Constr. Approx.

CONSTRUCTIVE APPROXIMATION

Polynomial Wavelets on the Interval

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Abstract. We investigate a polynomial wavelet decomposition of the $L^2(-1,1)$-space with Chebyshev-weight, where the wavelets fulfill certain interpolatory conditions. For this approach we obtain the two-scale relations and decomposition formulas. Dual functions and Riesz-stability will be discussed.

1. Introduction

The wavelet approach has been proved already as an efficient tool to analyze functions (compare [1], [4], [6]). Starting from considerations on $\mathbb{R}^n$ there are also well-known results on bounded intervals and in the periodic case. In particular, in the periodic case a wavelet analysis by trigonometric polynomials is obtained for the first time by C.K. Chui, H.N. Mhaskar [2]. A corresponding approach where the wavelets and scaling functions fulfill interpolatory conditions is outlined in [11] and [12] and is based on results of A.A. Privalov [13]. Here we use algebraic polynomials to obtain orthogonality with respect to the Chebyshev-weight between the wavelets and corresponding scaling functions. These underlying functions fulfill interpolatory conditions on certain Chebyshev-nodes. Thus they are bounded and localized. In fact, we use here the extrema of the Chebyshev polynomials T_{2^j} as nodes which guarantee the nesting property from one level to the next (see e.g. [7], [5]).

The organization of the paper is as follows. In Section 2 we introduce the necessary preliminaries and definitions. With Lemmas 2.1 and 2.2, we give a helpful representation of our scaling functions and wavelets which is in particular reasonable for efficient calculations. Here we also formulate our basic result, namely the orthogonality between the scaling functions and the wavelets from the same level. In Section 3 we consider algorithms for the basis transformations; i.e., we represent scaling functions and wavelets from the level j in the basis of scaling functions from the level $j+1$, named two-scale relations, and backwards, called decomposition.

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Dual scaling functions and wavelets are studied by C.K. Chui and J.Z. Wang in [3]. Here we obtain representations for the dual functions of our algebraic polynomial approach, which are again well-localized. In Section 5 we investigate the Riesz stability of our bases. Therefore we state Marcinkiewicz-Zygmund-type inequalities for weighted L^p-spaces. As an easy consequence we obtain error estimates for the interpolatory projection into the space of scaling functions using the concept of the best one-sided approximation by algebraic polynomials.

2. Scaling functions and wavelets

In the sequel we denote by Π_n the set of all polynomials of degree at most n. Furthermore let T_n and U_n be the Chebyshev polynomials of first and second kind, respectively; i.e.,

$$T_n(x) := \cos n \arccos x \quad \text{and} \quad U_{n-1}(x) := \frac{\sin(n \arccos x)}{\sin(\arccos x)}.$$

Important identities for our purposes are

$$U_{n-1}(x) = \frac{1}{n} T_n'(x)$$

and

$$U_{2n-1}(x) = 2U_{n-1}(x)T_n(x).$$

Furthermore we denote by ω_j the product

$$\omega_j(x) := (1-x^2)U_{2^j-1}(x) \quad , \qquad j = 0, 1, 2, \ldots.$$

The zeros of ω_j are

$$\cos \frac{k\pi}{2^j} \quad \text{for} \quad k = 0, \ldots, 2^j,$$

which are in fact nothing else but the extrema of T_{2^j}. Let us remark here that the zeros of ω_j are also zeros of ω_{j+1}. Now we are able to introduce the so-called scaling functions

$$\phi_{j,l}(x) := \frac{\omega_j(x)}{\omega_j'(\cos\frac{l\pi}{2^j})(x - \cos\frac{l\pi}{2^j})}, \quad l = 0, \ldots, 2^j,\ j = 0, 1, \ldots.$$

Easily we conclude the representation (see e.g. [14], p. 33)

$$\phi_{j,l}(x) = \frac{(1-x^2)U_{2^j-1}(x)}{2^j(-1)^{l+1}(x - \cos\frac{l\pi}{2^j})}\epsilon_{j,l}, \tag{2.1}$$

where, here and in the following,

$$\epsilon_{j,l} := \begin{cases} \frac{1}{2} \text{ if } l = 0 \text{ or } l = 2^j, \\ 1 \text{ if } l = 1, 2, \ldots, 2^j - 1. \end{cases}$$

Furthermore we have the interpolatory property

$$\phi_{j,l}(\cos\frac{k\pi}{2^j}) = \delta_{k,l} \quad , \qquad k,l = 0,\dots,2^j \tag{2.2}$$

and the equality $V_j = \Pi_{2^j}$, where V_j denotes, as usual, the space of scaling functions

$$V_j := \operatorname{span}\{\phi_{j,l} : l = 0,\dots,2^j\}.$$

Hence,

$$\dim V_j = 2^j + 1. \tag{2.3}$$

In the sequel we also need a somewhat different representation of the scaling functions $\phi_{j,l}$, which is in particular useful for the application of the fast Chebyshev transform to compute the interpolants (see also [5]).

