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POLYNOMIAL LYM INEQUALITIES

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For a Sperner family $\mathcal{A} \subseteq 2^{[n]}$ let $\mathcal{A}_i$ denote the family of all i-element sets in $\mathcal{A}$. We sharpen the LYM inequality $\sum_i |\mathcal{A}_i|/\binom{n}{i} \leq 1$ by adding to the LHS all possible products of fractions $|\mathcal{A}_i|/\binom{n}{i}$, with suitable coefficients. A corresponding inequality is established also for the linear lattice and the lattice of subsets of a multiset (with all elements having the same multiplicity).

1. Introduction

Let $[n]$ be the set $\{1,2,\dots,n\}$, and $2^{[n]}$ the powerset of $[n]$. For $i \in [n]$ we denote the set of all i-element subsets of $[n]$ by $\binom{[n]}{i}$. A set $\mathcal{A} \subseteq 2^{[n]}$ is called a *Sperner family* (or *antichain*) if there are no inclusions among the members of $\mathcal{A}$:

$$A \not\subseteq B \text{ for all } A, B \in \mathcal{A},\ A \neq B.$$

A classic theorem by Sperner [28] says that the maximum size of an antichain in $2^{[n]}$ is $\binom{n}{\lfloor n/2 \rfloor}$, the only families achieving this cardinality being $\binom{[n]}{\lfloor n/2 \rfloor}$ and $\binom{[n]}{\lceil n/2 \rceil}$.

For a set family $\mathcal{A} \subseteq 2^{[n]}$ we denote $\mathcal{A} \cap \binom{[n]}{i}$ by $\mathcal{A}_i$, and we call $(|\mathcal{A}_0|, |\mathcal{A}_1|, \dots, |\mathcal{A}_n|)$ the *profile* of the family $\mathcal{A}$. A generalization of Sperner's

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Theorem is the LYM inequality (Bollobás [3], Lubell [25], Meshalkin [26], Yamamoto [29]): If $\mathcal{A}\subseteq 2^{[n]}$ is Sperner, then

$$\sum_{i=0}^{n} \frac{|\mathcal{A}_i|}{\binom{n}{i}} \leq 1\,. \tag{1}$$

Equality holds in (1) if and only if $\mathcal{A}=\binom{[n]}{i}$ for some i.

This inequality and its analogue for partially ordered sets is well studied in Sperner theory. We refer to [23] and [11, ch. 4.5] for surveys.

It is easily seen that the LYM inequality together with the nonnegative constraints $|\mathcal{A}_i|\geq 0$, $i=0,\dots,n$, describe the convex hull of the set of Sperner profiles, i.e. the convex hull of the set $\{(|\mathcal{A}_0|,|\mathcal{A}_1|,\dots,|\mathcal{A}_n|) : \mathcal{A}\subseteq 2^{[n]}$ is Sperner$\}$. Determining or approximating such profile sets for various set families is a basic task in extremal set theory. The systematic study of convex hulls of these profile sets was initiated in [13]. We refer to [11, ch. 3] for a survey. The set of Sperner profiles has a complete numerical characterization [4, 9] using the shadow function in the well known Kruskal–Katona Theorem [22, 24] (see also Section 3 for the latter). However, this characterization is sometimes difficult to handle with. The LYM inequality (1), although weaker, is therefore a valuable and basic tool. Our objective is to replace the linear constraint for Sperner profiles given by the LYM inequality by a sharper polynomial one. In terms of the convex hull of Sperner profiles, this new constraint gives a description of how far the Sperner profiles are from the bordering hyperplane (1). The main theorem is the following.

Theorem 1. *Let $\mathcal{A}\subseteq 2^{[n]}$ be a Sperner family. Then we have*

$$\sum_{i=0}^{n} \frac{|\mathcal{A}_i|}{\binom{n}{i}} + \sum_{I=\{i_1,\dots,i_s\}\subseteq[n-1],s\geq 2} \left(\prod_{j=1}^{s-1} \frac{n(i_{j+1}-i_j)}{i_j(n-i_{j+1})}\right)\left(\prod_{i\in I}\frac{|\mathcal{A}_i|}{\binom{n}{i}}\right) \leq 1\,. \tag{2}$$

Here and elsewhere we make the convention that sets of natural numbers as $\{i_1,\dots,i_s\}$ are always displayed with their elements being increasingly ordered; $i_1<\dots<i_s$.

