

# On some cubature formulas on the sphere<sup>1</sup>

Jürgen Prestin<sup>a,\*</sup> Daniela Roşca<sup>b</sup>

<sup>a</sup>*Institute of Mathematics, University of Lübeck, Wallstrasse 40, D-23560 Lübeck,  
Germany*

<sup>b</sup>*Department of Mathematics, Technical University of Cluj-Napoca, str. Daicoviciu  
15, RO-400020 Cluj-Napoca, Romania*

---

**Abstract**

We construct interpolatory cubature rules on the two-dimensional sphere, using the fundamental system of points obtained by Laín Fernández in [2,3]. The weights of the cubature rules are calculated explicitly. We also discuss the cases when this cubature leads to positive weights. Finally, we study the possibility to construct spherical designs and the degree of exactness.

*Key words:* cubature formulas, spherical functions, Legendre Polynomials

*PACS:* 65D32, 43A90, 42C10

---

---

\* Corresponding author

<sup>1</sup> Research supported by the EU Research Training Network MINGLE, HPRN-CT-1999-00117.

<sup>2</sup> *E-mail addresses:*    prestini@math.uni-luebeck.de       (J.    Prestin),  
Daniela.Rosca@math.utcluj.ro (D. Roşca)

## 1 Introduction

Based on a fundamental system of points on the two-dimensional sphere constructed by N. Laín Fernández, we investigate corresponding interpolatory cubature rules. These point systems satisfy a certain grid structure which can be exploited successfully. The main point of interest in this paper is the question whether spherical designs, leading to cubature rules with equal weights, can be constructed out of these point systems. For  $(n+1)^2$  points with  $n = 1, 3, 5, 7, 9$  we can create such spherical designs. For larger  $n$  point systems of this type cannot be spherical designs. However, as a byproduct of our approach for arbitrary odd  $n$ , we obtain positive cubature rules. The crucial relation used in obtaining these cubature results is some simplification of the addition formula evaluated at these point systems presented in Lemma 2.

Let  $\mathbb{S}^2 = \{x \in \mathbb{R}^3 : \|x\|_2 = 1\}$  denote the unit sphere of the Euclidean space  $\mathbb{R}^3$  and let

$$\begin{aligned} \Psi : [0, \pi] \times [0, 2\pi) &\rightarrow \mathbb{S}^2, \\ (\rho, \theta) &\mapsto (\sin \rho \cos \theta, \sin \rho \sin \theta, \cos \rho) \end{aligned}$$

be its parametrization in spherical coordinates  $(\rho, \theta)$ . The coordinate  $\rho$  of a point  $\xi (\Psi(\rho, \theta)) \in \mathbb{S}^2$  is usually called the *latitude* of  $\xi$ . For given functions  $f, g : \mathbb{S}^2 \rightarrow \mathbb{C}$ , we consider the inner product and norm

$$\begin{aligned} \langle f, g \rangle &= \int_{\mathbb{S}^2} f(\xi) \overline{g(\xi)} d\omega(\xi), \\ \|f\| &= \sqrt{\langle f, f \rangle}, \end{aligned}$$

where  $d\omega(\xi)$  stands for the surface element of the sphere. Let  $P_k$ ,  $k = 0, 1, \dots$ , denote the Legendre polynomials of degree  $k$  on  $[-1, 1]$ , normalized within the condition  $P_k(1) = 1$  and let  $V_n$  be the space of spherical polynomials of degree less than or equal to  $n$ . The dimension of  $V_n$  is  $\dim V_n = (n+1)^2 = N$  and the reproducing kernel of  $V_n$  is defined by

$$K_n(\xi, \eta) = \sum_{k=0}^n \frac{2k+1}{4\pi} P_k(\xi \cdot \eta) = k_n(\xi \cdot \eta), \quad \xi, \eta \in \mathbb{S}^2.$$

Then for given  $n$  we consider a set of points  $\{\xi_i\}_{i=1, \dots, N} \subset \mathbb{S}^2$  and the polynomial functions  $\varphi_i^n : \mathbb{S}^2 \rightarrow \mathbb{C}$ ,  $i = 1, \dots, N$ , defined by

$$\varphi_i^n(\circ) = K_n(\xi_i, \circ) = \sum_{k=0}^n \frac{2k+1}{4\pi} P_k(\xi_i \cdot \circ), \quad i = 1, \dots, N.$$

These polynomials are called *scaling functions*. A set of points  $\{\xi_i\}_{i=1, \dots, N}$  for which the scaling functions  $\{\varphi_i^n\}_{i=1, \dots, N}$  constitute a basis for  $V_n$  is called a

*fundamental system* for  $V_n$ .

The Gram matrix  $\Phi_n$  associated to the scaling functions  $\{\varphi_i^n\}_{i=1,\dots,N}$  has the entries

$$\Phi_n(r, s) = \langle \varphi_r^n, \varphi_s^n \rangle = K_n(\xi_r, \xi_s)$$

and it is positive definite when  $\{\xi_i\}_{i=1,\dots,N}$  is a fundamental system for  $V_n$ . Given the fundamental system  $\{\varphi_i^n\}_{i=1,\dots,N}$  of  $V_n$ , we can construct unique spherical polynomials  $L_j^n : \mathbb{S}^2 \rightarrow \mathbb{C}$  in  $V_n$  satisfying the condition  $L_j^n(\xi_i) = \delta_{ij}$ . These functions are called Lagrangians and the set  $\{L_j^n\}_{j=1,\dots,N}$  constitutes a basis of  $V_n$ . Furthermore, any  $f \in V_n$  can be written as

$$f = \sum_{i=1}^N f(\xi_i) L_i^n. \quad (1)$$

It is easy to verify that the Gram matrix of the Lagrangians, defined by  $\mathbf{L}_n = (\langle L_i^n, L_j^n \rangle)_{i,j=1,\dots,N} \in \mathbb{C}^{N \times N}$ , satisfies

$$\Phi_n \mathbf{L}_n = \mathbf{I}_N.$$

Here  $\mathbf{I}_N$  denotes the  $N \times N$  dimensional identity matrix. This means that the Lagrangians  $\{L_j^n\}_{j=1,\dots,N}$  are the dual functions of the scaling functions  $\{\varphi_i^n\}_{i=1,\dots,N}$ .

We wish to find appropriate numerical procedures for approximating the value of the integral

$$I(F) = \int_{\mathbb{S}^2} F(\xi) d\omega(\xi), \quad F \in C(\mathbb{S}^2).$$

This can be achieved by considering cubature formulae of the type

$$I_L(F) = \sum_{i=1}^L w_i F(\zeta_i), \quad (2)$$

where the  $L$  pairwise different points  $\{\zeta_i\}_{i=1,\dots,L} \subset \mathbb{S}^2$  are the so-called *cubature nodes* and the coefficients  $\{w_i\}_{i=1,\dots,L}$  are called the *cubature weights*. The basic problem in numerical integration consists in choosing appropriate nodes and weights such that the sum (2) approximates  $I(F)$  for a large class of functions, as  $L \rightarrow \infty$ .

Let  $f \in V_n$  and let  $\{L_i^n\}_{i=1,\dots,N}$  be the Lagrangians associated to a fundamental system  $\{\xi_i\}_{i=1,\dots,N}$ . By integrating formula (1) we obtain

$$\int_{\mathbb{S}^2} f(\xi) d\omega(\xi) = \sum_{i=1}^N f(\xi_i) \int_{\mathbb{S}^2} L_i^n(\xi) d\omega(\xi).$$

Therefore, the weights can be defined as

$$w_i^n = \int_{\mathbb{S}^2} L_i^n(\xi) d\omega(\xi) = \langle L_i^n, 1 \rangle, \quad i = 1, \dots, N,$$

yielding the following cubature formula

$$\int_{\mathbb{S}^2} F(\xi) d\omega(\xi) \approx \sum_{i=1}^N w_i^n F(\xi_i). \quad (3)$$

On the other hand, taking  $f \equiv 1 \in V_n$  in (1), we obtain  $\sum_{i=1}^N L_i^n \equiv 1$  and therefore

$$w_i^n = \langle L_i^n, 1 \rangle = \langle L_i^n, \sum_{k=1}^N L_k^n \rangle = \sum_{k=1}^N \langle L_i^n, L_k^n \rangle.$$

This means that the weight  $w_i^n$  can be calculated as the sum of the entries of the  $i$ -th row of the matrix  $\mathbf{L}_n$ , which is the inverse of the Gram matrix  $\Phi_n$ . Consequently, we obtain

$$\Phi_n (w_1^n, w_2^n, \dots, w_N^n)^T = (1, 1, \dots, 1)^T. \quad (4)$$

Recently, Laín Fernández proved the following result.