Lemma 2.1. *The scaling functions can be written as*

$$\phi_{j,l}(x) = \frac{1}{2^{j-1}} \sum_{r=0}^{2^j} T_r(x) T_r(\cos\frac{l\pi}{2^j}) \epsilon_{j,r} \epsilon_{j,l}\,.$$

Proof. By the identity

$$(1-x^2) U_{2^j-1}(x) = T_{2^j-1}(x) - x T_{2^j}(x)\,,$$

one concludes from (2.1) that

$$\begin{aligned}
\phi_{j,l}(x) &= \frac{\epsilon_{j,l}}{2^j} \left\{ (-1)^l T_{2^j}(x) + \frac{(-1)^l \cos\frac{l\pi}{2^j} T_{2^j}(x) - (-1)^l T_{2^j-1}(x)}{x - \cos\frac{l\pi}{2^j}} \right\} \\
&= \frac{\epsilon_{j,l}}{2^j} \left\{ T_{2^j}(\cos\frac{l\pi}{2^j}) T_{2^j}(x) \right. \\
&= + \left. \frac{T_{2^j-1}(\cos\frac{l\pi}{2^j}) T_{2^j}(x) - T_{2^j}(\cos\frac{l\pi}{2^j}) T_{2^j-1}(x)}{x - \cos\frac{l\pi}{2^j}} \right\},
\end{aligned}$$

from which the assertion follows directly by the Christoffel-Darboux formula (see e.g. [14], p. 35). □

Now it is straightforward to define a corresponding interpolation operator L_j for given functions on $[-1,1]$ by the Lagrange formula

$$L_j f := \sum_{l=0}^{2^j} f(\cos\frac{l\pi}{2^j}) \phi_{j,l}$$

with the simple properties

$$L_j f(\cos\frac{l\pi}{2^j}) = f(\cos\frac{l\pi}{2^j}) \quad , \quad l = 0, \ldots, 2^j,$$

$$L_j f \in V_j$$

and

$$L_j f = f \quad \text{for all} \quad f \in V_j .$$

Let us continue with the following definition of our wavelets $\psi_{j,l}$ for $l = 0, \ldots, 2^j - 1$ and $j = 0, 1, 2, \ldots$, namely

$$\begin{aligned}\psi_{j,l}(x) &:= \frac{T_{2^j}(x)}{2^j(x - \cos\frac{(2l+1)\pi}{2^{j+1}})}\left(2\omega_j(x) - \omega_j(\cos\frac{(2l+1)\pi}{2^{j+1}})\right)\\ &= \frac{T_{2^j}(x)}{2^j(x - \cos\frac{(2l+1)\pi}{2^{j+1}})}\left(2(1-x^2)U_{2^j-1}(x) - (-1)^l \sin\frac{(2l+1)\pi}{2^{j+1}}\right).\end{aligned}$$

Again we have an interpolatory property for the wavelets, not at the original zeros of ω_j, but at the zeros of ω_{j+1} which are not zeros of ω_j; i.e.,

$$\psi_{j,l}(\cos\frac{(2k+1)\pi}{2^{j+1}}) = \delta_{k,l} \quad \text{for} \quad k, l = 0, \ldots, 2^j - 1 . \tag{2.4}$$

Hence, we have, for the wavelet space

$$W_j := \operatorname{span}\{\psi_{j,l} : l = 0, \ldots, 2^j - 1\} ,$$

the properties

$$\dim W_j = 2^j \tag{2.5}$$

and

$$W_j \subset \Pi_{2^{j+1}} . \tag{2.6}$$

Note that in this approach W_j and V_j do not have the same dimension. Before stating the fundamental orthogonality condition between the scaling functions and the wavelets with respect to the inner product

$$(f, g) := \int_{-1}^{1} f(x) g(x) \frac{dx}{\sqrt{1-x^2}} ,$$

let us derive a representation of the wavelets in terms of Chebyshev polynomials, which is also very useful for computational purposes.

Lemma 2.2. *The wavelets can be written as*

$$\psi_{j,k}(x) = \frac{1}{2^{j-1}} \sum_{r=2^j+1}^{2^{j+1}} T_r(x) T_r(\cos\frac{(2k+1)\pi}{2^{j+1}}) \epsilon_{j+1,r}\,.$$

Proof. The combination of the two Christoffel-Darboux formulas

$$2\sum_{r=0}^{2^j} T_r(x) T_r(\cos\frac{(2k+1)\pi}{2^{j+1}}) \epsilon_{j,r} = \frac{T_{2^j}(x) \sin\frac{(2k+1)\pi}{2^{j+1}} (-1)^{k+1}}{x - \cos\frac{(2k+1)\pi}{2^{j+1}}}$$

and

$$2\sum_{r=0}^{2^{j+1}} T_r(x) T_r(\cos\frac{(2k+1)\pi}{2^{j+1}}) \epsilon_{j+1,r}$$

$$= \frac{T_{2^{j+1}}(x)\cos\frac{(2k+1)\pi}{2^{j+1}} - T_{2^{j+1}+1}(x)}{x - \cos\frac{(2k+1)\pi}{2^{j+1}}} + T_{2^{j+1}}(x) = \frac{(1-x^2)U_{2^{j+1}-1}(x)}{x - \cos\frac{(2k+1)\pi}{2^{j+1}}}$$

yields the desired result. □

By Lemmas 2.1 and 2.2, the orthogonality of the scaling functions and wavelets from the same level now follows directly from the orthogonality of the Chebyshev polynomials.

