Simultaneous Approximation by Bernstein Operators in Hölder Norms

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In this paper we discuss approximation of continuous functions f on $[0,1]$ in Hölder norms including simultaneous approximation of derivatives of f.

1 Introduction

This article deals with special approximation properties of the classical Bernstein operators

$$B_n(f,x) := \sum_{k=0}^{n} f(\frac{k}{n}) \cdot \binom{n}{k} x^k (1-x)^{n-k}.$$

As is well known, these constitute an approximation process giving for any function $f \in C[0,1]$ — the space of continuous real-valued function on the compact interval $[0,1]$ — a sequence of algebraic polynomials $B_n f$ which approximate f arbitrarily well with respect to the sup norm $\|\cdot\|$. The latter symbol will denote this norm throughout our present note.

Here it is our aim to discuss the rate of approximation of Hölder continuous functions by Bernstein operators, measured by Hölder norms with different exponents and orders of derivatives of the function to be approximated. Generally the class $\text{Lip}_L\alpha$ of *Hölder (Lipschitz) continuous* [1] functions on $[0,1]$ with exponent α for some $0 < \alpha \le 1$ and constant L consists of all functions which obey the inequality

$$|f(x) - f(y)| \le L|x-y|^\alpha, \qquad x, y \in [0,1]. \tag{1.1}$$

The first known result concerning approximation of a function $f \in \text{Lip}_L\alpha$ by the Bernstein operator was proved by Kac ([17], [18]) in 1938 and later reproved by Mathé in [22] in 1999 in a different way. This result is the following

Theorem A *If the function $f : [0,1] \to \mathbb{R}$ is Hölder continuous with exponent α and constant L, then*

$$|f(x) - B_n(f,x)| \le L \left(\frac{x(1-x)}{n} \right)^{\frac{\alpha}{2}}, \quad n \in \mathbb{N}.$$

A result on smoothness preservation by the Bernstein operator was given in 1965 by Hajek in [14].

Theorem B *Let $f \in Lip_L 1$. Then $B_n f \in Lip_L 1$.*

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[1] Otto Ludwig Hölder, 1859 (Stuttgart) – 1937 (Leipzig); Rudolf Otto Sigismund Lipschitz, 1832 (Königsberg) – 1903 (Bonn).

This result was later generalized by Lindvall [20] and Brown, Elliott, Paget in [3] who showed that the same statement is also true if $\mathrm{Lip}_L 1$ is replaced by $\mathrm{Lip}_L \alpha, 0 < \alpha \leq 1$. This means that if global smoothness of a function $f \in C[0,1]$ is expressed by stating that it satisfies a certain Lipschitz condition, then the same is true for its approximant $B_n f$. Theorem B was generalized in 1991 by Anastassiou, Cottin and Gonska [1] in

Theorem C *For the Bernstein operators B_n one has for all $C[0,1]$ and $\delta \geq 0$*

$$\omega_1(B_n f;\delta) \leq 1 \cdot \tilde{\omega}_1(f;\delta) \leq 2 \cdot \omega_1(f,\delta).$$

Here $\tilde{\omega}_1(f;\cdot)$ denotes the least concave majorant of the modulus of continuity $\omega_1(f;\cdot)$, given by

$$\omega_1(f;\delta) = \sup\{|f(x) - f(y)| \ : \quad |x-y| \leq \delta\}.$$

The constants 1 and 2 are best possible.

We introduce the following two seminorms: For each $0 < \delta \leq 1$ put

$$\Theta_\alpha(f,\delta) := \sup_{x,y\in[0,1],0<|x-y|\leq\delta} \frac{|f(x)-f(y)|}{|x-y|^\alpha}$$

and

$$\Theta_\alpha(f) := \sup_{0<\delta\leq 1} \Theta_\alpha(f,\delta). \tag{1.2}$$

It is clear that the function $f : [0,1] \to \mathbb{R}$ satisfies a Hölder (Lipschitz) condition of order $\alpha, \alpha \in (0,1]$, if $\Theta_\alpha(f) < \infty$. It is easy to verify that in this case L in (1.1) is chosen to be $\Theta_\alpha(f)$ or any upper bound of the latter.

Given the classical first-order modulus of continuity it is easy to prove that, for $0 < \delta \leq 1$, one has

$$\Theta_\alpha(f,\delta) = \sup_{0<t\leq\delta} \frac{\omega_1(f;t)}{t^\alpha}. \tag{1.3}$$

In the sequel we shall consider the spaces $\mathrm{Lip}_\alpha[0,1] := \{f \in C[0,1] : \Theta_\alpha(f) < \infty\}$ and $\mathrm{lip}_\alpha([0,1]) := \{f \in \mathrm{Lip}_\alpha[0,1] : \Theta_\alpha(f,\delta) \to 0$ as $\delta \to 0\}$.