**Proposition 1** [2,3] *Let  $n \in \mathbb{N}$  be an odd number,  $\alpha \in (0, 2)$  and let  $0 < \rho_1 < \rho_2 < \dots < \rho_{\frac{n+1}{2}} < \pi/2$ ,  $\rho_{n+2-j} = \pi - \rho_j$ ,  $j = 1, \dots, (n+1)/2$ , denote a system of symmetric latitudes. Then the set of points*

$$S_n(\alpha) = \left\{ \xi_{j,k} = \Psi(\rho_j, \theta_k^j) : j, k = 1, \dots, n+1 \right\},$$

where

$$\theta_k^j = \begin{cases} \frac{2k\pi}{n+1}, & \text{if } j \text{ is odd,} \\ \frac{2(k-1)+\alpha}{n+1}\pi, & \text{if } j \text{ is even,} \end{cases}$$

constitutes a fundamental system for  $V_n$ .

In the following we will study the cubature formula (2), for odd  $n$ , with the nodes in  $S_n(\alpha)$ .

## 2 The matrix $\Phi_n$

Due to the symmetry of the fundamental system of points in  $S_n(\alpha)$  we expect equal weights on each latitude. However, for further considerations, we need to calculate these weights explicitly.

We consider the following numbering of the points of  $S_n(\alpha)$ :

$$\begin{aligned} \eta_1 &= \xi_{1,1}, \eta_2 = \xi_{1,2}, \dots, \eta_{n+1} = \xi_{1,n+1}, \\ \eta_{n+2} &= \xi_{2,1}, \eta_{n+3} = \xi_{2,2}, \dots, \eta_{2(n+1)} = \xi_{2,n+1}, \\ &\dots \\ \eta_{(n+1)n+1} &= \xi_{n+1,1}, \eta_{(n+1)n+2} = \xi_{n+1,2}, \dots, \eta_{(n+1)^2} = \xi_{n+1,n+1}. \end{aligned}$$

For a given point  $\xi_{i,j} \in S_n(\alpha)$ , the corresponding numbered point is  $\eta_{(n+1)(j-1)+k}$ . Reciprocally, given the point  $\eta_l$ ,  $l$  can be uniquely written as  $l = k + (n+1)(j-1)$ , with  $k, j \in \mathbb{N}$ ,  $0 \leq k < n+1$ , and thus  $\eta_l = \xi_{j,k}$ . With this numbering, the matrix  $\Phi_n$  can be regarded as a block matrix with circulant blocks of dimension  $m \times m$ , with  $m = n+1$ ,

$$\Phi_n = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1m} \\ \vdots & \vdots & & \vdots \\ A_{m1} & A_{m2} & \dots & A_{mm} \end{pmatrix}.$$

Indeed, the entries of the block  $A_{ij}$  are

$$(A_{ij})_{r,s} = K_n \left( \eta_{(i-1)m+r}, \eta_{(j-1)m+s} \right) = k_n \left( \eta_{(i-1)m+r} \cdot \eta_{(j-1)m+s} \right),$$

where

$$\begin{aligned} \eta_{(i-1)m+r} &= \xi_{i,r} = \Psi \left( \rho_i, \theta_r^i \right), \\ \eta_{(j-1)m+s} &= \xi_{j,s} = \Psi \left( \rho_j, \theta_s^j \right) \end{aligned}$$

and

$$\eta_{(i-1)m+r} \cdot \eta_{(j-1)m+s} = \sin \rho_i \sin \rho_j \cos \left( \theta_r^i - \theta_s^j \right) + \cos \rho_i \cos \rho_j. \quad (5)$$

Evaluating  $\theta_r^i - \theta_s^j$  we obtain

$$\theta_r^i - \theta_s^j = \begin{cases} \frac{2(r-s)\pi}{m} & \text{for } i-j \text{ even,} \\ \frac{2(r-s)\pi}{m} + (-1)^i \frac{(\alpha-2)\pi}{m} & \text{for } i-j \text{ odd.} \end{cases}$$

Thus, for fixed  $i$  and  $j$ , the inner product (5) depends only on the difference  $r-s$ , so we can denote it by  $E_{ij}(r-s)$ . It is immediate that

$$E_{ij}(r-s-mk) = E_{ij}(r-s) \text{ for all } k \in \mathbb{Z}.$$

With these considerations, the entries of the matrix  $A_{ij}$  are

$$(A_{ij})_{r,s} = (k_n \circ E_{ij})(r-s),$$

and therefore the matrix  $A_{ij}$  is circulant.

Consider now the Fourier matrix  $F_m \in \mathbb{C}^{m \times m}$ , with the entries

$$(F_m)_{jk} = \frac{1}{\sqrt{m}} \omega^{-(j-1)(k-1)} \text{ where } \omega = \exp(2\pi i/m).$$

The matrix  $\Phi_n$  can be written as

$$\begin{aligned}
\mathbf{\Phi}_n &= (I_m \otimes F_m)^* \mathbf{\Lambda} (I_m \otimes F_m) \\
&= \text{diag}(F_m^*, \dots, F_m^*) \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \dots & \Lambda_{1m} \\ \vdots & \vdots & & \vdots \\ \Lambda_{m1} & \Lambda_{m2} & \dots & \Lambda_{mm} \end{pmatrix} \text{diag}(F_m, \dots, F_m) \\
&= \mathbf{F}_N^* \mathbf{\Lambda} \mathbf{F}_N,
\end{aligned}$$

with  $\mathbf{F}_N = \text{diag}(F_m, \dots, F_m)$ ,  $A_{ij} = F_m^* \Lambda_{ij} F_m$  and  $\Lambda_{ij} = \text{diag}(\lambda_1(A_{ij}), \dots, \lambda_m(A_{ij}))$ . Here  $\lambda_k(A_{ij})$ ,  $k = 1, \dots, m$ , denote the eigenvalues of the circulant matrix  $A_{ij}$  and they are given by

$$\lambda_k(A_{ij}) = p_{A_{ij}}(\omega^{k-1}),$$

with

$$p_{A_{ij}}(x) = \sum_{\mu=1}^m (A_{ij})_{1,\mu} x^{\mu-1} = \sum_{\mu=1}^m K_n \left( \eta_{(i-1)m+1}, \eta_{(j-1)m+\mu} \right) x^{\mu-1}.$$

Consider the permutation matrix  $\mathbf{P}$  obtained from  $\mathbf{I}_N$  by re-ordering its columns  $\mathbf{c}_i$ ,  $i = 1, \dots, N$ , as follows

$$\begin{aligned}
\mathbf{P} &= \left( \mathbf{c}_1, \mathbf{c}_{m+1}, \mathbf{c}_{2m+1}, \dots, \mathbf{c}_{(m-1)m+1}, \mathbf{c}_2, \mathbf{c}_{m+2}, \right. \\
&\quad \left. \dots, \mathbf{c}_{(m-1)m+2}, \dots, \mathbf{c}_m, \mathbf{c}_{2m}, \mathbf{c}_{3m}, \dots, \mathbf{c}_{m^2} \right).
\end{aligned}$$

