin. Therefore, if f is analytic in the disc $\{z \in \mathbb{C} \,:\, |z - \cos\theta| < d\}$ for some $\theta \in (0,\pi)$ and $0 < d < 1$, then f° is analytic in discs of radius $d/2$ around $e^{i\theta}$ and $e^{-i\theta}$. Thus, (4) implies (19).

Our assumption about the one-sided derivatives of f implies that the only singularities of f° are at those points $e^{\pm i\theta}$, where f has a singularity at $\cos\theta$, $\theta \in (0,\pi)$. Moreover, all the angular derivatives of f° at these points will have at most jump discontinuities of the same order as the corresponding singularity of f. We have also assumed the existence of all the one-sided derivatives of f at ± 1. The function f° is then infinitely differentiable at the points ± 1 on U. The estimate (5) then leads to (20). □

4. Numerical Results

We consider the function

$$f(z) := \frac{1}{2}\left\{\Gamma_1(ze^{0.8i}) + \Gamma_3(ze^{-i})\right\},$$

which has a singularity of order 1 at $z = z_1 = e^{-0.8i}$ and of order 3 at $z = z_3 = e^{i}$. In the following figures, the quantity $\arg z$ is on the x-axis, and the y-axis denotes the values of the operators in question. The results of our work [5] indicate that the operators

$$V_{1024}(f,z) := v_{512}(f,z) - v_{256}(f,z)$$

are also able to detect the singularity at z_1 and perhaps also at z_3. We observe that both $\tau_{1024}(f)$ and $V_{1024}(f)$ are in $\mathbb{H}_{1023}$, and the operator V_{1024} is obviously much easier to compute. On the other hand, Figure 2 shows the summability factors $\tilde{g}_{1024,k}$ (respectively $\tilde{h}_{256,k}$) for the operators τ_{1024} (respectively V_{1024}). It is clear that while both of them are formally of the same order, the real contribution to τ_{1024} actually comes from a polynomial of order less than 700.

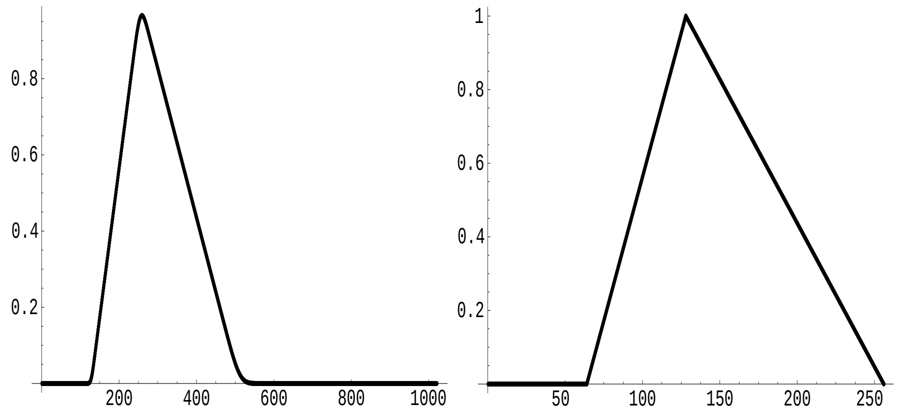

Fig. 2. Left: $\tilde{g}_{1024,k}$, Right: $\tilde{h}_{256,k}$.

Figure 3 shows that τ_{1024} and V_{1024} are both sufficiently well localized for the detection of the singularity at z_1.

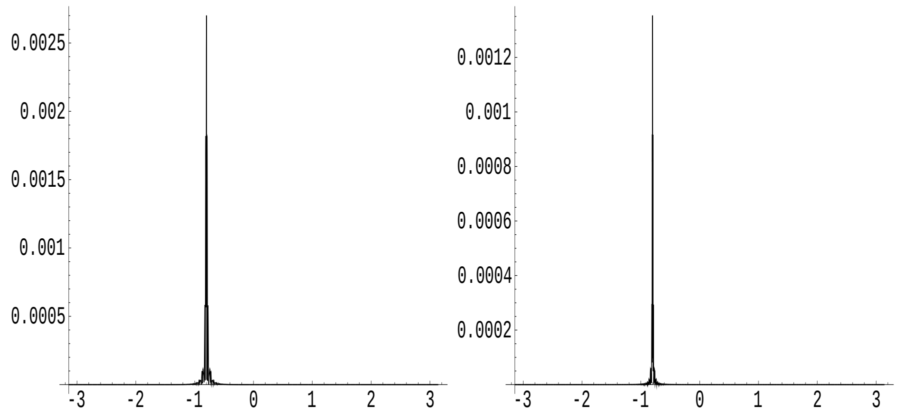

Fig. 3. Left: $|\tau_{1024}(f)|$, Right: $|V_{1024}(f)|$ on $[-\pi, \pi]$.

However, as Figure 4 shows, in the presence of this singularity z_1, V_{1024} is not at all able to detect the higher-order singularity at z_3. The operator τ_{1024} is able to do so.