Note that in (2) the coefficients of all products $\prod_{i\in I}|\mathcal{A}_i|/\binom{n}{i}$ are positive, thus (2) is indeed a sharpening of the LYM inequality (1). An immediate corollary is the equality characterization in Sperner's Theorem.

Note further that the coefficient of $\prod_{i\in I}|\mathcal{A}_i|/\binom{n}{i}$ is just the product of all quadratic coefficients belonging to pairs of consecutive indices $\{i_j,i_{j+1}\}\subseteq I$. In fact, (2) will follow from the validity of all quadratic inequalities

$$\frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}} + \frac{|\mathcal{A}_k|}{\binom{n}{k}} + \frac{n(k-\ell)}{\ell(n-k)}\frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}}\frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1\,,$$

where $0<\ell<k<n$. This works quite general for ranked partially ordered sets (posets): quadratic LYM inequalities for pairs of consecutive levels in a ranked poset imply a polynomial LYM inequality. This result is proved in Section 2. It is applied in later sections, where we establish quadratic (and hence polynomial) LYM inequalities for the Boolean lattice (Section 3), the linear lattice (Section 5) and the lattice of subsets of a multiset in which all elements have the same multiplicity (Section 6). In Section 4 some applications of Theorem 1 are presented.

We would like to draw the reader's attention also to the following three sharpenings of the LYM inequality.

In [8], the proportion of sets in $2^{[n]}$ which are incomparable to all sets of the Sperner family $\mathcal{A}$ is added to the LHS of (1): let $\mathcal{B}=\{B\subseteq[n]: B\not\subseteq A \text{ and } B\not\supseteq A \text{ for all } A\in\mathcal{A}\}$, then

$$\sum_{i=0}^{n}\frac{|\mathcal{A}_i|}{\binom{n}{i}}+\frac{|\mathcal{B}|}{2^n}\leq 1\,.$$

This inequality follows by applying the LYM inequality to the Sperner family $\mathcal{A}\cup\mathcal{B}_i$, where i is such that $|\mathcal{B}|/2^n\leq|\mathcal{B}_i|/\binom{n}{i}$.

In [1], the LYM inequality is lifted to an identity:

$$\sum_{A\in\mathcal{A}}\frac{1}{\binom{n}{|A|}}+\sum_{\emptyset\neq B\in 2^{[n]}\setminus\mathcal{A}}\frac{|\bigcap_{B\supseteq A\in\mathcal{A}}A|}{|B|\binom{n}{|B|}}=1\,.$$

Both sharpenings are not expressible in the profile of $\mathcal{A}$.

In [15], a sequence of inequalities is established, each of which sharpens the LYM inequality by raising the coefficients $1/\binom{n}{i}$ in (1) depending on the profile of the family $\mathcal{A}$. The first and simplest of these inequalities is as follows. Let s be the smallest k for which $\sum_{i\leq k}|\mathcal{A}_i|/\binom{n-1}{i-1}>1$. Then

$$\sum_{i\leq s}\frac{s}{i}\,\frac{|\mathcal{A}_i|}{\binom{n}{i}}+\sum_{i>s}\frac{n-s}{n-i}\,\frac{|\mathcal{A}_i|}{\binom{n}{i}}\leq 1\,.$$

This result follows from Theorem 1 (see Section 4). Note that here different profiles $(|\mathcal{A}_0|,\dots,|\mathcal{A}_n|)$ yield in general different sharpenings.

2. Polynomial LYM posets

Let P be a ranked partially ordered set of rank n. For $i=0,\dots,n$ we denote by P_i the i-th level of P, that is the set of all rank i elements. Also, we denote the i-th Whitney number $|P_i|$ of P by $W_i=W_i(P)$.

We always suppose that P is graded, i.e. P_0 resp. P_n is the set of minimal resp. maximal elements of P.

A family $\mathcal{A}\subseteq P$ is called *Sperner*, if no two elements of $\mathcal{A}$ are comparable in P. We call $\mathcal{A}\subseteq P$ a 2-*level Sperner family* if $\mathcal{A}$ is Sperner and $\mathcal{A}$ has nonempty intersection with exactly two levels of P, i.e. $\mathcal{A}=\mathcal{A}_\ell\cup\mathcal{A}_k$ for some ℓ,k ($\mathcal{A}_\ell,\mathcal{A}_k\neq\emptyset$), where we put $\mathcal{A}_i=\mathcal{A}\cap P_i$ for $i=0,\dots,n$.