The product  $\mathbf{P} \mathbf{\Lambda} \mathbf{P}$  will be the diagonal block matrix  $\mathbf{D}_N = \text{diag}(D_1, D_2, \dots, D_m)$ , with

$$(D_k)_{ij} = p_{A_{ij}}(\omega^{k-1}), \quad k = 1, 2, \dots, m.$$

For the cubature formulae we are interested in evaluating the vector

$$(w_1^n, w_2^n, \dots, w_N^n)^T = \mathbf{\Phi}_n^{-1} \mathbf{u}_N,$$

where  $\mathbf{u}_N = (1, 1, \dots, 1)^T \in \mathbb{R}^N$ . We have shown that the matrix  $\mathbf{\Phi}_n$  can be written as

$$\mathbf{\Phi}_n = \mathbf{F}_N^* \mathbf{\Lambda}_N \mathbf{F}_N = \mathbf{F}_N^* \mathbf{P} \mathbf{D}_N \mathbf{P} \mathbf{F}_N$$

and thus, using the property  $\mathbf{P}^{-1} = \mathbf{P}$ , we obtain

$$\mathbf{\Phi}_n^{-1} = \mathbf{F}_N^* \mathbf{P} \mathbf{D}_N^{-1} \mathbf{P} \mathbf{F}_N.$$

Then, with the notation  $\mathbf{e}_1 = (1, 0, \dots, 0) \in \mathbb{R}^m$  we compute

$$\begin{aligned}
F_m \mathbf{u}_m &= \frac{1}{\sqrt{m}} (m, 0, \dots, 0)^T = \sqrt{m} \mathbf{e}_1^T, \\
\mathbf{F}_N \mathbf{u}_N &= \left( \sqrt{m}, 0, \dots, 0 \mid \dots \mid \sqrt{m}, 0, \dots, 0 \right)^T = \sqrt{m} (\mathbf{e}_1 \mid \mathbf{e}_1 \mid \dots \mid \mathbf{e}_1)^T, \\
\mathbf{P} \mathbf{F}_N \mathbf{u}_N &= \sqrt{m} \left( \underbrace{1, 1, \dots, 1}_m \mid \underbrace{0, 0, \dots, 0}_m \mid \dots \mid \underbrace{0, 0, \dots, 0}_m \right)^T = \sqrt{m} \left( \mathbf{u}_m^T \mid \mathbf{0} \mid \dots \mid \mathbf{0} \right)^T, \\
\mathbf{D}_N^{-1} \mathbf{P} \mathbf{F}_N \mathbf{u}_N &= \sqrt{m} \operatorname{diag} \left( D_1^{-1}, D_2^{-1}, \dots, D_m^{-1} \right) \left( \mathbf{u}_m^T \mid \mathbf{0} \mid \dots \mid \mathbf{0} \right)^T \\
&= \sqrt{m} \left( (D_1^{-1} \mathbf{u}_m)^T \mid \mathbf{0} \mid \dots \mid \mathbf{0} \right)^T.
\end{aligned}$$

Denoting  $D_1^{-1} \mathbf{u}_m = \mathbf{s} = (s_1, s_2, \dots, s_m)^T$ , we further obtain

$$\mathbf{P} \mathbf{D}_N^{-1} \mathbf{P} \mathbf{F}_N \mathbf{u}_N = \sqrt{m} \left( \underbrace{s_1, 0, \dots, 0}_m \mid \underbrace{s_2, 0, \dots, 0}_m \mid \dots \mid \underbrace{s_m, 0, \dots, 0}_m \right)^T,$$

and finally

$$\Phi_n^{-1} \mathbf{u}_N = \mathbf{F}_N^* \mathbf{P} \mathbf{D}_N^{-1} \mathbf{P} \mathbf{F}_N \mathbf{u}_N = \left( \underbrace{s_1, s_1, \dots, s_1}_m \mid \underbrace{s_2, s_2, \dots, s_2}_m \mid \dots \mid \underbrace{s_m, s_m, \dots, s_m}_m \right)^T.$$

In conclusion, the weights  $(w_1^n, w_2^n, \dots, w_N^n)$  take at most  $m$  distinct values, contained in the vector  $\mathbf{s} = (s_1, s_2, \dots, s_m)^T = D_1^{-1} \mathbf{u}_m$ .

### 3 The matrix $D_1^{-1}$

We focus now on the matrix  $D_1$  having the entries

$$\begin{aligned}
(D_1)_{ij} &= \sum_{k=1}^m K_n \left( \eta_{(i-1)m+1}, \eta_{(j-1)m+k} \right) \\
&= \sum_{k=1}^m \sum_{l=0}^n \frac{2l+1}{4\pi} P_l \left( \eta_{(i-1)m+1} \cdot \eta_{(j-1)m+k} \right) \\
&= \frac{1}{4\pi} \sum_{l=0}^n (2l+1) \sum_{k=1}^m P_l \left( \eta_{(i-1)m+1} \cdot \eta_{(j-1)m+k} \right).
\end{aligned}$$

At this point we prove the following result.

**Lemma 2** *Let  $\{\xi_k\}_{k=0, \dots, n}$  be  $n+1$  equidistant points on a circle of  $\mathbb{S}^2$ , situated at the latitude  $\rho^*$ . Then for every point  $\xi(\Psi(\theta, \rho)) \in \mathbb{S}^2$  we have*

$$\sum_{k=0}^n P_l(\xi \cdot \xi_k) = (n+1) P_l(\cos \rho) P_l(\cos \rho^*) \text{ for } l = 0, 1, \dots, n.$$



**Proof.** The spherical coordinates of the point  $\xi_k$  are  $(\theta_k, \rho^*)$ , with  $\theta_k = (\beta + 2k\pi)/(n+1)$ ,  $\beta \in [0, 2\pi)$ ,  $k = 0, 1, \dots, n$ . Using the associated Legendre functions  $P_k^j$ , defined by

$$P_k^j(t) = \left( \frac{(k-j)!}{(k+j)!} \right)^{1/2} (1-t^2)^{j/2} \frac{d^j}{dt^j} P_k(t), \quad j = 0, \dots, k, \quad t \in [-1, 1],$$

we can write

$$\begin{aligned} P_l(\xi \cdot \xi_k) &= \sum_{s=-l}^l P_l^{|s|}(\cos \rho) P_l^{|s|}(\cos \rho^*) e^{is\theta - is\theta_k}, \\ \sum_{k=0}^n P_l(\xi \cdot \xi_k) &= \sum_{s=-l}^l P_l^{|s|}(\cos \rho) P_l^{|s|}(\cos \rho^*) e^{is\theta - is\frac{\beta}{n+1}} \sum_{k=0}^n \left( e^{-is\frac{2\pi}{n+1}} \right)^k. \end{aligned}$$

Since  $-n \leq -l \leq s \leq l \leq n$ , it follows that the only non-zero term of the sum  $\sum_{s=-l}^l$  is the one corresponding to  $s = 0$ . In this case, the sum  $\sum_{k=0}^n$  equals  $n+1$  and thus

$$\sum_{k=0}^n P_l(\xi \cdot \xi_k) = (n+1) P_l(\cos \rho) P_l(\cos \rho^*) \quad \text{for } l = 0, 1, \dots, n.$$

■

As an immediate consequence we prove the following result.

**Lemma 3** *Let  $\{\xi_k, \tilde{\xi}_k, k = 0, 1, \dots, n\}$  be  $n+1$  equidistant points situated on circles which are symmetric with respect to the equator, at the latitudes  $\rho^*$  and  $\pi - \rho^*$ , respectively. Then*

$$\sum_{k=0}^n \left( P_{2p+1}(\xi \cdot \xi_k) + P_{2p+1}(\xi \cdot \tilde{\xi}_k) \right) = 0 \quad \text{for } p = 0, 1, \dots, (n+1)/2 - 1.$$

**Proof.** Since  $P_{2p+1}$  is an odd polynomial, we obtain

$$P_{2p+1}(\cos \rho^*) + P_{2p+1}(-\cos \rho^*) = 0.$$

Then, multiplying it by  $P_{2p+1}(\cos \rho)$ , the conclusion follows immediately. ■

Let us come back to the matrix  $D_1$ . Its entries are

$$(D_1)_{ij} = \frac{n+1}{4\pi} \sum_{l=0}^n (2l+1) P_l(\cos \rho_i) P_l(\cos \rho_j).$$