Theorem 2.3. *For* $l = 0, \ldots, 2^j$ *and* $k = 0, \ldots, 2^j - 1$,

$$(\phi_{j,l}, \psi_{j,k}) = 0.$$

As an easy consequence of Theorem 2.3, (2.3) and (2.5)-(2.6), we deduce an orthogonal decomposition of the spaces V_j.

Corollary 2.4. *For all* $j = 0, 1, 2, \ldots$,

$$V_{j+1} = V_j \oplus W_j \tag{2.7}$$

and by defining W_{-1} *as* V_0 *we obtain*

$$\Pi_{2^{j+1}} = V_{j+1} = \bigoplus_{k=-1}^{j} W_k\,.$$

For further development we also need the following important relationship between function values of the scaling functions and the wavelets, which can be obtained from the following well-known quadrature formula of Lobatto-Markov type ([14], p. 50)

$$\int_{-1}^{1} p(x)\,\frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{n}\sum_{m=1}^{n-1} p(\cos\frac{m\pi}{n}) + \frac{\pi}{2n}(p(1) + p(-1)), \tag{2.8}$$

which holds for all polynomials $p \in \Pi_{2n-1}$. To apply (2.8) for $p \in \Pi_{2n}$ we write $p = p_{2n-1} + \gamma T_n^2$ and obtain

$$\int_{-1}^{1} p(x) \frac{dx}{\sqrt{1-x^2}} = \frac{\pi}{n} \sum_{m=1}^{n-1} p(\cos\frac{m\pi}{n}) + \frac{\pi}{2n}(p(1)+p(-1)) - \frac{\pi}{2}\gamma. \tag{2.9}$$

Lemma 2.5. *For $l = 0, \ldots, 2^j$ and $k = 0, \ldots, 2^j - 1$,*

$$\phi_{j,l}(\cos\frac{(2k+1)\pi}{2^{j+1}}) = -\psi_{j,k}(\cos\frac{l\pi}{2^j})\epsilon_{j,l}. \tag{2.10}$$

Proof. Using the quadrature formula (2.8) with $n = 2^{j+1}$ and the interpolatory conditions (2.2), (2.4) for the scaling functions and the wavelets, respectively, we get

$$\begin{aligned}\int_{-1}^{1} \phi_{j,l}(x)\psi_{j,k}(x) \frac{dx}{\sqrt{1-x^2}} &= \frac{\pi}{2^{j+1}} \sum_{m=0}^{2^{j+1}} \phi_{j,l}(\cos\frac{m\pi}{2^{j+1}})\psi_{j,k}(\cos\frac{m\pi}{2^{j+1}})\epsilon_{j+1,m} \\ &= \frac{\pi}{2^{j+1}}\phi_{j,l}(\cos\frac{(2k+1)\pi}{2^{j+1}}) + \frac{\pi}{2^{j+1}}\psi_{j,k}(\cos\frac{l\pi}{2^j})\epsilon_{j,l}.\end{aligned}$$

Now the proof is complete by using the orthogonality stated in Theorem 2.3. □

Note that Lemma 2.5 can also be used as a proof of Theorem 2.3 if one computes the corresponding function-values from the definitions of the scaling functions and wavelets. Trigonometric identities yield

$$\begin{aligned}\phi_{j,l}(\cos\frac{(2k+1)\pi}{2^{j+1}})\epsilon_{j,l}^{-1} &= -\psi_{j,k}(\cos\frac{l\pi}{2^j}) \\ &= \frac{(-1)^{k+l}\sin^2\frac{(2k+1)\pi}{2^{j+1}}}{2^{j+1}\sin\frac{(2k+2l+1)\pi}{2^{j+2}}\sin\frac{(2k-2l+1)\pi}{2^{j+2}}}.\end{aligned}$$

For completeness we repeat here the interpolatory conditions

$$\phi_{j,l}(\cos\frac{k\pi}{2^j}) = \psi_{j,k}(\cos\frac{(2l+1)\pi}{2^{j+1}}) = \delta_{k,l}.$$

3. Two-scale relations and decomposition

Considering (2.7), one is interested in representing the wavelets from W_j and the scaling functions from V_j in terms of the basis from V_{j+1}; i.e., determine the coefficients $c_{j,n,l}$ and $d_{j,n,l}$ in the so-called two-scale relations

$$\phi_{j,n} = \sum_{l=0}^{2^{j+1}} c_{j,n,l}\phi_{j+1,l}\ , \quad n = 0, \ldots, 2^j,$$

and

$$\psi_{j,n} = \sum_{l=0}^{2^{j+1}} d_{j,n,l}\phi_{j+1,l}\ , \quad n = 0, \ldots, 2^j - 1.$$