Given a family $\mathcal{A}\subseteq P$, let $\Delta_\ell(\mathcal{A})$ denote the lower shadow of $\mathcal{A}$ in level ℓ, i.e. the set of all elements in P_ℓ which are less or equal to some element in $\mathcal{A}$. Similary, $\nabla_\ell(\mathcal{A})$ denotes the upper shadow of $\mathcal{A}$ in level ℓ.

Recall that P is called a *LYM poset* [23], if for all Sperner families $\mathcal{A}\subseteq P$

$$\sum_{i=0}^{n}\frac{|\mathcal{A}_i|}{W_i}\leq 1 \tag{LYM}$$

holds. Equivalently, for all $\ell<k$ and $\mathcal{A}_k\subseteq P_k$ it holds the *normalized matching property* [18]

$$\frac{|\Delta_\ell(\mathcal{A}_k)|}{W_\ell}\geq\frac{|\mathcal{A}_k|}{W_k}. \tag{NMP}$$

Let $c:2^{[n]}\to\mathbb{R}_{\geq 0}$, $\{i_1,\dots,i_s\}\mapsto c_{i_1,\dots,i_s}$, be a function from the Boolean lattice to the nonnegative real numbers. We always assume that $c_i=1$ for $i\in[n]$.

We say that the poset P satisfies a *polynomial LYM inequality* with coefficients c, if for all Sperner families $\mathcal{A}\subseteq P$

$$\sum_{I\subseteq[n],\,|I|\geq 1} c_I\left(\prod_{i\in I}\frac{|\mathcal{A}_i|}{W_i}\right)\leq 1 \tag{PLYM}$$

holds. In particular, if $\mathcal{A}=\mathcal{A}_\ell\cup\mathcal{A}_k$ is a 2-level Sperner family, $\ell<k$, then (PLYM) reads

$$\frac{|\mathcal{A}_\ell|}{W_\ell}+\frac{|\mathcal{A}_k|}{W_k}+c_{\ell,k}\frac{|\mathcal{A}_\ell|}{W_\ell}\frac{|\mathcal{A}_k|}{W_k}\leq 1\,. \tag{QLYM}$$

This quadratic inequality has an equivalent formulation in terms of shadows: If $\mathcal{A}_k\subseteq P_k$ and $\ell<k$, then

$$\frac{|\Delta_\ell(\mathcal{A}_k)|}{W_\ell}\geq\frac{|\mathcal{A}_k|}{W_k}\frac{1+c_{\ell,k}}{\left(1+c_{\ell,k}\frac{|\mathcal{A}_k|}{W_k}\right)}. \tag{QNMP}$$

Indeed, $\mathcal{A}_\ell\cup\mathcal{A}_k$ is Sperner iff $\mathcal{A}_\ell\subseteq P_\ell\setminus\Delta_\ell(\mathcal{A}_k)$.

Lemma 2. *Let $0 \le \ell < k < m \le n$ and $(1+c_{\ell,k})(1+c_{k,m}) \ge 1+c_{\ell,m}$. If (PLYM) holds for all 2-level Sperner families $\mathcal{A}_\ell \cup \mathcal{A}_k$ and $\mathcal{A}_k \cup \mathcal{A}_m$, then it also holds for all 2-level Sperner families $\mathcal{A}_\ell \cup \mathcal{A}_m$.*

Proof. Let $\mathcal{A}_\ell \cup \mathcal{A}_m$ be a 2-level Sperner family, $\mathcal{A}_\ell \subseteq P_\ell$, $\mathcal{A}_m \subseteq P_m$. Put $\mathcal{A}_k := \Delta_k(\mathcal{A}_m)$ and let $\alpha_i = |\mathcal{A}_i|/W_i$ for $i \in \{\ell, k, m\}$. Applying (PLYM) to the 2-level Sperner families $\mathcal{A}_\ell \cup \mathcal{A}_k$ and $(P_k \setminus \mathcal{A}_k) \cup \mathcal{A}_m$ gives

$$\begin{aligned}(1-\alpha_\ell)(1-\alpha_k) &\ge (1+c_{\ell,k})\alpha_\ell\alpha_k\,,\\ \alpha_k(1-\alpha_m) &\ge (1+c_{k,m})(1-\alpha_k)\alpha_m\,.\end{aligned}$$

Multiplying these inequalities shows that $\mathcal{A}_\ell \cup \mathcal{A}_m$ satisfies (PLYM). ∎

Note that (LYM) follows by applying (NMP) successively to pairs of consecutive levels. This argument extends to our setting and shows that quadratic LYM inequalities for pairs of consecutive levels imply a global PLYM inequality.