With the notations  $n + 1 = m$ ,  $q = (n + 1)/2$  and  $\cos \rho_i = r_i$ ,  $i = 1, \dots, q$ , it can be written as  $D_1 = \mathbf{G} \cdot \mathbf{G}^T$ , with

$$\mathbf{G} = \sqrt{\frac{m}{4\pi}} \begin{pmatrix} P_0(r_1) & \sqrt{3}P_1(r_1) & \sqrt{5}P_2(r_1) & \cdots & \sqrt{2n+1}P_n(r_1) \\ P_0(r_2) & \sqrt{3}P_1(r_2) & \sqrt{5}P_2(r_2) & \cdots & \sqrt{2n+1}P_n(r_2) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ P_0(r_q) & \sqrt{3}P_1(r_q) & \sqrt{5}P_2(r_q) & \cdots & \sqrt{2n+1}P_n(r_q) \\ \hline P_0(-r_q) & \sqrt{3}P_1(-r_q) & \sqrt{5}P_2(-r_q) & \cdots & \sqrt{2n+1}P_n(-r_q) \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ P_0(-r_2) & \sqrt{3}P_1(-r_2) & \sqrt{5}P_2(-r_2) & \cdots & \sqrt{2n+1}P_n(-r_2) \\ P_0(-r_1) & \sqrt{3}P_1(-r_1) & \sqrt{5}P_2(-r_1) & \cdots & \sqrt{2n+1}P_n(-r_1) \end{pmatrix}.$$

Consider the matrix  $\mathbf{P}_1$ , obtained from the identity matrix  $\mathbf{I}_m$  by the transformations row  $(n + 1) \leftarrow \text{row } (n + 1) - \text{row } 1$ , row  $n \leftarrow \text{row } n - \text{row } 2$ ,  $\dots$ , row  $(q + 1) \leftarrow \text{row } (q + 1) - \text{row } q$ . Then the matrix  $\mathbf{G}_1 = \mathbf{P}_1 \mathbf{G}$  has the form

$$\mathbf{G}_1 = \sqrt{\frac{m}{4\pi}} \begin{pmatrix} P_0(r_1) & \sqrt{3}P_1(r_1) & \sqrt{5}P_2(r_1) & \cdots & \sqrt{2n+1}P_n(r_1) \\ \vdots & \vdots & \vdots & & \vdots \\ P_0(r_q) & \sqrt{3}P_1(r_q) & \sqrt{5}P_2(r_q) & \cdots & \sqrt{2n+1}P_n(r_q) \\ \hline 0 & -2\sqrt{3}P_1(r_q) & 0 & \cdots & -2\sqrt{2n+1}P_n(r_q) \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & -2\sqrt{3}P_1(r_1) & 0 & \cdots & -2\sqrt{2n+1}P_n(r_1) \end{pmatrix}.$$

By multiplying it by a permutation matrix  $\mathbf{P}_2$  we can write

$$\mathbf{G}_2 = \mathbf{G}_1 \mathbf{P}_2 = c_n \left( \begin{array}{c|c} \mathbf{X}_q & \mathbf{P}_3 \mathbf{Y}_q \\ \hline \mathbf{0}_q & -2\mathbf{Y}_q \end{array} \right),$$

where  $c_n = \sqrt{\frac{n+1}{4\pi}}$ ,

$$\mathbf{X}_q = \begin{pmatrix} 1 & \sqrt{5}P_2(r_1) & \cdots & \sqrt{2n-1}P_{n-1}(r_1) \\ \vdots & \vdots & & \vdots \\ 1 & \sqrt{5}P_2(r_q) & \cdots & \sqrt{2n-1}P_{n-1}(r_q) \end{pmatrix}, \quad (6)$$

$$\mathbf{Y}_q = \begin{pmatrix} \sqrt{3}P_1(r_q) & \sqrt{7}P_3(r_q) & \dots & \sqrt{2n+1}P_n(r_q) \\ \vdots & \vdots & & \vdots \\ \sqrt{3}P_1(r_1) & \sqrt{7}P_3(r_1) & \dots & \sqrt{2n+1}P_n(r_1) \end{pmatrix},$$

and  $\mathbf{P}_3 \in \mathbb{R}^{q \times q}$  is a permutation matrix which changes the rows  $\left(1, \frac{n+1}{2}\right)$ ,  $\left(2, \frac{n-1}{2}\right), \dots, \left(\frac{q}{2}, \frac{q+1}{2}\right)$ .

We wish to evaluate  $D_1^{-1}\mathbf{u}_m$ , so we write

$$\begin{aligned} D_1 &= \mathbf{G}\mathbf{G}^T = \mathbf{P}_1^{-1}\mathbf{G}_1 \left(\mathbf{P}_1^{-1}\mathbf{G}_1\right)^T = \mathbf{P}_1^{-1}\mathbf{G}_1\mathbf{G}_1^T \left(\mathbf{P}_1^T\right)^{-1} \\ &= \mathbf{P}_1^{-1}\mathbf{G}_2\mathbf{P}_2^T \left(\mathbf{G}_2\mathbf{P}_2^T\right)^T \left(\mathbf{P}_1^{-1}\right)^T = \mathbf{P}_1^{-1}\mathbf{G}_2\mathbf{G}_2^T \left(\mathbf{P}_1^{-1}\right)^T \end{aligned}$$

and finally

$$D_1^{-1} = \mathbf{P}_1^T \left(\mathbf{G}_2^{-1}\right)^T \mathbf{G}_2^{-1}\mathbf{P}_1.$$

Then  $\mathbf{P}_1\mathbf{u}_m = (\mathbf{u}_q | \mathbf{0}_q)^T$ , and

$$\begin{aligned} \mathbf{G}_2^{-1} &= \frac{1}{c_n} \begin{pmatrix} \mathbf{X}_q & \mathbf{P}_3\mathbf{Y}_q \\ \mathbf{0}_q & -2\mathbf{Y}_q \end{pmatrix}^{-1} \\ &= \frac{1}{c_n} \begin{pmatrix} \mathbf{X}_q^{-1} & \frac{1}{2}\mathbf{X}_q^{-1}(\mathbf{P}_3\mathbf{Y}_q)\mathbf{Y}_q^{-1} \\ \mathbf{0}_q & -\frac{1}{2}\mathbf{Y}_q^{-1} \end{pmatrix} = \frac{1}{c_n} \begin{pmatrix} \mathbf{X}_q^{-1} & \frac{1}{2}\mathbf{X}_q^{-1}\mathbf{P}_3 \\ \mathbf{0}_q & -\frac{1}{2}\mathbf{Y}_q^{-1} \end{pmatrix}. \end{aligned}$$

Moreover, we obtain

$$\begin{aligned} \mathbf{G}_2^{-1}\mathbf{P}_1\mathbf{u}_m &= \frac{1}{c_n} \begin{pmatrix} \mathbf{X}_q^{-1}\mathbf{u}_q \\ \mathbf{0}_q \end{pmatrix}, \\ \left(\mathbf{G}_2^{-1}\right)^T \mathbf{G}_2^{-1}\mathbf{P}_1\mathbf{u}_m &= \frac{1}{c_n^2} \begin{pmatrix} \left(\mathbf{X}_q^{-1}\right)\mathbf{X}_q^{-1}\mathbf{u}_q \\ \frac{1}{2}\left(\mathbf{X}_q^{-1}\mathbf{P}_3\right)^T \mathbf{u}_q \end{pmatrix} \\ &= \frac{1}{c_n^2} \begin{pmatrix} \left(\mathbf{X}_q\mathbf{X}_q^T\right)^{-1}\mathbf{u}_q \\ \frac{1}{2}\mathbf{P}_3\left(\mathbf{X}_q\mathbf{X}_q^T\right)^{-1}\mathbf{u}_q \end{pmatrix} \end{aligned}$$

and

$$\mathbf{P}_1^T \left(\mathbf{G}_2^{-1}\right)^T \mathbf{G}_2^{-1}\mathbf{P}_1\mathbf{u}_m = \frac{1}{c_n^2} \begin{pmatrix} \frac{1}{2}\left(\mathbf{X}_q\mathbf{X}_q^T\right)^{-1}\mathbf{u}_q \\ \frac{1}{2}\mathbf{P}_3\left(\mathbf{X}_q\mathbf{X}_q^T\right)^{-1}\mathbf{u}_q \end{pmatrix} = D_1^{-1}\mathbf{u}_m.$$

Denoting

$$\mathbf{a} = (a_1, a_2, \dots, a_q)^T = (\mathbf{X}_q \mathbf{X}_q^T)^{-1} \mathbf{u}_q,$$

the vector  $\mathbf{P}_3 (\mathbf{X}_q \mathbf{X}_q^T)^{-1} \mathbf{u}_p$  is equal to  $(a_q, \dots, a_2, a_1)^T$  and thus

$$D_1^{-1} \mathbf{u}_m = \frac{2\pi}{m} (a_1, a_2, \dots, a_q, a_q, a_{q-1}, \dots, a_2, a_1).$$

In conclusion, the cubature weights  $w_i^n$  take at most  $q$  distinct values and in order to evaluate them it is enough to evaluate the components of the vector  $\mathbf{a} = (\mathbf{X}_q \mathbf{X}_q^T)^{-1} \mathbf{u}_q$ .