This can be easily done by applying the interpolation operator L_{j+1}. One obtains, by using the interpolation properties (2.2) and (2.4),

$$\begin{aligned}\phi_{j,n} = L_{j+1}\phi_{j,n} &= \sum_{l=0}^{2^{j+1}} \phi_{j,n}(\cos\frac{l\pi}{2^{j+1}})\phi_{j+1,l} \\ &= \phi_{j+1,2n} + \sum_{l=0}^{2^j-1} \phi_{j,n}(\cos\frac{(2l+1)\pi}{2^{j+1}})\phi_{j+1,2l+1}\end{aligned}$$

and

$$\begin{aligned}\psi_{j,n} = L_{j+1}\psi_{j,n} &= \sum_{l=0}^{2^{j+1}} \psi_{j,n}(\cos\frac{l\pi}{2^j})\phi_{j+1,l} \\ &= \phi_{j+1,2n+1} + \sum_{l=0}^{2^j} \psi_{j,n}(\cos\frac{l\pi}{2^j})\phi_{j+1,2l}.\end{aligned}$$

Let us illustrate the basis transformation now in matrix representation. Therefore we reorder the ϕ_{j+1} and take at first the $\phi_{j+1,2l}$ with even index and then the $\phi_{j+1,2l+1}$ with odd index. For further development, we shall denote the corresponding permutation matrix by $\mathbf{P}_j$.

Here we get as a matrix $\mathbf{C}_j$:

	$\phi_{j+1,0} \cdots \phi_{j+1,2l} \cdots \phi_{j+1,2^{j+1}}$	$\phi_{j+1,1} \cdots \phi_{j+1,2l+1} \cdots \phi_{j+1,2^{j+1}-1}$
$\phi_{j,0}$ . . . $\phi_{j,n}$ . . . $\phi_{j,2^j}$	$\delta_{n,l}$	$\phi_{j,n}(\cos\frac{(2l+1)\pi}{2^{j+1}})$
$\psi_{j,0}$ . . . $\psi_{j,n}$ . . . $\psi_{j,2^j-1}$	$\psi_{j,n}(\cos\frac{l\pi}{2^j})$	$\delta_{n,l}$

Denoting the identity matrix of dimension 2^j by $\mathbf{I}_{2^j}$ and defining the diagonal matrix

$$\mathbf{E}_j := (\delta_{k,l}\epsilon_{j,k})_{k,l=0,\ldots,2^j}$$

and the function value matrix

$$\mathbf{K}_j := \left(\psi_{j,n}(\cos\frac{l\pi}{2^j})\right)_{\substack{n=0,\ldots,2^j-1\\ l=0,\ldots,2^j}},$$

we can summarize the results. Note, that by Lemma 2.5 we can write $-\mathbf{E}_j\mathbf{K}_j^T$ for the right upper block of the matrix $\mathbf{C}_j$.

Theorem 3.1. *For the matrix of the two-scale relations it holds*

$$\mathbf{C}_j = \left(\begin{array}{c|c} \mathbf{I}_{2^j+1} & -\mathbf{E}_j\mathbf{K}_j^T \\ \hline \mathbf{K}_j & \mathbf{I}_{2^j} \end{array}\right).$$

Then the reconstruction algorithm for a sum

$$f = \sum_{l=0}^{2^j} f_{j,l}\phi_{j,l} + \sum_{l=0}^{2^j-1} g_{j,l}\psi_{j,l}$$

reads as follows

$$f = \sum_{l=0}^{2^{j+1}} f_{j+1,l}\phi_{j+1,l}$$

with

$$\begin{pmatrix} f_{j+1,0} \\ f_{j+1,1} \\ \cdot \\ \cdot \\ \cdot \\ f_{j+1,2^{j+1}} \end{pmatrix} = \mathbf{P}_j^T\mathbf{C}_j^T \begin{pmatrix} f_{j,0} \\ \vdots \\ f_{j,2^j} \\ g_{j,0} \\ \vdots \\ g_{j,2^j-1} \end{pmatrix}.$$

Now we are interested in the more interesting case of decomposing a given function from V_{j+1} into wavelets from W_j and scaling functions from V_j. Therefore we have to determine a and b in the equation

$$\phi_{j+1,n} = \sum_{l=0}^{2^j} a_{j,n,l}\phi_{j,l} + \sum_{l=0}^{2^j-1} b_{j,n,l}\psi_{j,l} \tag{3.11}$$

Then we can explain the inverse $\mathbf{C}_j^{-1}$ of the matrix $\mathbf{C}_j$ of the two-scale relations as the following basis transformation:

	$\phi_{j,0} \cdots \phi_{j,l} \cdots \phi_{j,2^j}$	$\psi_{j,0} \cdots \psi_{j,l} \cdots \psi_{j,2^j-1}$
$\phi_{j+1,0}$ $\vdots$ $\phi_{j+1,2n}$ $\vdots$ $\phi_{j+1,2^{j+1}}$	$a_{j,2n,l}$	$b_{j,2n,l}$
$\phi_{j+1,1}$ $\vdots$ $\phi_{j+1,2n+1}$ $\vdots$ $\psi_{j+1,2^{j+1}-1}$	$a_{j,2n+1,l}$	$b_{j,2n+1,l}$

