Uniqueness of best proximity pairs and rigidity of metrics

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The main results of the present talk were proved in
O. Dovgoshey and R. Shanin, *Uniqueness of best proximity pairs and rigidity of semimetric spaces*, arXiv:2201.04380v2[mathGN], 2 April 2022.

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Definition 1

Let (X, d) be a metric space. A set $A \subseteq X$ is said to be *proximinal* in (X, d) if, for every $x \in X$, there exists $a_0 \in A$ such that

$$d(x, a_0) = \inf\{d(x, a) \colon a \in A\}.$$

The point a_0 is called a *best approximation* to x in A.

In *The Theory of Best Approximation and Functional Analysis* (1974) I. Singer wrote:
"The term «proximinal» ... (a combination of «proximity» and «minimal») was proposed by R. Killgrove and used first by R.R. Phelps (1957) "Convex sets and nearest points", *Proc. Amer. Math. Soc.*, *8*(4), 790–797."

For nonempty subsets A and B of a metric space (X, d), we define a distance from A to B as

$$\operatorname{dist}(A, B) := \inf\{d(a, b) \colon a \in A \text{ and } b \in B\}.$$

If A is a one-point set, $A = \{a\}$, then we write $\operatorname{dist}(a, B)$ instead of $\operatorname{dist}(\{a\}, B)$.

Definition 2

Let (X, d) be a metric space, and let A, $B \subseteq X$ be nonempty. A pair $(a_0, b_0) \in A \times B$ is called a *best proximity pair* for the sets A and B if $d(a_0, b_0) = \text{dist}(A, B)$.

Thus, for the case $A = \{a\}$, a pair (a, b) is a best proximity pair for A and B if and only if b is a best approximation to a in B.

The main goal of the first part of present talk is to describe conditions for the uniqueness of best approximations in metric spaces.

The complete description of metric spaces having at most one best proximity pair (a_0, b_0) for all disjoint nonempty A, B will be given in the second part of the talk.

We will use some concepts from Graph Theory.

A *graph* is a pair (V, E) consisting of a nonempty set V and a set E whose elements are unordered pairs $\{u, v\}$ of different elements u, $v \in V$.

For a graph $G = (V, E)$, the sets $V = V(G)$ and $E = E(G)$ are called the *set of vertices* and the *set of edges*, respectively.

A graph whose edge set is empty is called a *null graph*. Two vertices u, $v \in V$ are *adjacent* if $\{u, v\}$ is an edge in G. The *degree* of a vertex v_0 in a graph G, denoted $\deg(v_0) = \deg_G(v_0)$, is the number of all vertices which are adjacent with v_0 in G.

Definition 3

A graph G is *bipartite* if the vertex set $V(G)$ can be partitioned into two nonvoid disjoint sets, or *parts*, in such a way that no edge has both ends in the same part.

Example 4

An important subclass of bipartite graphs is formed by the so-called stars. We shall say that a graph S is a *star* if $|V(S)| \geqslant 2$ and there is a vertex $c \in V(S)$, the *center* of S, such that c is adjacent with every $v \in V(S) \setminus \{c\}$.

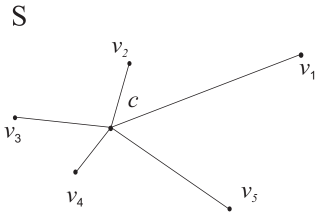

Figure 1.

The graph S is the star with the center c and the vertex set $\{c, v_1, \ldots, v_5\}$.

Definition 5

A bipartite graph G with fixed parts A and B is *proximinal* if there exists a metric space (X, d) such that A and B are disjoint proximinal subsets of X, and the equivalence

$$\big(\{a, b\} \in E(G)\big) \Leftrightarrow \big(d(a, b) = \operatorname{dist}(A, B)\big)$$

is valid for every $a \in A$ and every $b \in B$. In this case we write $G = G_{X,d}(A, B)$ and say that G is proximinal for (X, d).

Example 6

Let G be a star with a center c. Write $A = \{c\}$ and $B = V(G) \setminus \{c\}$. Then G is proximinal with the parts A and B.

The following theorem was obtained in (2022), K. Chaira, O. Dovgoshey, S. Lazaiz, "Bipartite graphs and best proximity pairs," *Journal of Mathematical Sciences, 264*(4), 369–388.

Theorem 7

Let G be a bipartite graph with fixed parts A and B. Then following statements are equivalent.

(i) G is proximinal for a metric space.

(ii) Either G is not a null graph or G is a null graph but A and B are infinite.

Definition 8

A metric space (X, d) is said to be *strongly rigid* if $d(x, y) = d(u, v) \neq 0$ implies $\{x, y\} = \{u, v\}$ for all x, y, u, $v \in X$.

Definition 9

A metric space (X, d) is *weakly rigid* if every three-point subspace of (X, d) is strongly rigid.