#### 4 The weights of the cubature formula

The matrix  $\mathbf{X}_q$  given in (6) can be written as  $\mathbf{X} = \mathbf{X}_q = \mathbf{V} \cdot \mathbf{U}$ , with

$$\mathbf{V} = \begin{pmatrix} 1 & r_1^2 & r_1^4 & \dots & r_1^{n-1} \\ 1 & r_2^2 & r_2^4 & \dots & r_2^{n-1} \\ \vdots & & & & \vdots \\ 1 & r_q^2 & r_q^4 & \dots & r_q^{n-1} \end{pmatrix}, \quad (7)$$

$$\mathbf{U} = \begin{pmatrix} b_{00} & \sqrt{5}b_{02} & \sqrt{9}b_{04} & \dots & \sqrt{2n-1}b_{0,n-1} \\ & \sqrt{5}b_{12} & \sqrt{9}b_{14} & \dots & \sqrt{2n-1}b_{1,n-1} \\ & & & \vdots & \\ & & & & \sqrt{2n-1}b_{q-1,n-1} \end{pmatrix},$$

the entries of the matrix  $\mathbf{U}$  being the coefficients of the Legendre polynomials

$$P_{2l}(x) = \sum_{k=0}^l b_{k,2l} x^{2k}.$$

Since the vector  $\mathbf{a}$  is the solution of the system

$$\mathbf{X} \mathbf{X}^T \mathbf{a} = \mathbf{u}_q, \quad (8)$$

using the Cramer's rule, the component  $a_i$  can be written as

$$a_i = \frac{\det(\mathbf{Y}_i)}{\det(\mathbf{X} \mathbf{X}^T)},$$

where  $\mathbf{Y}_i$  is the matrix obtained by replacing in  $\mathbf{Y} = \mathbf{X} \mathbf{X}^T$  the column  $i$  by the vector  $\mathbf{u}_q$ . One can prove that the matrix  $\mathbf{Y}_i$  can be written as  $\mathbf{X} \mathbf{X}_i^T$ ,

where  $\mathbf{X}_i^T$  is the matrix obtained from  $\mathbf{X}^T$  by replacing the column  $i$  by the column vector  $\mathbf{e}_1^T = (1, 0, \dots, 0)$ . Consequently,

$$a_i = \frac{\det(\mathbf{X}) \cdot \det(\mathbf{X}_i^T)}{\det(\mathbf{X}) \cdot \det(\mathbf{X}^T)} = \frac{\det(\mathbf{X}_i^T)}{\det(\mathbf{X}^T)} = \frac{\det(\mathbf{Z}_i^T)}{\det(\mathbf{Z}^T)},$$

where

$$\mathbf{Z} = \begin{pmatrix} 1 & P_2(r_1) & \dots & P_{n-1}(r_1) \\ 1 & P_2(r_2) & \dots & P_{n-1}(r_2) \\ \vdots & \vdots & & \dots \\ 1 & P_2(r_q) & \dots & P_{n-1}(r_q) \end{pmatrix}$$

and the matrix  $\mathbf{Z}_i^T$  is obtained from the matrix  $\mathbf{Z}^T$  by replacing the column  $i$  by the vector  $(1, 0, \dots, 0)^T$ . Thus, the vector  $\mathbf{a}$  becomes the solution of the system

$$\mathbf{Z}^T \cdot \mathbf{a} = \mathbf{e}_1,$$

which implies  $\mathbf{a} = (\mathbf{Z}^T)^{-1} \mathbf{e}_1$ . The matrix  $\mathbf{Z}^T$  can be written as  $\mathbf{Z}^T = \mathbf{L} \cdot \mathbf{V}^T$ , where  $\mathbf{V}$  is defined in (7) and

$$\mathbf{L} = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ b_{02} & b_{12} & 0 & \dots & 0 \\ b_{04} & b_{14} & b_{24} & \dots & 0 \\ \vdots & & & & \\ b_{0,n-1} & b_{1,n-1} & b_{2,n-1} & \dots & b_{q-1,n-1} \end{pmatrix}.$$

So,  $\mathbf{a} = (\mathbf{V}^T)^{-1} \mathbf{L}^{-1} \mathbf{e}_1$ , the vector  $\mathbf{g} = \mathbf{L}^{-1} \mathbf{e}_1$  being the first column of the matrix  $\mathbf{L}^{-1}$ .

In order to calculate the first column of  $\mathbf{L}^{-1}$  we use the orthogonality property of Legendre polynomials. Since  $P_{2l}$  is an even polynomial, orthogonal to  $P_0$  on the interval  $[-1, 1]$  for  $l \neq 0$ , we can write

$$0 = \int_0^1 P_{2l}(x) dx = \int_0^1 \sum_{k=0}^l b_{k,2l} x^{2k} dx = \sum_{k=0}^l \frac{1}{2k+1} b_{k,2l}.$$

Hence, the row  $l$ ,  $l \neq 1$  of the matrix  $\mathbf{L}$  is orthogonal to the vector  $\mathbf{v} = \left(1, \frac{1}{3}, \frac{1}{5}, \dots, \frac{1}{2q-1}\right)^T$  and thus the first column of  $\mathbf{L}^{-1}$  will be  $\mathbf{v}$ .

We have now to evaluate the vector  $\mathbf{a} = \mathbf{V}^{-1} \mathbf{v}$ . Using again the Cramer's rule, we can calculate  $a_i$  by replacing the vector  $\mathbf{v}$  into the  $i$ th column of  $\mathbf{V}^T$ , as follows:

$$\begin{aligned}
a_i &= \frac{1}{\det(\mathbf{V})} \cdot \begin{vmatrix} 1 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1 \\ r_1^2 & r_2^2 & \cdots & r_{i-1}^2 & \frac{1}{3} & r_{i+1}^2 & \cdots & r_q^2 \\ r_1^4 & r_2^4 & \cdots & r_{i-1}^4 & \frac{1}{5} & r_{i+1}^4 & \cdots & r_q^4 \\ \vdots & & & & & & & \\ r_1^{n-1} & r_2^{n-1} & \cdots & r_{i-1}^{n-1} & \frac{1}{2q-1} & r_{i+1}^{n-1} & \cdots & r_q^{n-1} \end{vmatrix} \\
&= (-1)^{i+1} \left( V_0^i - \frac{1}{3}V_1^i + \frac{1}{5}V_2^i + \cdots + (-1)^{q-1} \frac{1}{2q-1} V_q^i \right),
\end{aligned}$$

where  $V_k^i$ ,  $k = 0, 1, \dots, q-1$ , denote the lacunary Vandermonde determinants, obtained from the matrix  $V$  by eliminating the row  $k+1$  together with the column  $i$ . Regarding the lacunary Vandermonde determinants, the following result is known.