The space $\mathrm{Lip}_\alpha[0,1]$ is complete under the so-called Hölder norm

$$\|\cdot\|_{0,\alpha} := \|\cdot\| + \Theta_\alpha(\cdot). \tag{1.4}$$

In 2000 it was proved by Bustamante and Jiménez-Pozo [8] that the following holds.

Theorem D *For all $f \in lip_\alpha[0,1]$,*

$$\|B_n f - f\|_{0,\alpha} \to 0 \ \text{ as } \ n \to \infty.$$

This result was extended by Cárdenas-Morales, Jiménez-Pozo and Muñoz-Delgado in 2006 [11] who proved

Theorem E *There exists a constant K such that, for every $f \in lip_\alpha[0,1]$*

$$\|B_n f - f\|_{0,\alpha} \leq K\Theta_\alpha(f, n^{-1/2}).$$

In 2006 Bustamante and Roldan [10] modified these results by using the second order Ditzian-Totik modulus of smoothness, but we do not deal with this question here. We note, however, that the two authors mentioned also established inverse results.

Not only the approximation by (special) linear positive operators, but also by trigonometric polynomials in Hölder (or Lipschitz) norm is mostly recent. Reasons of space do not allow us to summarize related results. Instead, we only recall articles of Kalandiya [19], Ioakimidis [15], [16], Prestin (see [24], [25], [26], [27]), Bustamante (see [4], [5], [6], [7], [8], [9], [10]) and their co-authors.

A survey of the main known results related to quantitative polynomial approximation in Hölder norms can be found in [9].

To formulate our main result we introduce the following notation: For $m \in \mathbb{N}_0, 0 \le \alpha \le 1$, define

$$C^{m,\alpha}[0,1] := \left\{ f \in C^m[0,1] : \|f\|_{m,\alpha} := \sum_{k=0}^{m} \|f^{(k)}\| + {}^1|f^{(m)}|_\alpha < \infty \right\},$$

where the seminorm ${}^1|\cdot|_\alpha$ is given by

$$^1|g|_\alpha := \sup_{h>0} h^{-\alpha} \cdot \max_{x\in[0,1-h]} |g(x+h) - g(x)|. \tag{1.5}$$

The standard example for a function $g \in C^{0,\alpha}[0,1]$ is $g(x) = x^\beta$ with $\beta \ge \alpha$. Then, ${}^1|g|_\alpha = 1$ for $\beta \le 1$ and, e.g., ${}^1|g|_\alpha = 2^{2-\alpha}(1-\alpha)^{1-\alpha}(2-\alpha)^{\alpha-2}$ for $\beta = 2$. In the next section we prove that

$$^1|f|_\alpha = \Theta_\alpha(f).$$

It is also possible to define a seminorm equivalent to ${}^1|f|_\alpha$ via a corresponding K-functional. All these three definitions will be used in the proof of our main result. Further we define

$$\begin{aligned} \tilde{C}^{m,\alpha} &:= \left\{ g \in C^{m,\alpha} : \lim_{h\to 0+} h^{-\alpha} \max_{x\in[0,1-h]} |g^{(m)}(x+h) - g^{(m)}(x)| = 0 \right\} \\ &= \left\{ g \in C^{m,\alpha} : \omega_1(g^{(m)};\delta) = o(\delta^\alpha), \delta \to 0+ \right\}. \end{aligned}$$

Our main result is the following

Theorem 1 *Let $r, m \in \mathbb{N}_0, 0 \le \alpha, \beta \le 1, r \le m$, $r + \beta \le m + \alpha$. Then for $f \in C^{m,\alpha}$ and $n > m+1$ one has*

$$\|B_n f - f\|_{r,\beta} \le C_r \cdot (n-r-1)^{\max\left\{-1, \frac{r+\beta-m-\alpha}{2}\right\}} \cdot \|f\|_{m,\alpha}. \tag{1.6}$$

Here C_r is a constant depending only on r.

An immediate corollary for the case $r = m$ and $\alpha = \beta$ is

Corollary 1 *(Cottin and Gonska* [12]*) For $f \in C^{r,\alpha}[0,1], r \in \mathbb{N}_0, 0 \le \alpha \le 1$ we have*

$$\|B_n f - f\|_{r,\alpha} \le C_r \|f\|_{r,\alpha}. \tag{1.7}$$

In Section 2 we give auxiliary results. In Section 3 we give the proof of Theorem 1.