Theorem 3. *Let P be a graded poset of rank n. Let $c_{i-1,i}$, $i = 1, \ldots, n$, be nonnegative real numbers, and put $c_i := 1$ for all $i \in [n]$,*

$$c_{\ell,k} := \left(\prod_{i=\ell+1}^{k} (1 + c_{i-1,i})\right) - 1 \tag{3}$$

for all $0 \le \ell < k \le n$ with $k - \ell > 1$, and

$$c_{i_1,\ldots,i_s} := \prod_{j=1}^{s-1} c_{i_j,i_{j+1}} \tag{4}$$

for all $\{i_1, \ldots, i_s\} \subseteq [0,n]$, $s \ge 3$.

If (PLYM) holds for all 2-level Sperner families of P which are located at consecutive levels, then (PLYM) holds for all Sperner families of P.

Proof. Note that

$$(1 + c_{\ell,k})(1 + c_{k,m}) = (1 + c_{\ell,m}) \tag{5}$$

for all $0 \le \ell < k < m \le n$. Thus, by Lemma 2, (PLYM) holds for all 2-level Sperner families of P.

Let $\mathcal{A} \subseteq P$ be a Sperner family. We may assume that $\mathcal{A} = \mathcal{A}_0 \cup \cdots \cup \mathcal{A}_n$ with $\mathcal{A}_i \neq \emptyset$ for all $i = 0, \ldots, n$ (replace P with its restriction on those levels which $\mathcal{A}$ has nonempty intersection with).

We argue by induction on n. The polynomial LYM inequality for the family $\mathcal{A}$ will follow from the corresponding inequalities for the Sperner families $\nabla_1(\mathcal{A}_0)\cup\mathcal{A}_1\cup\cdots\cup\mathcal{A}_n$ and $\mathcal{A}_0\cup\mathcal{A}_1$. To see this, put $\alpha_i=|\mathcal{A}_i|/W_i$, $\delta=|\nabla_1(\mathcal{A}_0)|/W_1$, and let

$$g(x_0,\ldots,x_n)=\sum_{I\subseteq[0,n],\,I\neq\emptyset} c_I\left(\prod_{i\in I} x_i\right)$$

and $f(x_0,x_1)=g(x_0,x_1,\alpha_2,\ldots,\alpha_n)$. Note that

$$f(x_0,x_1+y_1)=f(x_0,x_1)+y_1(f(x_0,1)-f(x_0,0))\,.$$

We claim that

$$(1+c_{01})(f(0,1)-1)=f(1,0)-1\,. \tag{6}$$

Indeed, for $I=\{i_1,\ldots,i_s\}\subseteq[2,n]$, $I\neq\emptyset$, the coefficient of $\prod_{i\in I}\alpha_i$ on the LHS of (6) is $(1+c_{01})(1+c_{1i_1})c_I$, which is by (5) equal to $(1+c_{0i_1})c_I$, the corresponding coefficient on the RHS of (6).

Let us now prove (PLYM) for the family $\mathcal{A}$. By induction and the upper shadow version of (QNMP) we have

$$\begin{aligned}1 &\geq f(0,\alpha_1+\delta)=f(0,\alpha_1)+\delta\,(f(0,1)-f(0,0))\\ &\geq f(0,\alpha_1)+\frac{\alpha_0(1+c_{01})}{1+c_{01}\alpha_0}\,(f(0,1)-f(0,0))\,.\end{aligned}$$