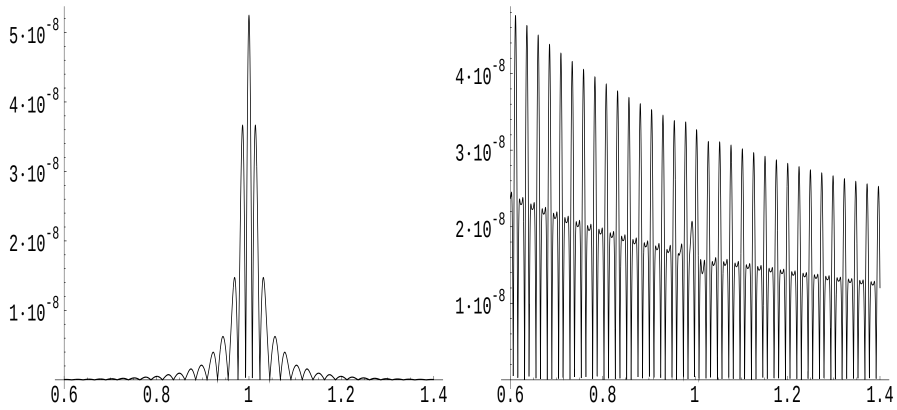

Fig. 4. Left: $|\tau_{1024}(f)|$, Right: $|V_{1024}(f)|$ on $[0.6, 1.4]$.

In Figure 5, we examine the behavior of both near z_1. As we go away from this point, the operator τ_{1024} seems to decay slower in the beginning, but then decreases much more rapidly as the theory predicts.

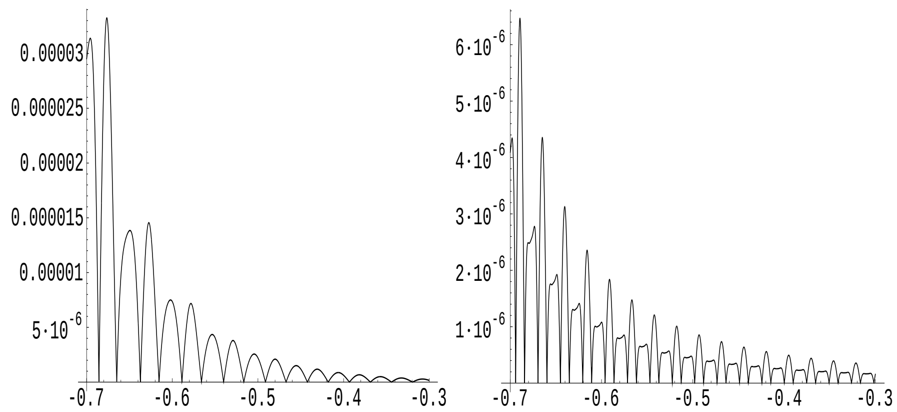

Fig. 5. Left: $|\tau_{1024}(f)|$, Right: $|V_{1024}(f)|$ on $[-0.7, -0.3]$.

In Figure 6, it is clear that the oscillations in τ_{1024} have already "died" long before we come to z_3, while those of v_{1024} are still not sufficiently small to affect the behavior significantly at z_3.

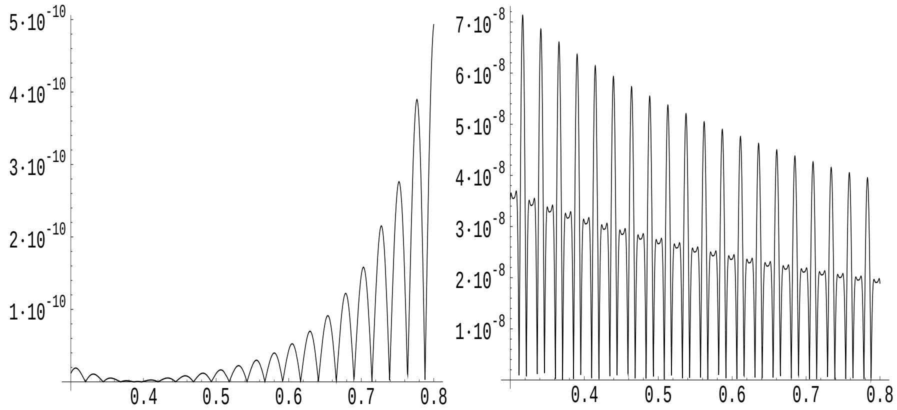

Fig. 6. Left: $|\tau_{1024}(f)|$, Right: $|V_{1024}(f)|$ on $[0.3, 0.8]$.

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Department of Mathematics
California State University
Los Angeles, CA 90032, U.S.A.
E-mail address: `hmhaska@calstatela.edu`

Institute for Biomathematics and Biometry
GSF – National Research Institute for Environment and Health
D-85764, Neuherberg, Germany
E-mail address: `prestin@gsf.de`