2 Auxiliary Results

In the proof of our theorem we use the following seminorm:

$$^2|g|_\alpha := \sup_{h>0} h^{-\alpha} \cdot \omega_1(g,h), \quad \text{for } 0 \le \alpha \le 1. \tag{2.1}$$

Moreover we have

Lemma 1 *For all $f \in C^{0,\alpha}[0,1]$ one has*

$$^1|g|_\alpha = {}^2|g|_\alpha = \Theta_\alpha(f).$$

Proof: It is clear that

$$^1|f|_\alpha \le {}^2|f|_\alpha.$$

Further, for some $0 < t \le h$ we have

$$h^{-\alpha} \cdot \omega_1(f,h) = h^{-\alpha} \cdot \max_{x\in[0,1-t]} |f(x+t) - f(x)| \le t^{-\alpha} \cdot \max_{x\in[0,1-t]} |f(x+t) - f(x)|. \tag{2.2}$$

We take $\sup\limits_{h>0}$ on the left side of (2.2) and after this take $\sup\limits_{t>0}$ on the right side to obtain

$$^2|f|_\alpha \le {}^1|f|_\alpha.$$

Next we establish

$$^2|f|_\alpha = \Theta_\alpha(f).$$

We have, for some $t \in (0,1]$,

$$\begin{aligned} ^2|f|_\alpha &= t^{-\alpha} \cdot \omega_1(f,t) \\ &= t^{-\alpha} \cdot \sup_{0<|x-y|\le t} |f(y) - f(x)| \\ &= t^{-\alpha} \cdot |f(y_0) - f(x_0)| \qquad (\text{ for some } x_0, y_0 \in [0,1],\ 0 < |x_0 - y_0| \le t) \\ &\le \frac{|f(y_0) - f(x_0)|}{|y_0 - x_0|^\alpha} \\ &\le \sup_{0<|x-y|\le t} \frac{|f(y) - f(x)|}{|y-x|^\alpha} \\ &\le \sup_{t>0} \sup_{x,y\in[0,1],\,0<|x-y|\le t} \frac{|f(y) - f(x)|}{|y-x|^\alpha}. \end{aligned}$$

Therefore

$$^2|g|_\alpha \le \Theta_\alpha(f).$$

Further we have

$$\Theta_\alpha(f,h) = \sup_{x,y\in[0,1],0<|x-y|\le h} \frac{|f(x) - f(y)|}{|x-y|^\alpha} = \frac{|f(x_0) - f(y_0)|}{|x_0 - y_0|^\alpha}, \qquad \text{for some } |x_0 - y_0| \le h.$$

Consequently

$$\begin{aligned} \Theta_\alpha(f) &= \sup_{h>0} \Theta_\alpha(f,h) \\ &\le \sup_{h>0} \frac{\omega_1(f;|x_0-y_0|)}{|x_0-y_0|^\alpha} \qquad \text{with } |x_0 - y_0| \le h \\ &\le \sup_{h>0} \frac{\omega_1(f;h)}{h^\alpha} = {}^2|f|_\alpha. \end{aligned}$$

The proof of Lemma 1 is complete. □

Due to the equivalence between $\omega_1(f,h)$ and the K-functional below we also use the following seminorm

$$^3|g|_\alpha := \sup_{h>0} h^{-\alpha} \cdot K(g,h;C,C^1), \tag{2.3}$$

where

$$K(g,h) := K(g,h;C,C^1) := \inf_{G\in C^1[0,1]} \{\|g - G\| + h\|G'\|\}.$$

It follows that

$$^1|g|_\alpha \sim {}^3|g|_\alpha.$$

We also need the following three well-known lemmas which are, in fact, theorems of the authors in parentheses.

Lemma 2 *(Popoviciu* [23]*) For $f \in C[0,1]$ we have*

$$\|f - B_n f\| \le \frac{5}{4} \cdot \omega_1\left(f, \frac{1}{\sqrt{n}}\right). \tag{2.4}$$

Lemma 3 *(Lorentz* [21]*) For $f \in C^1[0,1]$ we have*

$$\|f - B_n f\| \le \frac{3}{4} \cdot \frac{1}{\sqrt{n}} \cdot \omega_1\left(f', \frac{1}{\sqrt{n}}\right). \tag{2.5}$$

Lemma 4 *(Brown, Elliot, Paget* [3]*) For $f \in C^{0,\alpha}[0,1]$ we have*

$$\max_{x\in[0,1-h]} |\Delta_h B_n f(x)] \le h^\alpha \cdot {}^1|f|_\alpha. \tag{2.6}$$

We will also need the following two auxiliary results.