Theorem 3.2. *For the decomposition matrix* $\mathbf{C}_j^{-1}$,

$$\mathbf{C}_j^{-1} = \frac{1}{2} \left(\begin{array}{c|c} \mathbf{I}_{2^{j+1}} + \mathbf{E}_j \mathbf{B}_j & \mathbf{E}_j \mathbf{K}_j^T \\ \hline -\mathbf{K}_j & \mathbf{I}_{2^j} \end{array} \right) \tag{3.12}$$

with

$$\mathbf{B}_j := \left(\frac{(-1)^{r-s}}{2^j} \right)_{r,s=0,\ldots,2^j} .$$

Then the decomposition algorithm for any

$$f = \sum_{l=0}^{2^{j+1}} f_{j+1,l} \phi_{j+1,l}$$

is given by

$$f = \sum_{l=0}^{2^j} f_{j,l} \phi_{j,l} + \sum_{l=0}^{2^j-1} g_{j,l} \psi_{j,l} \ ,$$

where

$$\begin{pmatrix} f_{j,0} \\ \vdots \\ f_{j,2^j} \\ g_{j,0} \\ \vdots \\ g_{j,2^j-1} \end{pmatrix} = (\mathbf{C}_j^{-1})^T \mathbf{P}_j \begin{pmatrix} f_{j+1,0} \\ f_{j+1,1} \\ \cdot \\ \cdot \\ \cdot \\ f_{j+1,2^{j+1}} \end{pmatrix}.$$

Proof. We have to show that (3.12) is the inverse of $\mathbf{C}_j$, which means that

$$\mathbf{I}_{2^{j+1}+1} = \frac{1}{2}\left(\begin{array}{c|c} \mathbf{I}_{2^j+1} + \mathbf{E}_j\mathbf{B}_j + \mathbf{E}_j\mathbf{K}_j^T\mathbf{K}_j & \mathbf{E}_j\mathbf{K}_j^T - \mathbf{E}_j\mathbf{K}_j^T \\ \hline \mathbf{K}_j + \mathbf{K}_j\mathbf{E}_j\mathbf{B}_j - \mathbf{K}_j & \mathbf{K}_j\mathbf{E}_j\mathbf{K}_j^T + \mathbf{I}_{2^j} \end{array}\right).$$

This is equivalent to

$$\mathbf{K}_j\mathbf{E}_j\mathbf{B}_j = \mathbf{0} \tag{3.13}$$

$$\mathbf{K}_j\mathbf{E}_j\mathbf{K}_j^T = \mathbf{I}_{2^j} \tag{3.14}$$

$$\mathbf{E}_j\mathbf{K}_j^T\mathbf{K}_j = \mathbf{I}_{2^j+1} - \mathbf{E}_j\mathbf{B}_j, \tag{3.15}$$

where (3.13) follows immediately from (3.14) and (3.15). To prove (3.14) and (3.15), we use Lemmas 2.1 and 2.5. For the first assertion, we obtain

$$\begin{aligned} \mathbf{K}_j\mathbf{E}_j\mathbf{K}_j^T &= \left(-\sum_{t=0}^{2^j} \psi_{j,n}(\cos\frac{t\pi}{2^j})\phi_{j,t}(\cos\frac{(2l+1)\pi}{2^{j+1}})\right)_{n,l=0,\ldots,2^j-1} \\ &= \left(\sum_{t=0}^{2^j} \phi_{j,t}(\cos\frac{(2n+1)\pi}{2^{j+1}})\phi_{j,t}(\cos\frac{(2l+1)\pi}{2^{j+1}})\epsilon_{j,t}^{-1}\right)_{n,l=0,\ldots,2^j-1} \\ &= \left(2^{2-2j}\sum_{t=0}^{2^j}\sum_{r=0}^{2^j} \epsilon_{j,r}\epsilon_{j,t}T_r(\cos\frac{(2n+1)\pi}{2^{j+1}})T_r(\cos\frac{t\pi}{2^j})\right. \\ &\qquad \left.\sum_{s=0}^{2^j} \epsilon_{j,s}T_s(\cos\frac{(2l+1)\pi}{2^{j+1}})T_s(\cos\frac{t\pi}{2^j})\right)_{n,l=0,\ldots,2^j-1} \\ &= \left(2^{2-2j}\sum_{r=0}^{2^j}\sum_{s=0}^{2^j} \epsilon_{j,r}\epsilon_{j,s}T_r(\cos\frac{(2n+1)\pi}{2^{j+1}})T_s(\cos\frac{(2l+1)\pi}{2^{j+1}})\right. \\ &\qquad \left.\sum_{t=0}^{2^j} T_r(\cos\frac{t\pi}{2^j})T_s(\cos\frac{t\pi}{2^j})\epsilon_{j,t}\right)_{n,l=0,\ldots,2^j-1}. \end{aligned}$$

Now the last sum with respect to t can be easily computed by using the quadrature formula (2.8) and the orthogonality of the Chebyshev polynomials, namely

$$\sum_{t=0}^{2^j} T_r(\cos\frac{t\pi}{2^j})T_s(\cos\frac{t\pi}{2^j})\epsilon_{j,t} = 2^{j-1}\delta_{r,s}\epsilon_{j,r}^{-1}.$$