Applying (6) then yields

$$1\geq f(0,\alpha_1)+c_{01}\alpha_0\,(f(0,\alpha_1)-f(0,0))+\alpha_0\,(f(1,0)-f(0,0))=f(\alpha_0,\alpha_1)\,,$$

which completes the proof. ∎

Remark 4. The best possible coefficient $c_{\ell,k}$ in (QLYM) is the minimum of

$$\frac{W_\ell W_k}{|\mathcal{A}_\ell||\mathcal{A}_k|}-\frac{W_\ell}{|\mathcal{A}_\ell|}-\frac{W_k}{|\mathcal{A}_k|},$$

taken over all 2-level Sperner families $\mathcal{A}_\ell\cup\mathcal{A}_k$. Equivalently, in terms of shadows, (QLYM) holds iff $1+c_{\ell,k}$ is not larger than

$$\left(\frac{W_k}{|A_k|}-1\right)\Big/\left(\frac{W_\ell}{|\Delta_\ell(\mathcal{A}_k)|}-1\right) \tag{7}$$

for all $\mathcal{A}_k\neq\emptyset$ with $\Delta_\ell(\mathcal{A}_k)\neq P_\ell$. Of course, the quadratic coefficients of the theorem given by (3) are in general strictly smaller than these optimal coefficients.

3. The Boolean lattice

In this section we prove Theorem 1 by establishing the corresponding quadratic LYM inequalities for 2-level Sperner families.

Lemma 5. *If $\mathcal{A}\subseteq 2^{[n]}$ is a 2-level Sperner family, $\mathcal{A}=\mathcal{A}_\ell\cup\mathcal{A}_k$, $0<\ell<k<n$, then we have*

$$\frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}}+\frac{|\mathcal{A}_k|}{\binom{n}{k}}+\frac{n(k-\ell)}{\ell(n-k)}\,\frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}}\,\frac{|\mathcal{A}_k|}{\binom{n}{k}}\le 1. \tag{8}$$

Up to permutations, the only 2-level Sperner family which attains equality is $\{L\in\binom{[n]}{\ell}:n\in L\}\cup\{K\in\binom{[n-1]}{k}\}$.

In the proof we use the Kruskal–Katona Theorem [22, 24]. It says that among all families $\mathcal{A}_k\subseteq\binom{[n]}{k}$ of the same cardinality $m=|\mathcal{A}_k|$, the one consisting of the first m k-sets in the antilexicographic ordering has minimum size of shadow in level ℓ. Here, for sets A and B, A preceds B in the antilexicographic ordering if the largest element in which A and B differ belongs to B. Moreover, for such an optimal family $\mathcal{A}_k$ the shadow $\Delta_\ell(\mathcal{A}_k)$ consists again of the first $|\Delta_\ell(\mathcal{A}_k)|$ ℓ-sets in the antilexicographic ordering.

Let $c^{(n)}_{\ell,k}$ denote the quadratic coefficient in (8).

Proof. By induction on n, the case $n=3$ being easily verified. By the Kruskal–Katona Theorem, we may assume that $\mathcal{A}_k$ is the family of the first $|\mathcal{A}_k|$ k-sets, and $\mathcal{A}_\ell$ is the family of the last $\binom{n}{\ell}-|\Delta_\ell(\mathcal{A}_k)|$ ℓ-sets.

First, if $\mathcal{A}_k=\binom{[n-1]}{k}$ then also $\Delta_\ell(\mathcal{A}_k)=\binom{[n-1]}{\ell}$, and $|\mathcal{A}_\ell|=\binom{n}{\ell}-\binom{n-1}{\ell}=\binom{n-1}{\ell-1}$. In this case (8) holds with equality.

Let us now consider the case $|\mathcal{A}_k|<\binom{n-1}{k}$. Then no set in $\mathcal{A}_k$ contains the element n. If $\mathcal{A}'_\ell\subseteq\mathcal{A}_\ell$ denotes the subfamily of all sets which do not contain n, then $|\mathcal{A}_\ell|=|\mathcal{A}'_\ell|+\binom{n-1}{\ell-1}$, and $\mathcal{A}'_\ell\cup\mathcal{A}_k$ is a 2-level Sperner family in $2^{[n-1]}$. Let ∇' denote the upper shadow operator in $2^{[n-1]}$. We will show that the LHS of (8) will not decrease if $\mathcal{A}_\ell$ is replaced by $\mathcal{A}_\ell\setminus\mathcal{A}'_\ell$ and $\mathcal{A}_k$ is replaced by $\mathcal{A}_k\cup\nabla'_k(\mathcal{A}'_\ell)$, thus reducing the proof to the previous case. The above replacement amounts to adding