**Lemma 4** *Let  $x_1, x_2, \dots, x_p \in \mathbb{C}$  and  $V = V(x_1, x_2, \dots, x_p) \in \mathbb{C}$  their Vandermonde determinant. Let  $V_k$  denote the following lacunary determinant:*

$$V_k = V_k(x_1, \dots, x_p) = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_p \\ \vdots & \vdots & & \vdots \\ x_1^{k-1} & x_2^{k-1} & \cdots & x_p^{k-1} \\ x_1^{k+1} & x_2^{k+1} & \cdots & x_p^{k+1} \\ \vdots & \vdots & & \vdots \\ x_1^p & x_2^p & \cdots & x_p^p \end{vmatrix}, \quad k = 0, 1, \dots, p.$$

Then  $V_k = V \cdot S_{p-k}$ , where  $S_k$ ,  $k = 0, 1, \dots, p$ , denote the Viète sums of the numbers  $x_1, \dots, x_p$ .

Using Lemma 4, the numbers  $a_i$  can be expressed as

$$\begin{aligned}
a_i &= (-1)^{i+1} \frac{V(r_1^2, \dots, r_{i-1}^2, r_{i+1}^2, \dots, r_q^2)}{V(r_1^2, \dots, r_q^2)} \\
&\quad \times \left( S_q^i - \frac{1}{3}S_{q-1}^i + \frac{1}{5}S_{q-2}^i - \cdots + \frac{(-1)^{q-2}}{2q-3} S_1^i + \frac{(-1)^{q-1}}{2q-1} \right),
\end{aligned}$$

where  $S_k^i$ ,  $k = 1, \dots, q$ , are the Viète sums of the numbers  $r_1^2, \dots, r_{i-1}^2, r_{i+1}^2, \dots, r_q^2$ . Replacing the values  $\frac{1}{2l+1}$ ,  $l = 0, 1, \dots, q-1$ , with  $\int_0^1 x^{2l} dx$ , we further obtain

$$\begin{aligned}
a_i &= \frac{(-1)^{i+1}(-1)^{q-i}(-1)^{q-1}}{(r_i^2 - r_1^2) \dots (r_i^2 - r_{i-1}^2)(r_i^2 - r_{i+1}^2) \dots (r_i^2 - r_q^2)} \\
&\times \int_0^1 \left( x^{2q-2} - S_1^i x^{2q-4} + S_2^i x^{2q-6} - \dots + (-1)^{q-2} x^2 S_{q-1}^i + (-1)^{q-1} S_q^i \right) dx \\
&= \int_0^1 \frac{(x^2 - r_1^2) \dots (x^2 - r_{i-1}^2)(x^2 - r_{i+1}^2) \dots (x^2 - r_q^2)}{(r_i^2 - r_1^2) \dots (r_i^2 - r_{i-1}^2)(r_i^2 - r_{i+1}^2) \dots (r_i^2 - r_q^2)} dx \\
&= \int_0^1 l_i(x^2) dx,
\end{aligned}$$

where  $l_i(x)$  are the fundamental Lagrange polynomials associated to the points  $r_1^2, \dots, r_q^2$ .

Further, the vector  $\mathbf{1} = (l_1(x^2), l_2(x^2), \dots, l_q(x^2))^T$  can be regarded as the solution of the system

$$\mathbf{V}^T \mathbf{1} = \left( 1, x^2, x^4, \dots, x^{2(q-1)} \right)^T, \quad (9)$$

with  $\mathbf{V}$  given in (7). By integrating from 0 to 1 the equations of the system (9) we obtain that the weights  $\{a_i, i = 1, \dots, q\}$  and the cosines of the latitudes,  $\{r_j, j = 1, \dots, q\}$ , should satisfy the following conditions

$$\begin{aligned}
a_1 + a_2 + \dots + a_q &= 1, \\
a_1 r_1^2 + a_2 r_2^2 + \dots + a_q r_q^2 &= 1/3, \\
a_1 r_1^4 + a_2 r_2^4 + \dots + a_q r_q^4 &= 1/5, \\
&\dots \\
a_1 r_1^{2(q-1)} + a_2 r_2^{2(q-1)} + \dots + a_q r_q^{2(q-1)} &= 1/(2q-1).
\end{aligned} \quad (10)$$

Thus, for the quadrature formula

$$\int_0^1 f(x) dx \approx \sum_{i=1}^q a_i f(r_i), \quad (11)$$

the equations of the system (10) are equivalent to the requirements that the quadrature formula (11) is exact for the monomials  $1, x^2, x^4, \dots, x^{2q-2}$ .

On the other hand, if we denote  $v_1 = -r_q, v_2 = -r_{q-1}, \dots, v_q = -r_1, v_{q+1} = r_1, v_{q+2} = r_2, \dots, v_{2q} = r_q$ , then the numbers  $v_i$  can be regarded as the nodes of the quadrature formula

$$\int_{-1}^1 f(x) dx \approx \sum_{i=1}^{2q} a_i f(v_i), \quad (12)$$

with  $a_{n+2-i} = a_i$  for  $i = 1, \dots, q$ , and the system (10) is equivalent to the requirement that the quadrature formula (12) is exact for the monomials  $1, x, x^2, \dots, x^{2q-1}$ .

## 5 Cubature formulae with positive weights

We are particularly interested in the positivity of the weights  $w_i^n = \frac{2\pi}{n+1}a_i$ ,  $i = 1, \dots, q$ . Consider the interpolating cubature formula

$$\int_{\mathbb{S}^2} F(\xi) d\omega(\xi) \approx \sum_{i=1}^{n+1} \sum_{k=1}^{n+1} w_{i,k}^n F(\xi_{i,k}), \quad (13)$$

with  $\xi_{i,k} \in S_n(\alpha)$ , for which we have proved the equalities  $w_{i,k}^n = w_i^n$  for  $i, k \in \{1, \dots, n\}$  and the fact that the weights  $w_i^n$ ,  $i = 1, \dots, n+1$  take  $q$  distinct values denoted  $\frac{2\pi}{n+1}a_i$ ,  $i = 1, \dots, q$ . Let us denote  $r_i = \cos \rho_i$  for  $i = 1, \dots, q$ ,  $v_1 = -r_q, v_2 = -r_{q-1}, \dots, v_q = -r_1, v_{q+1} = r_1, v_{q+2} = r_2, \dots, v_{2q} = r_q$ . The following theorem describes three possible cases in which the weights are positive.

**Theorem 5** *With the above notations, the cubature formula (13) is positive in the following cases.*

1. The numbers  $v_i$  are the roots of the Legendre polynomial  $P_{n+1}$  and

$$a_i = \frac{2}{(1 - v_i^2) (P'_{n+1}(v_i))^2}, \text{ for } i = 1, \dots, q.$$

2. The numbers  $r_i$  are taken as  $r_i = \frac{t_i+1}{2}$ , where  $t_i$  are the roots of the Legendre polynomial  $P_q$ , and

$$a_i = \frac{1}{(1 - t_i^2) (P'_q(t_i))^2}, \text{ for } i = 1, \dots, q.$$

3. More general, the numbers  $r_i$  are taken as  $r_i = \frac{t_i+1}{2}$ , where  $t_i$  are the roots of the polynomial  $Q_q = P_q + \rho P_{q-1}$ , with  $\rho \in (-1, 1)$  and

$$a_i = \frac{1}{(1 - t_i^2) (P'_q(t_i))^2}, \text{ for } i = 1, \dots, q.$$

**Proof.** 1. Choosing the latitudes  $\{\rho_i, i = 1, \dots, n+1\}$  such that  $\cos \rho_i = v_i$ , the matrix  $D_1$  becomes diagonal with positive entries and therefore the vector  $D_1^{-1} \mathbf{u}_m$  has positive components.

2. With the notation  $g(t) = \frac{1}{2}f\left(\frac{t+1}{2}\right)$  we have

$$\begin{aligned} \int_0^1 f(x) dx &= \frac{1}{2} \int_{-1}^1 f\left(\frac{t+1}{2}\right) dt = \int_{-1}^1 g(t) dt \approx \sum_{i=1}^q \alpha_i g(t_i) \\ &= \frac{1}{2} \sum_{i=1}^q \alpha_i f\left(\frac{t_i+1}{2}\right) = \sum_{i=1}^q a_i f(r_i). \end{aligned}$$



Thus, the quadrature formula (11) has positive weights given by

$$a_i = \frac{\alpha_i}{2} = \frac{1}{(1 - t_i^2) \left(P'_q(t_i)\right)^2}$$

and it is exact for  $f \in \mathbb{P}_{2q-1}$ , since the quadrature formula

$$\int_{-1}^1 g(t) dt \approx \sum_{i=1}^q \alpha_i g(t_i)$$

is exact for  $g \in \mathbb{P}_{2q-1}$ .