By the Christoffel-Darboux formula and its limit case

$$\sum_{r=0}^{2^j} T_r^2(x)\epsilon_{j,r} = \frac{1}{4}(U_{2^{j+1}}(x) - 2T_{2^j}^2(x) + 2^{j+1} + 1) \tag{3.16}$$

we conclude that

$$\begin{aligned}
\mathbf{K}_j\mathbf{E}_j\mathbf{K}_j^T &= \left(2^{1-j}\sum_{r=0}^{2^j}\epsilon_{j,r}T_r(\cos\frac{(2n+1)\pi}{2^{j+1}})T_r(\cos\frac{(2l+1)\pi}{2^{j+1}})\right)_{n,l=0,\dots,2^j-1} \\
&= \left(2^{1-j}\frac{\delta_{n,l}}{4}\left(U_{2^{j+1}}(\cos\frac{(2n+1)\pi}{2^{j+1}}) + 2^{j+1} + 1\right)\right)_{n,l=0,\dots,2^j-1} \\
&= (\delta_{n,l})_{n,l=0,\dots,2^j-1} \; .
\end{aligned}$$

Analogously, we continue with

$$\begin{aligned}
\mathbf{E}_j\mathbf{K}_j^T\mathbf{K}_j &= \left(-\sum_{t=0}^{2^j-1}\phi_{j,n}(\cos\frac{(2t+1)\pi}{2^{j+1}})\psi_{j,t}(\cos\frac{l\pi}{2^j})\right)_{n,l=0,\dots,2^j} \\
&= \left(\sum_{t=0}^{2^j-1}\phi_{j,n}(\cos\frac{(2t+1)\pi}{2^{j+1}})\phi_{j,l}(\cos\frac{(2t+1)\pi}{2^{j+1}})\epsilon_{j,l}^{-1}\right)_{n,l=0,\dots,2^j} \\
&= \Bigg(2^{2-2j}\sum_{t=0}^{2^j-1}\sum_{r=0}^{2^j}\epsilon_{j,r}\epsilon_{j,n}T_r(\cos\frac{(2t+1)\pi}{2^{j+1}})T_r(\cos\frac{n\pi}{2^j}) \\
&\qquad \sum_{s=0}^{2^j}\epsilon_{j,s}T_s(\cos\frac{(2t+1)\pi}{2^{j+1}})T_s(\cos\frac{l\pi}{2^j})\Bigg)_{n,l=0,\dots,2^j} \\
&= \Bigg(2^{2-2j}\epsilon_{j,n}\sum_{r=0}^{2^j}\sum_{s=0}^{2^j}\epsilon_{j,r}\epsilon_{j,s}T_r(\cos\frac{n\pi}{2^j})T_s(\cos\frac{l\pi}{2^j}) \\
&\qquad \sum_{t=0}^{2^j-1}T_r(\cos\frac{(2t+1)\pi}{2^{j+1}})T_s(\cos\frac{(2t+1)\pi}{2^{j+1}})\Bigg)_{n,l=0,\dots,2^j} \; .
\end{aligned}$$

Using the quadrature rule (2.8) twice, with 2^j and 2^{j+1} nodes, respectively, (see also [14], p. 49), namely:

$$\sum_{t=0}^{2^j-1} T_r(\cos\frac{(2t+1)\pi}{2^{j+1}})T_s(\cos\frac{(2t+1)\pi}{2^{j+1}}) = 2^{j-1}\epsilon_{j,r}^{-1}\delta_{r,s}(1-\delta_{r,2^j}),$$

we obtain, again by the Christoffel-Darboux formula and (3.16),

$$\begin{aligned}\mathbf{E}_j\mathbf{K}_j^T\mathbf{K}_j &= \left(\epsilon_{j,n}2^{1-j}\sum_{r=0}^{2^j-1}\epsilon_{j,r}T_r(\cos\frac{l\pi}{2^j})T_r(\cos\frac{n\pi}{2^j})\right)_{n,l=0,\ldots,2^j}\\ &= \left(\delta_{l,n}-\epsilon_{j,n}\frac{(-1)^{l+n}}{2^j}\right)_{n,l=0,\ldots,2^j},\end{aligned}$$

which proves the theorem. □

4. Dual scaling functions and wavelets

With the results for decomposition and reconstruction we are able to construct the dual functions.

For $l=0,\ldots,2^j$, the dual scaling functions $\tilde{\phi}_{j,l}\in V_j$ are uniquely defined by

$$(\phi_{j,k},\tilde{\phi}_{j,l})=\delta_{k,l}\qquad\text{for all}\quad k=0,\ldots,2^j.$$

Analogously, for $l=0,\ldots,2^j-1$, the dual wavelets $\tilde{\psi}_{j,l}\in W_j$ are uniquely defined by

$$(\psi_{j,k},\tilde{\psi}_{j,l})=\delta_{k,l}\qquad\text{for all}\quad k=0,\ldots,2^j-1.$$

Hence the dual functions are characterized by their coefficients with respect to the original basis in V_j and W_j, respectively.