$$\frac{|\nabla'_k(\mathcal{A}'_\ell)|}{\binom{n}{k}}-\frac{|\mathcal{A}'_\ell|}{\binom{n}{\ell}}+\frac{\binom{n-1}{\ell-1}\cdot|\nabla'_k(\mathcal{A}'_\ell)|-|\mathcal{A}'_\ell|\cdot|\mathcal{A}_k|}{\binom{n}{\ell}\binom{n}{k}}c^{(n)}_{\ell,k}$$

to the LHS of (8), which is nonnegative if and only if

$$\frac{|\nabla'_k(\mathcal{A}'_\ell)|}{\binom{n-1}{k}}\ge\frac{|\mathcal{A}'_\ell|}{\binom{n-1}{\ell}}\left(1+c^{(n)}_{\ell,k}\frac{|\mathcal{A}_k|}{\binom{n}{k}}\right).$$

Since $|\mathcal{A}_k| \leq \binom{n-1}{k} - |\nabla'_k(\mathcal{A}'_\ell)|$, the last inequality follows from

$$\frac{|\nabla'_k(\mathcal{A}'_\ell)|}{\binom{n-1}{k}} \geq \frac{|\mathcal{A}'_\ell|}{\binom{n-1}{\ell}} \frac{1 + c^{(n)}_{\ell,k} \frac{\binom{n-1}{k}}{\binom{n}{k}}}{1 + c^{(n)}_{\ell,k} \frac{\binom{n-1}{k}}{\binom{n}{k}} \frac{|\mathcal{A}'_\ell|}{\binom{n-1}{\ell}}},$$

which itself follows from the induction hypothesis (the upper shadow version of (QNMP) applied to $\mathcal{A}'_\ell$ and $\nabla'_k(\mathcal{A}'_\ell)$) and

$$c^{(n-1)}_{\ell,k} \geq c^{(n)}_{\ell,k} \frac{\binom{n-1}{k}}{\binom{n}{k}}.$$

The remaining case $|\mathcal{A}_k| > \binom{n-1}{k}$ can be proved analogously, or follows from the previous one by taking complements and observing that $c_{\ell,k} = c_{n-k,n-\ell}$.

To prove uniqueness of the extremal family we remark that the above exchange operations strictly increase the LHS of (8). Thus, if a 2-level Sperner family $\mathcal{A}_\ell \cup \mathcal{A}_k$ attains equality in (8) then $|\mathcal{A}_k| = \binom{n-1}{k}$ and $|\mathcal{A}_\ell| = \binom{n-1}{\ell-1}$. It is known ([17, 27]) that up to permutations the only family $\mathcal{A}_k$ with $|\mathcal{A}_k| = \binom{n-1}{k}$ and $|\Delta_\ell(\mathcal{A}_k)| = \binom{n-1}{\ell}$ is $\mathcal{A}_k = \binom{[n-1]}{k}$. ∎

Note that the coefficients

$$c_{\ell,k} = \frac{n(k-\ell)}{\ell(n-k)}$$

satisfy $(1+c_{\ell,k})(1+c_{k,m}) = 1+c_{\ell,m}$ for all $0<\ell<k<m<n$. Thus, Theorem 1 follows from Lemma 5 and Theorem 3. We should remark that equality holds in (2) only for full levels $\mathcal{A} = \binom{[n]}{i}$, $i = 0, \ldots, n$, and for the optimal 2-level Sperner families from Lemma 5. Indeed, the proof of Theorem 3 and the uniqueness statement of Lemma 5 show that no Sperner family meeting more than two levels attains equality in (2). It is therefore an interesting task to replace the cubic and higher coefficients in (2) by larger ones. Optimal cubic coefficients have been determined, but will be published elsewhere.

4. Some applications

In this section we reprove some known facts from extremal set theory related to Sperner and intersecting set families using Theorem 1. In all cases the linear and certain quadratic terms of the polynomial LYM inequality will suffice to deduce the results.

We start with some implications for intersecting families. Recall that two nonempty families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are called *cross-intersecting* if for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$. A family $\mathcal{A} \subseteq 2^{[n]}$ is called *intersecting* if $\mathcal{A}$ is cross-intersecting with itself.