Example 10

Let $R = \{z_1, z_2, z_3, z_4\}$ be the four-point subset of the complex plane,

$$z_1 = 0 + 0i, \quad z_2 = 0 + 3i, \quad z_3 = 4 + 3i, \quad z_4 = 4 + 0i$$

and d be the restriction of the usual Euclidean metric on $R \times R$. The equalities $d(z_1, z_2) = d(z_3, z_4) = 3$ imply that (R, d) in not strongly rigid, but it can be proved directly that (R, d) is weakly rigid.

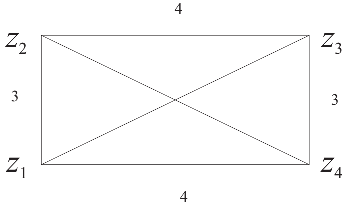

Figure 2. The rectangle (R, d) is not strongly rigid but weakly rigid.

To the best of my knowledge, the weakly rigid metric spaces were firstly introduced in the preprint O. Dovgoshey and R. Shanin, *Uniqueness of best proximity pairs and rigidity of semimetric spaces*, arXiv:2201.04380v2[mathGN] 2 April 2022.

Theorem 11

Let G be a bipartite graph with fixed parts A and B. Then following statement are equivalent.

(i) G is proximinal for a strongly rigid metric space.

(ii) The following conditions are simultaneously fulfilled:

(ii_1) The inequalities $|E(G)| \leqslant 1$ and $|V(G)| \leqslant \mathfrak{c}$ hold, where $\mathfrak{c}$ is the cardinality of the continuum.

(ii_2) If G is a null graph, then A and B are infinite.

Theorem 12

Let G be a bipartite graph with parts A and B. Then following statements are equivalent:

- *(i) G is proximinal for a weakly rigid metric space.*
- *(ii) The following conditions are simultaneously fulfilled:*
  - *(ii_1) The inequality $|V(G)| \leqslant \mathfrak{c}$ holds and $\deg_G(v) \leqslant 1$ for every $v \in V(G)$.*
  - *(ii_2) If G is a null graph, then A and B are infinite.*

Let A and B be disjoint proximinal subsets of a metric space (X, d). In the next theorem we denote by A_0 and B_0 the sets defined as

$$A_0 := \{a \in A\colon\ \operatorname{dist}(a, B) = \operatorname{dist}(A, B)\}$$

and

$$B_0 := \{b \in B\colon\ \operatorname{dist}(b, A) = \operatorname{dist}(A, B)\}.$$

Theorem 13

Let (X, d) *be a metric space. Then the following statements are equivalent.*

(i) The inequality $\deg_G(v) \leqslant 1$ *holds for every vertex* v *of every proximinal graph* $G = G_{X,d}(A, B)$.

(ii) The equality

$$|A_0| = |B_0|$$

holds for all disjoint proximinal subsets A, B *of* X.

(iii) For every proximinal $A \subseteq X$ *and every* $x \in X$ *there exists the unique best approximation to* x *in* A.

(iv) For every $Y \subseteq X$ *and every* $x \in X$ *there exists at most one best approximation to* x *in* Y.

(v) (X, d) *is weakly rigid.*

A metric space (Y, ρ) is *ultrametric* iff the *strong triangle inequality*

$$d(x, y) \leqslant \max\{d(x, z), d(z, y)\}$$

holds for all x, y, $z \in Y$. For every ultrametric space X, each triangle in X is isosceles with the base being no greater than the legs. The converse statement is also valid: If Z is a metric space and each triangle in Z is isosceles with the base no greater than the legs, then Z is an ultrametric space. The situation is opposite for weakly rigid spaces. A metric space is weakly rigid if and only if it does not contain any isosceles triangles. However, there is a close interconnection between proximinal subsets in weakly rigid spaces and in ultrametric ones. To describe this we introduce the concept of *best approximations graph*.

Definition 14

Let (X, d) be a metric space and let A be a proximinal subset of X with $X \setminus A \neq \varnothing$. The graph *of best approximations by points of* A is a graph $\Gamma = \Gamma_{X,d}(A)$ such that Γ is bipartite with parts A and $X \setminus A$ and points $x \in X \setminus A$ and $a \in A$ are adjacent if and only if $d(x, a) = \operatorname{dist}(x, A)$.

Proposition 15

Let (X, d) be a weakly rigid metric space, let A be a proximinal subset of X such that $X \setminus A \neq \varnothing$, and let $\Gamma_{X,d}(A)$ be the graph of best approximations by points of A. Then there is an ultrametric $\rho\colon X \times X \to [0, \infty)$ such that

$$\Gamma_{X,d}(A) = G_{X,\rho}(A, B)$$

with $B = X \setminus A$.

Conjecture 16

Let Γ *be a bipartite graph with fixed parts* A *and* B. *Then the following statements are equivalent.*

(i) There is a weakly rigid metric d *on* $X = A \cup B$ *such that* A *is proximinal in* (X, d), *and* $\Gamma = \Gamma_{X,d}(A)$ *is the graph of best approximations by points of* A.

(ii) Every connected component of Γ *is a star with a center* $a \in A$, *and the equality* $\deg_{\Gamma} b = 1$ *holds for every* $b \in B$, *and* $|V(\Gamma)| \leqslant \mathfrak{c}$.

Thank you for your attention!