Theorem 4.1. *For the dual scaling functions $\tilde{\phi}_{j,l}$, $l=0,\ldots,2^j$,*

$$\tilde{\phi}_{j,l}=\sum_{t=0}^{2^j}\alpha_{j,l,t}\phi_{j,t}$$

with

$$\alpha_{j,l,t}:=\frac{2^j}{\pi}\delta_{l,t}\epsilon_{j,t}^{-1}+\frac{(-1)^{l-t}}{\pi}.$$

Analogously, for the dual wavelets $\tilde{\psi}_{j,l}$, $l=0,\ldots,2^j-1$,

$$\tilde{\psi}_{j,l}=\sum_{t=0}^{2^j-1}\beta_{j,l,t}\psi_{j,t}$$

with

$$\beta_{j,l,t}:=\frac{2^j}{\pi}\delta_{l,t}+\frac{1}{\pi}.$$

Proof. At first we remark that from the definition of the dual functions and from the decomposition formula (3.11), we have

$$\sum_{t=0}^{2^j} \alpha_{j,l,t}(\phi_{j+1,2s}, \phi_{j,t}) = (\phi_{j+1,2s}, \tilde{\phi}_{j,l})$$
$$= \sum_{r=0}^{2^j} a_{j,2s,r}(\phi_{j,r}, \tilde{\phi}_{j,l}) + \sum_{r=0}^{2^j-1} b_{j,2s,r}(\psi_{j,r}, \tilde{\phi}_{j,l})$$
$$= a_{j,2s,l}$$

and

$$\sum_{t=0}^{2^j-1} \beta_{j,l,t}(\phi_{j+1,2s+1}, \psi_{j,t}) = (\phi_{j+1,2s+1}, \tilde{\psi}_{j,l})$$
$$= \sum_{r=0}^{2^j} a_{j,2s+1,r}(\phi_{j,r}, \tilde{\psi}_{j,l}) + \sum_{r=0}^{2^j-1} b_{j,2s+1,r}(\psi_{j,r}, \tilde{\psi}_{j,l})$$
$$= b_{j,2s+1,l} \, .$$

Now we compute the inner products on the lefthand-side by the quadrature rule (2.8) and (2.9), respectively, namely:

$$(\phi_{j+1,2s}, \phi_{j,t}) = \frac{\pi}{2^{j+1}} \sum_{r=0}^{2^{j+1}} \phi_{j+1,2s}(\cos \frac{r\pi}{2^{j+1}}) \phi_{j,t}(\cos \frac{r\pi}{2^{j+1}}) \epsilon_{j+1,r}$$
$$= \frac{\pi}{2^{j+1}} \delta_{s,t} \epsilon_{j,s}$$

and

$$(\phi_{j+1,2s+1}, \psi_{j,t}) = \frac{\pi}{2^{j+1}} \sum_{r=0}^{2^{j+1}} \phi_{j+1,2s+1}(\cos \frac{r\pi}{2^{j+1}}) \psi_{j,t}(\cos \frac{r\pi}{2^{j+1}}) \epsilon_{j+1,r} - \frac{\pi}{2} 2^{-2j}$$
$$= \frac{\pi}{2^{j+1}} \delta_{s,t} - \frac{\pi}{2^{2j+2}} \, .$$

Denoting

$$\mathbf{G}_j := (\alpha_{j,l,t})_{l,t=0,\ldots,2^j} \quad \text{and} \quad \mathbf{H}_j := (\beta_{j,l,t})_{l,t=0,\ldots,2^j-1} \, ,$$

we can summarize

$$\mathbf{G}_j(\frac{\pi}{2^{j+1}} \mathbf{I}_{2^j+1} \mathbf{E}_j) = \frac{1}{2}(\mathbf{I}_{2^j+1} + \mathbf{E}_j \mathbf{B}_j)$$

and

$$\mathbf{H}_j(\frac{\pi}{2^{j+1}} \mathbf{I}_{2^j} - \frac{\pi}{2^{2j+2}} \mathbf{J}_j) = \frac{1}{2} \mathbf{I}_{2^j},$$

where $\mathbf{J}_j$ is the $2^j \times 2^j$ matrix with constant entries 1. Simple matrix operations yield

$$\mathbf{G}_j = \frac{2^j}{\pi}(\mathbf{E}_j^{-1} + \mathbf{B}_j)$$

and

$$\mathbf{H}_j = \frac{2^j}{\pi}\mathbf{I}_{2^j} + \frac{1}{\pi}\mathbf{J}_j,$$

which proves the Theorem. □

5. Riesz stability and error estimates

In a recent paper, G. Min [7] considered the Lagrange interpolation based on the zeros of

$$(1 - x^2)T_n(x)T_n'(x) \tag{5.17}$$

and proved the weighted L^p-convergence for all continuous functions but without any quantitative knowledge of the error with respect to the smoothness of the underlying function. Note that this is also discussed by P. Nevai ([9], Theorem 6) and the first named author [10].

Here we generalize this result to obtain weighted L^p-convergence for all Riemann integrable functions. Furthermore we describe the error in terms of the best one-sided approximation in the weighted L^p-space. To agree with our former definitions we formulate the results only for the interpolation operators L_j, built from a dyadic nesting of nodes. The case of general n in (5.17) can be handled analogously.