Theorem 6. *Let $\mathcal{A}_\ell \subseteq \binom{[n]}{\ell}$ and $\mathcal{B}_k \subseteq \binom{[n]}{k}$ be cross-intersecting. Then, if $\ell + k \leq n$,*

$$\frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}} + \frac{|\mathcal{B}_k|}{\binom{n}{k}} + \frac{n(n-\ell-k)}{\ell k} \frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}} \frac{|\mathcal{B}_k|}{\binom{n}{k}} \leq 1.$$

Equality holds for $\ell + k < n$ if and only if $\mathcal{A}_\ell$ and $\mathcal{B}_k$ consist of all ℓ-element resp. k-element sets containing a fixed element.

If $\ell + k < n$, this is just a reformulation of Lemma 5 (note that $L \cap K \neq \emptyset$ iff $L \not\subseteq ([n] \setminus K)$. If $\ell + k = n$, Theorem 6 follows from the fact that at most one set out of each pair of complementary sets can be contained in $\mathcal{A}_\ell \cup \mathcal{B}_k$.

Theorem 7 ([12],[14]). *Let $\mathcal{A} \subseteq 2^{[n]}$ be intersecting. Then*

$$|\mathcal{A}_\ell| \leq \binom{n-1}{\ell-1}$$

for $\ell \leq n/2$, and

$$\frac{n-k}{n} \frac{|\mathcal{A}_\ell|}{\binom{n-1}{\ell-1}} + \frac{|\mathcal{A}_k|}{\binom{n}{k}} \leq 1$$

for $k > n/2$, $\ell + k \leq n$.

The first inequality is the well-known Erdős–Ko–Rado theorem [12]. Both sets of inequalities in the theorem describe the convex hull of all profiles of intersecting families [14], [11, p. 95].

Proof. Let $\ell \leq n/2$. By Theorem 6 (with $\mathcal{A}_\ell = \mathcal{B}_k$),

$$\left(1 + \frac{n(n-2\ell)}{\ell^2} \frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}}\right)^2 \leq 1 + \frac{n(n-2\ell)}{\ell^2},$$

from which the first inequality easily follows.

Let $k > n/2$ and $\ell + k \leq n$. Since $|\mathcal{A}_\ell| \leq \binom{n-1}{\ell-1}$, the second inequality is obviously true if $|\mathcal{A}_k|/\binom{n}{k} < k/n$. If $|\mathcal{A}_k| \geq \binom{n-1}{k-1}$ then the LHS of the inequality we want to prove is not greater than the LHS of the quadratic inequality in Theorem 6. ∎

We continue with some applications for Sperner families. We will use the following weaker version of the PLYM inequality: For every Sperner family $\mathcal{A}\subseteq 2^{[n]}$ and every (real) number s we have

$$\begin{aligned} 1 &\geq \sum_i \frac{|\mathcal{A}_i|}{\binom{n}{i}} + \sum_{\ell\leq s\leq k} \frac{k-\ell}{n} \frac{|\mathcal{A}_\ell|}{\binom{n-1}{\ell-1}} \frac{|\mathcal{A}_k|}{\binom{n-1}{k}} \\ &= \sum_{\ell<s} \frac{|\mathcal{A}_\ell|}{\binom{n-1}{\ell-1}} \left(\frac{\ell}{n} + \sum_{k\geq s} \frac{s-\ell}{n} \frac{|\mathcal{A}_k|}{\binom{n-1}{k}} \right) \\ &\qquad + \sum_{k\geq s} \frac{|\mathcal{A}_k|}{\binom{n-1}{k}} \left(\frac{n-k}{n} + \sum_{\ell\leq s} \frac{k-s}{n} \frac{|\mathcal{A}_\ell|}{\binom{n-1}{\ell-1}} \right). \end{aligned} \tag{9}$$

Theorem 8 ([15]). *Let $\mathcal{A}\subseteq 2^{[n]}$ be a Sperner family and let s be the smallest integer k for which*

$$\sum_{i\leq k} \frac{|\mathcal{A}_i|}{\binom{n-1}{i-1}} > 1 \tag{10}$$

holds. Then

$$\sum_{i<s} \frac{s}{i} \frac{|\mathcal{A}_i|}{\binom{n}{i}} + \sum_{i\geq s} \frac{n-s}{n-i} \frac{|\mathcal{A}_i|}{\binom{n}{i}} \leq 1\,.$$