Lemma 5 *For $f \in C^k[0,1]$ and $k = 1, 2, \ldots, n-1$ one has*

$$\|(f - B_n f)^{(k)}\| \le \|f^{(k)} - B_{n-k}(f^{(k)})\| + \min\{1, \frac{(k-1)^2}{n}\}\|f^{(k)}\| + \omega_1\left(f^{(k)}; \frac{k}{n}\right). \tag{2.7}$$

Proof: It is known that

$$B_n^{(k)}(f,x) = n(n-1)\ldots(n-k+1) \cdot \sum_{\nu=0}^{n-k} \Delta^k f\left(\frac{\nu}{n}\right) \cdot \binom{n-k}{\nu} \cdot x^\nu \cdot (1-x)^{n-\nu-k} \tag{2.8}$$

with

$$\Delta f\left(\frac{\nu}{n}\right) = f\left(\frac{\nu+1}{n}\right) - f\left(\frac{\nu}{n}\right).$$

By the mean value theorem we obtain

$$B_n^{(k)}(f,x) = \prod_{i=1}^{k-1}(1 - \frac{i}{n}) \cdot \sum_{\nu=0}^{n-k} f^{(k)}\left(\frac{\nu}{n} + \theta_\nu \frac{k}{n}\right) \cdot \binom{n-k}{\nu} x^\nu (1-x)^{n-\nu-k},$$

where $0 < \theta_\nu < 1$. Therefore,

$$\begin{aligned} B_n^{(k)}(f,x) &= \prod_{i=1}^{k-1}(1 - \frac{i}{n}) \\ &\quad \times \left\{ B_{n-k}(f^{(k)}, x) + \sum_{\nu=0}^{n-k}\left[f^{(k)}\left(\frac{\nu}{n} + \theta_\nu\frac{k}{n}\right) - f^{(k)}\left(\frac{\nu}{n}\right)\right]\binom{n-k}{\nu} x^\nu (1-x)^{n-k-\nu}\right\}. \end{aligned}$$

It is easy to observe that

$$1 - \prod_{i=1}^{k-1}(1 - \frac{i}{n}) \le 1 - \left(1 - \frac{k-1}{n}\right)^{k-1} \le \begin{cases} \dfrac{(k-1)^2}{n}, \\ 1. \end{cases} \tag{2.9}$$

Hence

$$
\begin{aligned}
f^{(k)}(x) - B_n^{(k)}(f,x) &= f^{(k)}(x) - \prod_{i=1}^{k-1}(1-\frac{i}{n}) \cdot B_{n-k}(f^{(k)},x) \\
&\quad - \prod_{i=1}^{k-1}(1-\frac{i}{n}) \cdot \sum_{\nu=0}^{n-k} \left[f^{(k)}\left(\frac{\nu}{n}+\theta_\nu \cdot \frac{k}{n}\right) - f^{(k)}\left(\frac{\nu}{n}\right)\right] \binom{n-k}{\nu} x^\nu (1-x)^{n-k-\nu} \\
&= f^{(k)}(x) - B_{n-k}(f^{(k)},x) + \left[1 - \prod_{i=1}^{k-1}(1-\frac{i}{n})\right] B_{n-k}(f^{(k)},x) \\
&\quad - \prod_{i=1}^{k-1}(1-\frac{i}{n}) \cdot \sum_{\nu=0}^{k-1} \left[f^{(k)}\left(\frac{\nu}{n}+\theta_\nu \cdot \frac{k}{n}\right) - f^{(k)}\left(\frac{\nu}{n}\right)\right] \binom{n-k}{\nu} x^\nu (1-x)^{n-k-\nu}.
\end{aligned}
$$

Finally we arrive at

$$
\|f^{(k)} - B_n^{(k)}(f)\| \le \|f^{(k)} - B_{n-k}(f^{(k)})\| + \min\left\{1, \frac{(k-1)^2}{n}\right\} \|f^{(k)}\| + \omega_1\left(f^{(k)}; \frac{k}{n}\right),
$$

and the proof of Lemma 5 is complete. □

Lemma 6 *(Gonska* [13]*) Let* $f \in C^l[0,1]$, $l \in \mathbb{N}_0$. *For any* $h \in (0,1]$ *and* $s \in \mathbb{N}$ *there exists a function* $f_{h,l+s} \in C^{2l+s}[0,1]$ *with*

$$
(i) \qquad \|f^{(j)} - f_{h,l+s}^{(j)}\| \le C \cdot \omega_{l+s}(f^{(j)},h), \text{ for } 0 \le j \le l,
$$

$$
(ii) \qquad \|f_{h,l+s}^{(j)}\| \le C \cdot h^{-j} \cdot \omega_j(f,h), \text{ for } 0 \le j \le l+s.
$$

Here the constant C *depends only on* l *and* s.

We end this section with some preparations for Section 3. We apply Lemma 3 and Lemma 5 to obtain

$$
\|f^{(k)} - B_n^{(k)}(f)\| \le \frac{3}{4(n-k)}\|f^{(k+2)}\| + \frac{(k-1)^2}{n}\|f^{(k)}\| + \frac{k}{n}\|f^{(k+1)}\|. \tag{2.10}
$$

The last inequality we apply later for $0 \le k \le m-2$.

For the case $k = m-1$ the estimates (2.5) and (2.7) yield

$$
\|f^{(k)} - B_n^{(k)} f\| \le \frac{3}{4}(n-k)^{-\frac{1}{2}-\frac{\alpha}{2}}\Theta_\alpha(f^{(k+1)}) + \frac{(k-1)^2}{n}\|f^{(k)}\| + \frac{k}{n}\|f^{(k+1)}\| \tag{2.11}
$$

for $0 \le \alpha \le 1$.