3. It is known (see [4], Th. 3.3.4, [5]) that the polynomial  $Q_q$  has distinct zeros, all situated in  $(-1, 1)$  if and only if  $\rho \in (-1, 1)$ . In this case, the quadrature formula (11) has positive weights and is exact for  $f \in \mathbb{P}_{2q-2}$ . ■

## 6 Spherical designs

A spherical design is a set of points of  $\mathbb{S}^2$  which generates a cubature formula with equal weights, exact for spherical polynomials up to a certain degree. We try to find conditions on the latitudes  $\rho_i$ , which assure that the set  $S_n(\alpha)$  is a spherical design. So we suppose

$$w_1^n = w_2^n = \dots = w_N^n = w_n,$$

implying  $a_1 = a_2 = \dots = a_q = 1/q$ . We make again the notations  $\cos \rho_i = r_i$ ,  $r_i^2 = \gamma_i$ , for  $i = 1, \dots, q$ . In the trivial case  $n = 1$ , we obtain a spherical design if and only if  $w_1 = \pi$  and  $\rho_1 = \pi/6$ .

Next we focus on the case  $n \geq 3$ . In this case the numbers  $\gamma_i$  should satisfy the following conditions:

$$\begin{aligned} \gamma_1 + \gamma_2 + \dots + \gamma_q &= q/3, \\ \gamma_1^2 + \gamma_2^2 + \dots + \gamma_q^2 &= q/5, \\ &\dots \\ \gamma_1^{q-1} + \gamma_2^{q-1} + \dots + \gamma_q^{q-1} &= q/n, \end{aligned} \tag{14}$$

equivalent to the requirement that the quadrature formula (cf. (12))

$$\int_{-1}^1 f(x) dx \approx \frac{1}{q} \sum_{i=1}^q (f(r_i) + f(-r_i)) \tag{15}$$

is exact for the monomials  $1, x, \dots, x^{2q-1}$ . Let us mention here that, unlike the classical Chebyshev quadrature, we impose in (15) for our purpose only  $q - 1$  conditions for  $q$  unknowns.

The purpose now is to show how to solve the system (14), containing  $q - 1$  equations and  $q$  unknowns. A way to do it is described in the following. We introduce the notations  $W_0 = q$ ,

$$W_k = \gamma_1^k + \gamma_2^k + \dots + \gamma_q^k, \quad k = 1, 2, \dots, q$$

and the parameter  $\beta = W_q$ . The solutions of the system (14) are the roots of the polynomial

$$T_q(x) = x^q + S_1x^{q-1} + S_2x^{q-2} + \dots + S_q,$$

where  $(-1)^k S_k$  denotes the Viète sums of order  $k$  associated to the numbers  $\gamma_1, \dots, \gamma_q$ . They are related to the sums  $W_k$  through the following relations:

$$\begin{aligned} W_1 + S_1 &= 0, \\ W_2 + S_1W_1 + 2S_2 &= 0, \\ W_3 + S_1W_2 + S_2W_1 + 3S_3 &= 0, \\ &\dots \\ W_q + S_1W_{q-1} + S_2W_{q-2} + \dots + S_qW_0 &= 0. \end{aligned} \tag{16}$$

Thus, in order to solve the system (14), we first calculate recursively the numbers  $S_1, S_2, \dots, S_q$  using the relations (16) and then determine the roots of the polynomial  $T_q$ . This polynomial depends on the parameter  $\beta$ , which appears only in the free term  $S_q$ .

For our purposes we need to find those values of  $\beta$  for which the polynomial  $T_q$  has all the roots located in  $(0, 1)$ . The following theorem discusses all the cases for which the set  $S_n(\alpha)$  defined in Theorem 1 constitutes a spherical design.

**Theorem 6** *Let  $n \in \mathbb{N}$ ,  $n \geq 3$ , be an odd number,  $q = (n+1)/2$  and consider the set  $S_n(\alpha)$ , defined in Proposition 1, with arbitrary  $\alpha \in (0, 2)$  and with the latitudes  $\{\rho_i, i = 1, \dots, q\}$  taken such that  $\cos \rho_i = \sqrt{\gamma_i}$ , where  $\gamma_i$  are the roots of the polynomial*

$$T_q(x) = x^q + S_1x^{q-1} + S_2x^{q-2} + \dots + S_q.$$

*The polynomial  $T_q$  has all the roots located in  $(0, 1)$  – and hence  $S_n(\alpha)$  constitutes a spherical design – in the following four cases.*

1. *For  $n = 3$ , the polynomial  $T_2(x) = x^2 - \frac{2x}{3} + \frac{1}{2} \left( \frac{4}{9} - \beta \right)$  has all the roots located in  $(0, 1)$  if and only if  $\beta \in \left( \frac{2}{9}, \frac{4}{9} \right)$ .*
2. *For  $n = 5$ , the polynomial  $T_3(x) = x^3 - x^2 + \frac{x}{5} + \frac{1}{3} \left( \frac{2}{5} - \beta \right)$  has all the roots located in  $(0, 1)$  if and only if  $\beta \in (0.4, 0.433996 \dots)$ .*

3. For  $n = 7$ , the polynomial  $T_4(x) = x^4 - \frac{4x^3}{3} + \frac{22x^2}{45} - \frac{148x}{2835} + \frac{1}{4} \left( \frac{18728}{42525} - \beta \right)$  has all the roots located in  $(0, 1)$  if and only if  $\beta \in (0.4336145 \dots, 0.4403997 \dots)$ .

4. For  $n = 9$ , the polynomial  $T_5(x) = x^5 - \frac{5x^4}{3} + \frac{8x^3}{9} - \frac{100x^2}{567} + \frac{17x}{1701} + \frac{1}{5} \left( \frac{2300}{5103} - \beta \right)$  has all the roots located in  $(0, 1)$  if and only if  $\beta \in (0.4507152 \dots, 0.4515677 \dots)$ .

For  $n \geq 11$ , the polynomial  $T_q$  cannot have all the roots located in  $(0, 1)$ .

**Proof.** The results in the first four cases are immediate consequences of the application of Rolle's sequence. In order to prove that for  $n \geq 11$  the system (14) has no real solution in  $(0, 1)$ , we use the results proved by Bernstein [1]. Here the author treated the quadrature formula

$$\int_0^1 f(x) dx \approx \frac{1}{n+1} \sum_{i=1}^{n+1} f(x_i) \quad (0 < x_i < 1). \quad (17)$$

By making the linear transform  $x = (y + 1)/2$ , it is immediate that (17) is equivalent to the quadrature formula

$$\int_{-1}^1 f(y) dy \approx \frac{2}{n+1} \sum_{i=1}^{n+1} f(y_i) \quad (-1 < y_i < 1). \quad (18)$$

If we denote  $\gamma_i = y_i^2$ ,  $i = 1, \dots, q$ , then the system (14) is equivalent to the requirement that (17) or (18) are exact for the monomials  $1, x, \dots, x^n$ .