As an essential step we need an Riesz stability argument for the scaling function basis in V_j. In our interpolatory approach this is nothing else than an Marcinkiewicz-Zygmund type inequality for polynomials because the point evaluations of a scaling function are nothing else than the coefficients in the basis representation. Therefore let us define, for $1 \le p < \infty$,

$$||g||_p := \left(\int_{-1}^{1} |g(x)|^p \frac{dx}{\sqrt{1-x^2}} \right)^{\frac{1}{p}}$$

and

$$||g||_{p,j} := \left(2^{-j} \sum_{k=0}^{2^j} \left| g(\cos \frac{k\pi}{2^j}) \right|^p \right)^{\frac{1}{p}}.$$

Theorem 5.1. *Let $g \in V_j$ and $1 < p < \infty$. Then there exist positive constants c_p and C_p, depending only on p, such that*

$$c_p||g||_{p,j} \le ||g||_p \le C_p||g||_{p,j}. \tag{5.18}$$

Proof. The assertion can be obtained from the corresponding trigonometric inequalities. Namely, we have for all trigonometric polynomials h of the form

$$h(t) = a_0 + \sum_{k=0}^{r-1} (a_k \cos kt + b_k \sin kt) + a_r \cos rt$$

the inequalities

$$c'_p \int_0^{2\pi} |h(t)|^p \, dt \le \frac{1}{2r} \sum_{k=0}^{2r-1} |h(\frac{k\pi}{r})|^p \le C'_p \int_0^{2\pi} |h(t)|^p \, dt \tag{5.19}$$

with certain positive constants c'_p, C'_p, depending only on p, $1 < p < \infty$. This can be deduced from [17], Theorem X.7.5, see also [16]. Now we only have to use the transformation $x = \cos t$ and the identity

$$\cos \frac{(2^{j+1} - k)\pi}{2^j} = \cos \frac{k\pi}{2^j} \quad \text{for all} \quad k = 1, \ldots, 2^j - 1$$

to obtain the equivalence between (5.18) and (5.19) for even h. □

With this fundamental equivalence result (5.18), we are able to state the convergence estimates. One of the main tools here is the best one-sided approximation, defined by

$$\tilde{E}_n(f, L^p) := \inf \{ ||Q_n - q_n||_p \ : \ q_n, Q_n \in \Pi_n \text{ with}$$

$$q_n(x) \le f(x) \le Q_n(x) \text{ for all } x \in [-1, 1] \} .$$

Theorem 5.2. *Let f be a given bounded and measurable function on $[-1, 1]$. Then for all $1 < p < \infty$,*

$$||f - L_j f||_p \le (1 + \frac{C_p}{c_p}) \tilde{E}_{2^j}(f, L^p) .$$

Proof. Let

$$\tilde{E}_{2^j}(f, L^p) = ||Q_{2^j} - q_{2^j}||_p .$$

Then, by applying (5.18) twice, we obtain

$$\begin{aligned} ||f - L_j f||_p &\le ||f - Q_{2^j}||_p + ||Q_{2^j} - L_j f||_p \\ &\le \tilde{E}_{2^j}(f, L^p) + C_p ||Q_{2^j} - L_j f||_{p,j} \\ &\le \tilde{E}_{2^j}(f, L^p) + C_p ||Q_{2^j} - q_{2^j}||_{p,j} \\ &\le (1 + \frac{C_p}{c_p}) \tilde{E}_{2^j}(f, L^p) \qquad . \end{aligned}$$

□

From Theorem 5.2 we can easily obtain convergence results for certain smoothness classes of functions f by the corresponding results for the best one-sided approximation (compare e.g. [8],[15]).

Corollary 5.3. *Let* $1 < p < \infty$ *and* $s = 0, 1, 2, \ldots$ *. Then for Riemann integrable* f*,*

$$\lim_{j \to \infty} \|f - L_j f\|_p = 0$$

and for $f^{(s)}$ *of bounded variation in* $[-1, 1]$*,*

$$\|f - L_j f\|_p \le C 2^{-\frac{sj}{p}} V(f^{(s)}).$$

6. Appendix

Let us illustrate with some examples that the special choice of Chebyshev nodes avoids oscillations near the boundary ± 1. At first we give some graphs of the scaling functions and wavelets.

Fig. 1. Graphs of $\phi_{6,32}$ and $\phi_{6,48}$. Graphs of $\psi_{6,32}$ and $\psi_{6,48}$.

At the end we decompose a cubic B-spline s with interior nodes at 0 and ± 0.5 into its wavelet parts. Therefore we interpolate s at 1025 points to obtain $L_{10}s$. Then we use our algorithms to compute the orthogonal projections $P_j^W(L_{10}s)$ into the wavelet spaces, which detect the discontinuities of the third derivative

Fig. 2. Graph of $P_j^V(L_{10}s), j = 4,3,2$. Graphs of $P_j^W(L_{10}s)$, $j = 9,8,3$.

of s. Furthermore we include graphs of some orthogonal projections $P_j^V(L_{10}s)$ into V_j.

Because of the very small maximum value of the wavelet parts one cannot see any difference in the graphs of s and $P_j^V(L_{10}s), \quad j = 4, 5, \ldots, 10$.

The local behaviour of the wavelet part in Figure 2 follows from localization properties of the error of the orthogonal and the interpolatory projection into V_j, namely $s - P_j^V s$ and $s - L_j s$. However, the quantitative description how the wavelet coefficients rely on local smoothness properties of the underlying function s remains open.

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