For $k = m$ we apply (2.4) and (2.7) to get

$$
\|f^{(k)} - B_n^{(k)} f\| \le \frac{5}{4}(n-k)^{-\frac{\alpha}{2}}\Theta_\alpha(f^{(k)}) + \frac{(k-1)^2}{n}\|f^{(k)}\| + \frac{k^\alpha}{n^\alpha}\Theta_\alpha(f^{(k)}). \tag{2.12}
$$

We summarize the estimates (2.11), (2.12) and (2.13) in the following:

$$
\sum_{k=0}^{r} \|f^{(k)} - B_n^{(k)} f\| \le C_r \begin{cases} \frac{1}{n-r}\|f\|_{r+2,0}, & r \le m-2, \\[2ex] \frac{1}{(n-r)^{\frac{1+\alpha}{2}}}\|f\|_{r+1,\alpha}, & r = m-1, \\[2ex] \frac{1}{(n-r)^{\frac{\alpha}{2}}}\|f\|_{r,\alpha}, & r = m. \end{cases} \tag{2.13}
$$

Here, C_r is a constant depending only on r. Now we are ready to start the proof of our main result.

3 Proof of the theorem

We consider four cases.

1st case: $r \le m-3$. It is clear that in this case

$$\max\left\{-1, \frac{r+\beta-m-\alpha}{2}\right\} = -1.$$

To prove (1.6) we only observe that

$$\Theta_\beta(f^{(r)} - B_n^{(r)} f) \le \|f^{(r+1)} - B_n^{(r+1)} f\|$$

and apply (2.14) - first line in the right hand side.

2nd case: $r = m-1$. Obviously

$$\frac{r+\beta-m-\alpha}{2} = \frac{\beta-\alpha-1}{2} \ge -1.$$

Hence

$$\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\beta} \max_{x\in[0,1-h]} |\Delta_h(f-B_nf)^{(r)}(x)| \le n^{\frac{\beta-1}{2}} \|f^{(r+1)} - B_n^{(r+1)} f\|.$$

We apply (2.13) to arrive at

$$\begin{aligned}
&\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\beta} \max_{x\in[0,1-h]} |\Delta_h(f-B_nf)^{(r)}(x)| \\
\le\ & n^{\frac{\beta-1}{2}} \left\{\frac{5}{4}(n-m)^{-\frac{\alpha}{2}}\Theta_\alpha(f^{(m)}) + \frac{(m-1)^2}{n}\|f^{(m)}\| + \frac{m^\alpha}{n^\alpha}\Theta_\alpha(f^{(m)})\right\} \\
\le\ & (n-m)^{\frac{-1-\alpha+\beta}{2}} \left\{\left(\frac{5}{4} + \frac{m^\alpha}{n^{\alpha/2}}\right)\Theta_\alpha(f^{(m)}) + \frac{(m-1)^2}{\sqrt{n}}\|f^{(m)}\|\right\} \\
\le\ & (n-m)^{\frac{-1-\alpha+\beta}{2}} (m-1)^{\frac{3}{2}} \left\{\Theta_\alpha(f^{(m)}) + \|f^{(m)}\|\right\} \\
=\ & (n-m)^{\frac{-1-\alpha+\beta}{2}} r^{3/2} \left\{\Theta_\alpha(f^{(m)}) + \|f^{(m)}\|\right\}.
\end{aligned} \tag{3.1}$$

Next we consider $\frac{1}{\sqrt{n}} \le h \le 1$. Consequently,

$$\begin{aligned}
&\sup_{\frac{1}{\sqrt{n}}\le h\le 1} h^{-\beta} \max_{x\in[0,1-h]} |\Delta_h(f-B_nf)^{(r)}(x)| \\
\le\ & 2n^{\frac{\beta}{2}} \|f^{(r)} - B_n^{(r)} f\| \qquad \text{(using (2.13))} \\
\le\ & n^{\frac{\beta}{2}} \cdot (n-r)^{\frac{-1-\alpha}{2}} \left\{\frac{3}{2}\Theta_\alpha(f^{(m)}) + 2\frac{(r-2)^2}{n^{\frac{1-\alpha}{2}}}\|f^{(m-1)}\| + \frac{2(r-1)}{n^{\frac{1-\alpha}{2}}}\|f^{(m)}\|\right\} \\
\le\ & C_r \cdot (n-r)^{\frac{-1-\alpha+\beta}{2}} \left\{\Theta_\alpha(f^{(m)}) + \|f^{(m-1)}\| + \|f^{(m)}\|\right\}.
\end{aligned} \tag{3.2}$$