In [1] the degree of exactness of (17) was denoted by  $M_{n+1}$ . For our purposes we have to prove that, for  $n$  odd,  $n \geq 11$ , we have  $M_{n+1} < n$ , implying the incompatibility of the system (14). The main result of [1] is the inequality

$$M_{n+1} < \pi \sqrt{2(n+1)}. \quad (19)$$

Since  $\pi \sqrt{2(n+1)} \leq n$  for  $n \geq 21$ , the system (14) will be incompatible for  $n \geq 21$ . Also, the following inequalities were proved

$$M_{12} < 12, \quad M_{14} < 13, \quad M_{16} < 15, \quad M_{18} < 15, \quad M_{20} < 20.$$

Since our goal is to prove that  $M_{n+1} < n$ , it remains to prove that  $M_{12} < 11$  and  $M_{20} < 19$ . In these cases, the associated polynomials  $T_q$  are

$$T_6(x) = x^6 - 2x^5 + \frac{7x^4}{5} - \frac{44x^3}{105} + \frac{9x^2}{175} - \frac{2x}{825} + \frac{1}{6} \left( \frac{92956}{202125} - \beta \right)$$

and

$$\begin{aligned}
T_{10}(x) = & x^{10} - \frac{10x^9}{3} + \frac{41x^8}{9} - \frac{1880x^7}{567} + \frac{2378x^6}{1701} - \frac{19556x^5}{56133} \\
& + \frac{2308918x^4}{45972927} - \frac{544840x^3}{137918781} + \frac{936619x^2}{7033857831} - \frac{384505294x}{25258583471121} \\
& + \frac{1}{10} \left( \frac{396816635954996}{833533254546993} - \beta \right),
\end{aligned}$$

respectively. An application of the Rolle's sequence shows that these polynomials cannot have all the roots located in  $(0, 1)$ . ■

## 7 The degree of exactness of the positive cubature formula

The cubature formula on the sphere (3) with the nodes in  $\xi_{j,k} \in S_n(\alpha)$  reduces to

$$\int_{\mathbb{S}^2} f(\xi) d\omega(\xi) \approx \sum_{j=1}^{n+1} w_j \sum_{k=1}^{n+1} f(\xi_{j,k}). \quad (20)$$

It is exact for  $f_n \in V_n$ , since it is an interpolatory one. The question which arises is whether it can be exact for all  $f \in V_{n+1}$ . The answer is given in the following theorem.

**Theorem 7** *Consider the interpolatory cubature formula (20), associated to the system of points  $S_n(\alpha)$  given in Proposition 1, which is exact for  $f \in V_n$  and suppose that the weights  $w_j$  are positive. Then the cubature formula is exact for  $f \in V_{n+1}$  if and only if  $\alpha = 1$  and*

$$\sum_{j=1}^{n+1} w_j P_{n+1}(\cos \rho_j) = 0, \quad (21)$$

where  $P_{n+1}$  denotes the Legendre polynomial of degree  $n + 1$ .

**Proof.** Let  $f \in V_{n+1}$ . Then  $f$  can be written as

$$f = f_n + g_n,$$

where  $g_n$  is an element of the wavelet space  $\text{Harm}_n = V_{n+1} \ominus V_n$ . It is well known that a basis in  $\text{Harm}_n$  is given by the set

$$\left\{ P_{n+1}^j(\cos \rho) \cos j\theta, j = 0, 1, \dots, n + 1 \right\} \cup \left\{ P_{n+1}^j(\cos \rho) \sin j\theta, j = 1, \dots, n + 1 \right\},$$

where  $P_{n+1}^j$  are the associated Legendre functions. Therefore the polynomial  $g_n$  can be written as

$$g_n = \sum_{j=0}^{n+1} P_{n+1}^j(\cos \rho) (b_j \cos j\theta + c_j \sin j\theta),$$

with  $b_j, c_j \in \mathbb{R}$ . We suppose the cubature formula (20) to be exact for  $f$ , meaning that

$$\int_{\mathbb{S}^2} f(\xi) d\omega(\xi) = \sum_{i=1}^N w_i^n f(\eta_i). \quad (22)$$

Since

$$\int_{\mathbb{S}^2} f(\xi) d\omega(\xi) = \int_{\mathbb{S}^2} f_n(\xi) d\omega(\xi),$$

formula (22) reduces to

$$\int_{\mathbb{S}^2} f_n(\xi) d\omega(\xi) = \sum_{i=1}^N w_i^n (f_n(\eta_i) + g_n(\eta_i)).$$

But  $f_n \in V_n$  implies

$$\int_{\mathbb{S}^2} f_n(\xi) d\omega(\xi) = \sum_{i=1}^N w_i^n f_n(\eta_i),$$

so we have

$$\sum_{i=1}^N w_i^n g_n(\eta_i) = 0.$$

Further, using the fact that the weights corresponding to the points situated at the same latitude  $\rho_j$  are equal, we obtain

$$\sum_{j=1}^{n+1} \sum_{k=1}^{n+1} w_j \sum_{l=0}^{n+1} P_{n+1}^l(\cos \rho_j) (b_l \cos l\theta_k^j + c_l \sin l\theta_k^j) = 0,$$

for all  $b_l, c_l \in \mathbb{R}$ , which is equivalent to

$$\begin{aligned} \sum_{j=1}^{n+1} w_j \sum_{k=1}^{n+1} P_{n+1}^l(\cos \rho_j) \cos l\theta_k^j &= 0, \text{ for } l = 0, 1, \dots, n+1, \\ \sum_{j=1}^{n+1} w_j \sum_{k=1}^{n+1} P_{n+1}^l(\cos \rho_j) \sin l\theta_k^j &= 0, \text{ for } l = 1, \dots, n+1. \end{aligned} \quad (23)$$

For  $l = 0$  it means that

$$\sum_{j=1}^{n+1} w_j P_{n+1}^0(\cos \rho_j) = 0, \quad (24)$$

and thus conditions (21) are satisfied. For  $l = 1, 2, \dots, n+1$  we have

$$\sum_{j=1}^{n+1} w_j \sum_{k=1}^{n+1} P_{n+1}^l(\cos \rho_j) e^{il\theta_k^j} = 0. \quad (25)$$

Replacing now

$$\theta_k^j = \frac{\beta_j + 2k\pi}{n+1} \text{ with } \beta_j = \begin{cases} 0, & \text{if } j \text{ is odd,} \\ (\alpha - 2)\pi, & \text{if } j \text{ is even,} \end{cases}$$

we get, for  $l = 1, 2, \dots, n + 1$ ,

$$\sum_{j=1}^{n+1} w_j P_{n+1}^l(\cos \rho_j) \sum_{k=1}^{n+1} e^{il \frac{\beta_j + 2k\pi}{n+1}} = 0.$$

The sum  $\sum_{k=1}^{n+1}$  will be zero for all  $l$ , except  $l = n + 1$ , when it equals  $(n + 1)e^{i\beta_j}$ . So we have

$$\sum_{j=1}^{n+1} w_j (\sin \rho_j)^{n+1} e^{i\beta_j} = 0,$$

implying

$$\begin{aligned} \sin \alpha\pi \sum_{j=1, j \text{ even}}^{n+1} w_j (\sin \rho_j)^{n+1} &= 0, \\ \sum_{j=1, j \text{ odd}}^{n+1} w_j (\sin \rho_j)^{n+1} + \cos \alpha\pi \sum_{j=1, j \text{ even}}^{n+1} w_j (\sin \rho_j)^{n+1} &= 0. \end{aligned}$$

One can obtain positive weights only in the case  $\alpha = 1$ , implying further that

$$\sum_{j=1, j \text{ odd}}^{n+1} w_j (\sin \rho_j)^{n+1} = \sum_{j=1, j \text{ even}}^{n+1} w_j (\sin \rho_j)^{n+1}.$$

This condition is satisfied for our cubature formula, since we have proved in Section 3 that the weights corresponding to the points situated on symmetric latitudes are equal.

Conversely, if  $\alpha = 1$  and  $\sum_{j=1}^{n+1} w_j P_{n+1}(\cos \rho_j) = 0$ , then relations (24) and (25) are satisfied, implying further that the cubature formula is exact for all  $f \in V_{n+1}$ . ■

## References

- [1] BERNSTEIN, S. N., *On a system of undetermined equations*, in: Selected Papers, vol. 2: Constructive function theory, 236-242 (1956) (in Russian).
- [2] LAÍN FERNÁNDEZ, N., *Polynomial Bases on the Sphere*, Logos-Verlag, Berlin, 2003.
- [3] LAÍN FERNÁNDEZ, N., *Localized Polynomial Bases on the Sphere*, Electron. Trans. Numer. Anal., **19**, 84-93 (2005).
- [4] SZEGÖ, G., *Orthogonal Polynomials*, vol. 23 of Amer. Math. Society Colloq. Publ., Rhode Island, 1975.

- [5] XU, Y., *Quasi-orthogonal polynomials, quadrature and interpolation*, J. Math. Anal. Appl. **182**, 779-799 (1994).