Now (3.1) and (3.2) complete the evaluation of the so-called Hölder term in the definition of $\|B_nf - f\|_{r,\beta}$. To estimate the norm terms, i.e., the sum in the lefthand side of (2.14) we simply apply the second line in the righthand side of (2.14). We conclude that in the case $r = m-1$ the estimate (1.6) holds with $C_r = \mathcal{O}(r^2)$. As it was observed in the evaluation of the Hölder term we divided the proof in two cases according to the position of $h : 0 \le h \le \frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}} < h \le 1$. The situation is similar in the remaining two cases:

3rd case: $r = m$. Therefore $\beta \le \alpha$, and

$$0 \ge \frac{r+\beta-m-\alpha}{2} = \frac{\beta-\alpha}{2} \ge -1.$$

To estimate the norm terms in $\|B_nf-f\|_{r,\beta}$ we apply the third line in the right-hand side of (2.14). Now we evaluate the Hölder term. We calculate as follows:

$$
\begin{aligned}
&\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\beta}\cdot\max_{x\in[0,1-h]}|\Delta_h(f-B_nf)^{(r)}(x)|\\
\le\;& n^{\frac{\beta-\alpha}{2}}\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\alpha}\cdot\max_{x\in[0,1-h]}\{|\Delta_h f^{(r)}(x)|+|\Delta_h(B_nf)^{(r)}(x)|\\
\le\;& n^{\frac{\beta-\alpha}{2}}\Theta_\alpha(f^{(m)})+n^{\frac{\beta-\alpha}{2}}\sup_{0<h\le\frac{1}{\sqrt{n}}} h^{-\alpha}\cdot\max_x|\Delta_h(B_nf)^{(r)}(x)|\\
\le\;& n^{\frac{\beta-\alpha}{2}}\Theta_\alpha(f^{(m)})+n^{\frac{\beta-\alpha}{2}}\sup_{0<h} h^{-\alpha}\cdot\max_x|\Delta_h(B_nf)^{(r)}(x)|\\
=\;& n^{\frac{\beta-\alpha}{2}}\Theta_\alpha(f^{(m)})+n^{\frac{\beta-\alpha}{2}}\Theta_\alpha((B_nf)^{(r)})\\
\le\;& 2n^{\frac{\beta-\alpha}{2}}\Theta_\alpha(f^{(m)}).
\end{aligned}\tag{3.3}
$$

Here the inequality

$$\Theta_\alpha((B_nf)^{(r)})\le\Theta_\alpha(f^{(r)})$$

follows from Corollary 3.3 in [12]:

$$f^{(r)}\in \mathrm{Lip}_L\alpha\Rightarrow(B_nf)^{(r)}\in\mathrm{Lip}_L\alpha.$$

Also we calculate

$$
\begin{aligned}
&\sup_{\frac{1}{\sqrt{n}}\le h\le 1} h^{-\beta}\cdot\max_{x\in[0,1-h]}|\Delta_h(f-B_nf)^{(r)}(x)|\\
\le\;& 2n^{\frac{\beta}{2}}\|(f-B_nf)^{(m)}\| \qquad\text{(using (2.14))}\\
\le\;& 2n^{\frac{\beta}{2}}\left\{\left[\frac{5}{4}(n-m)^{-\frac{\alpha}{2}}+\frac{m^\alpha}{n^\alpha}\right]\Theta_\alpha(f^{(m)})+\frac{(m-1)^2}{n}\|f^{(m)}\|\right\}\\
\le\;& 2n^{\frac{\beta-\alpha}{2}}\left\{\left[\frac{5}{4}\left(\frac{n}{n-m}\right)^{\frac{\alpha}{2}}+\frac{m^\alpha}{n^{\alpha/2}}\right]\Theta_\alpha(f^{(m)})+\frac{(m-1)^2}{1-\frac{\alpha}{2}}\|f^{(m)}\|\right\}\\
\le\;& C_1r^{1+\frac{\alpha}{2}}n^{\frac{\beta-\alpha}{2}}\left[\Theta_\alpha(f^{(m)})+\|f^{(m)}\|\right],
\end{aligned}\tag{3.4}
$$

where $C_1>0$ is an absolute constant independent on n and $r=m$. The estimates (3.3) and (3.4) complete the case $r=m$.

4th case: $r=m-2$.

4a) Let $\alpha=\beta$. Then $\frac{r+\beta-m-\alpha}{2}=-1$. Due to the first line in (2.14) we only need to evaluate the Hölder term in $\|B_nf-f\|_{r,\beta}$. Write $L_n:=I-B_n$, where I is the identity operator. Our goal is to establish

$$\Theta_\alpha(L_n^{(m-2)}f)\le\frac{C_1(m)}{n-m+1}\left[\Theta_\alpha f^m)+\|f^{(m)}\|+\|f^{(m+1)}\|\right].\tag{3.5}$$

Hence we need to show

$$\sup_{0<t\le1}t^{-\alpha}K(L_n^{(m-2)}f,t)\le\frac{C_1(m)}{n}\sup_{0<t\le1}t^{-\alpha}K(f^{(m)},t)+\frac{C_1(m)}{n}\left[\|f^{(m)}+f^{(m-1)}\|\right].\tag{3.6}$$

For $f\in C^{m,\alpha}[0,1]$ and any $g\in C^{m+1}[0,1]$ we write

$$K(L_n^{(m-2)}f,t)\le\|L_n^{(m-2)}(f-g)\|+t\|L_n^{(m-1)}g\|.\tag{3.7}$$

We apply (2.7) from Lemma 5 to estimate each of the summands in the right-hand side of the last inequality. First we have

$$\begin{aligned}\|L_n^{(m-2)}(f-g)\| &\le \|(f-g)^{(m-2)} - B_{n-m+2}((f-g)^{(m-2)})\| \\ &\quad + \frac{(m-2-1)^2}{n}\|(f-g)^{(m-2)}\| + \omega_1\left((f-g)^{(m-2)}; \frac{m-2}{n}\right) \\ &\le \frac{1}{n-m+2}\|(f-g)^{(m)}\| + \frac{(m-3)^2}{n}\|(f-g)^{(m-2)}\| + \frac{m-2}{n}\|(f-g)^{(m-1)}\|. \qquad (3.8)\end{aligned}$$

Moreover,

$$t\|L_n^{(m-1)}g\| \le \frac{t}{n-m+1}\|g^{(m+1)}\| + t\frac{(m-2)^2}{n}\|g^{(m-1)}\| + t\frac{m-1}{n}\|g^{(m)}\|. \qquad (3.9)$$

We sum up the right hand sides of (3.8) and (3.9) and get

$$\begin{aligned}K(L_n^{(m-2)}f, t) &\le \frac{C(m)}{n-m+1}\Big[\|(f-g)^{(m-2)}\| + t\|g^{(m-1)}\| \\ &\quad + \|(f-g)^{(m-1)}\| + t\|g^{(m)}\| + \|(f-g)^{(m)}\| + t\|g^{(m+1)}\|\Big], \qquad (3.10)\end{aligned}$$

where for $C(m)$ we can take $(m-1)^2 = (r+1)^2 = C(m)$. To choose an auxiliary function $g \in C^{m+1}[0,1]$ we apply Lemma 6. We use this lemma with $\ell = m$, $s = 1$ and define $g := f_{h,\ell+s} \in C^{2\ell+s}(I)$. The following inequalities are corollaries from Lemma 6 (i). For $j = m-2$ we get

$$\|f^{(m-2)} - g^{(m-2)}\| \le c\omega_{m+1}(f^{(m-2)}; h) \le ch\omega_m(f^{(m-1)}; h) \le ch2^m\|f^{(m-1)}\|. \qquad (3.11)$$

For $j = m-1$ we have

$$\|f^{(m-1)} - g^{(m-1)}\| \le c\omega_{m+1}(f^{(m-1)}; h) \le ch\omega_m(f^{(m)}; h) \le ch2^m\|f^{(m)}\|. \qquad (3.12)$$

For $j = m$ we get

$$\|f^{(m)} - g^{(m)}\| \le c\omega_{m+1}(f^{(m)}; h) \le c2^m\omega_1(f^{(m)}; h). \qquad (3.13)$$

The second condition of Lemma 6 (ii) implies the following inequalities:

$$\begin{aligned}\|g^{(m-1)}\| &\le ch^{-(m-1)}\omega_{m-1}(f, h) \le ch^{-(m-1)}h\omega_{m-2}(f', h) \\ &\le \ldots \le ch^{-(m-1)}h^{m-2}\omega_1(f^{(m-2)}; h) = ch^{-1}\omega_1(f^{(m-2)}; h) \\ &\le ch^{-1}h\|f^{(m-1)}\| = c\|f^{(m-1)}\|, \qquad (3.14)\end{aligned}$$

$$\|g^{(m)}\| \le ch^{-m}\omega_m(f, h) \le ch^{-m}h^{m-1}\omega_1(f^{(m-1)}; h) \le ch^{-1}h\|f^{(m)}\| = c\|f^{(m)}\|, \qquad (3.15)$$

$$\|g^{(m+1)}\| \le ch^{-(m+1)}\omega_{m+1}(f, h) \le ch^{-(m+1)}h^m\omega_1(f^{(m)}; h) \le h^{-1}\omega_1(f^{(m)}, h). \qquad (3.16)$$

In inequalities (3.11) to (3.16) we set $h := t$ and return to (3.10). Now from (3.8), (3.9), (3.10) and (3.7) we verify that

$$\begin{aligned}&t^{-\alpha}K(L_n^{(m-2)}f, t) \\ &\le \frac{C(m)\cdot c}{n-m+1}\Big[2^m\|f^{(m-1)}\| + 2^m\|f^{(m)}\| \\ &\quad + 2^m t^{-\alpha}\omega_1(f^{(m)}, t) + \|f^{(m-1)}\| + \|f^{(m)}\| + t^{-\alpha}\omega_1(f^{(m)}; t)\Big] \\ &\le \frac{c\cdot C(m)(2^m+1)}{n-m+1}\Big[t^{-\alpha}\omega_1(f^{(m)}; t) + \|f^{(m-1)}\| + \|f^{(m)}\|\Big] \\ &\le \frac{c\cdot C(m)(2^m+1)}{n-m+1}\Big[\Theta_\alpha(f^{(m)}) + \|f^{(m-1)}\| + \|f^{(m)}\|\Big],\end{aligned}$$

which is (3.5).

4b) We consider $0 \le \beta < \alpha \le 1, r = m-2$. So

$$\frac{r+\beta-m-\alpha}{2} = -1 - \frac{\alpha-\beta}{2}$$

and

$$\max\left\{-1, \frac{r+\beta-m-\alpha}{2}\right\} = -1.$$

We apply (3.5) to obtain

$$\begin{aligned}\sup_{0<t\le 1} t^{-\beta} K(L_n^{(m-2)} f, t) &\le \frac{c(m)}{n-m+1}\left[\|f^{(m)}\| + \|f^{(m-1)}\| + \sup_{0<t\le 1} t^{-\beta}\omega_1(f^{(m)};t)\right] \\ &\le \frac{c(m)}{n-m+1}\left[\|f^{(m)}\| + \|f^{(m-1)}\| + \sup_{0<t\le 1} t^{-\alpha}\omega_1(f^{(m)};t)\right].\end{aligned}$$

Hence we proved

$$\Theta_\beta(L_n^{(m-2)} f) \le \frac{c(m)}{n-m+1}\left[\|f^{(m)}\| + \|f^{(m-1)}\| + \Theta_\alpha(f^{(m)})\right]. \tag{3.17}$$

4c) Last we consider $0 \le \alpha < \beta \le 1, r = m-2$. So $\frac{r+\beta-m-\alpha}{2} = \frac{\beta-\alpha}{2} - 1 > -1$, and

$$\max\left\{-1, \frac{r+\beta-m-\alpha}{2}\right\} = \frac{\beta-\alpha}{2} - 1. \tag{3.18}$$

We need an appropriate estimate for the K-functional and again as earlier we should consider two cases according to the position of t.

Let first $0 < t \le \frac{1}{\sqrt{n}}$. Obviously

$$K(L_n^{(m-2)} f, t) \le t\|L_n^{(m-1)} f\|. \tag{3.19}$$

Hence

$$\begin{aligned}\sup_{0<t\le\frac{1}{\sqrt{n}}} t^{-\beta} K(L_n^{(m-2)} f, t) &\le \sup_{0<t\le\frac{1}{\sqrt{n}}} t^{1-\beta}\|L_n^{(m-1)} f\| \\ &\le \left(\frac{1}{\sqrt{n}}\right)^{1-\beta}\|L_n f\|_{m-1,0} \le C_1\left(\frac{1}{\sqrt{n}}\right)^{1-\beta} n^{-\frac{1}{2}-\frac{\alpha}{2}}\|f\|_{m,\alpha}.\end{aligned} \tag{3.20}$$

In estimate (3.18) we have used the statement of the theorem for the case $r = m-1$ established earlier (for $\beta = 0$). Thus the case $0 < t \le \frac{1}{\sqrt{n}}$ is completed. Next we have $\frac{1}{\sqrt{n}} \le t \le 1$. We multiply both sides of (3.15) by $t^{-\beta}$ and obtain

$$\begin{aligned}t^{-\beta} K(L_n^{(m-2)} f, t) &\le \frac{c(m)}{n-m+1}\left[t^{1-\beta}(\|f^{(m)}\| + \|f^{(m+1)}\|) + t^{-\beta}\omega_1(f^{(m)}, t)\right] \\ &\le \frac{c(m)}{n-m+1}\left[\|f^{(m)}\| + \|f^{(m-1)}\| + \left(\frac{1}{\sqrt{n}}\right)^{\alpha-\beta} t^{-\alpha}\omega_1(f^{(m)}, t)\right].\end{aligned}$$

Therefore,

$$\sup_{\frac{1}{\sqrt{n}}\le t\le 1} t^{-\beta} K(L_n^{(m-2)} f, t) \le \frac{c(m)}{n-m+1}\left[\|f^{(m)}\| + \|f^{(m-1)}\| + n^{-\frac{\alpha-\beta}{2}} \sup_{\frac{1}{\sqrt{n}}\le t\le 1} t^{-\alpha}\omega_1(f^{(m)}, t)\right].$$

The last estimate together with (3.20) completes the proof of 4c). □

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