Proof. Write the claimed inequality as

$$\sum_{i<s} \frac{s}{n} \frac{|\mathcal{A}_i|}{\binom{n-1}{i-1}} + \sum_{i\geq s} \frac{n-s}{n} \frac{|\mathcal{A}_i|}{\binom{n-1}{i}} \leq 1\,. \tag{11}$$

By the choice of s the first sum is not larger than s/n. Thus, we may assume that

$$\sum_{i\geq s} \frac{|\mathcal{A}_i|}{\binom{n-1}{i}} > 1\,. \tag{12}$$

But then (11) follows from (9) using the estimations (10) and (12). ∎

Theorem 9 ([7]). *Let $\mathcal{A}\subseteq 2^{[n]}$ be a Sperner family, and let s be a real number. Put*

$$bot(\mathcal{A}) = \sum_{i<s} \frac{|\mathcal{A}_i|}{\binom{n-1}{i-1}} + \frac{n-s}{n} \frac{|\mathcal{A}_s|}{\binom{n-1}{s-1}}\,, \quad top(\mathcal{A}) = \sum_{i>s} \frac{|\mathcal{A}_i|}{\binom{n-1}{i}} + \frac{s}{n} \frac{|\mathcal{A}_s|}{\binom{n-1}{s}}$$

(where $|\mathcal{A}_s|=0$ if s is not an integer).

Then $bot(\mathcal{A})\leq 1$ or $top(\mathcal{A})\leq 1$.

Proof. Assume that both $\mathrm{bot}(\mathcal{A}) > 1$ and $\mathrm{top}(\mathcal{A}) > 1$. Then, by lower estimating both inner sums in (9),

$$\begin{aligned}1 > \sum_{\ell<s} \frac{|\mathcal{A}_\ell|}{\binom{n-1}{\ell-1}} \left(\frac{s}{n} + \frac{s-\ell}{n} \frac{|\mathcal{A}_s|}{\binom{n}{s}} \right) \\ + \sum_{k\geq s} \frac{|\mathcal{A}_k|}{\binom{n-1}{k}} \left(\frac{n-s}{n} + \frac{(n-s)-(n-k)}{n} \frac{|\mathcal{A}_s|}{\binom{n}{s}} \right) \\ = \left(1 + \frac{|\mathcal{A}_s|}{\binom{n}{s}}\right) \left(\frac{s}{n} \sum_{\ell<s} \frac{|\mathcal{A}_\ell|}{\binom{n-1}{\ell-1}} + \frac{n-s}{n} \sum_{k\geq s} \frac{|\mathcal{A}_k|}{\binom{n-1}{k}} \right) - \frac{|\mathcal{A}_s|}{\binom{n}{s}} \sum_i \frac{|\mathcal{A}_i|}{\binom{n}{i}} .\end{aligned}$$

Lower estimating further with $\mathrm{bot}(\mathcal{A}) > 1$, $\mathrm{top}(\mathcal{A}) > 1$ and (1) yields the contradiction $1>1$. ∎

In [7] this theorem is stated for $s=n/2$. The proof method presented there even yields that Theorem 9 remains true if the factors $(n-s)/n$ and s/n in $\mathrm{bot}(\mathcal{A})$ and $\mathrm{top}(\mathcal{A})$ are replaced by arbitrary nonnegative real numbers whose sum is 1.

Theorem 9 implies that the maximum size of a *complemented* Sperner family $\mathcal{A} \subseteq 2^{[n]}$ (i.e. $A \in \mathcal{A}$ implies $[n] \setminus A \in \mathcal{A}$) is given by $2\binom{n-1}{\lfloor n/2\rfloor-1}$. The following result shows that the maximum size of a *complement-free* Sperner family $\mathcal{A} \subseteq 2^{[n]}$ (i.e. $A \in \mathcal{A}$ implies $[n] \setminus A \notin \mathcal{A}$) is given by $\binom{n}{n/2-1}$ if n is even.

Theorem 10 **([5]).** *Let $\mathcal{A} \subseteq 2^{[n]}$ be a Sperner family with $|\mathcal{A}_{n/2}| \leq \frac{1}{2}\binom{n}{n/2}$. Then*

$$\sum_{i\neq n/2} \frac{|\mathcal{A}_i|}{\binom{n}{i}} + \frac{|\mathcal{A}_{n/2}|}{\binom{n}{n/2-1}} \leq 1 .$$

Proof. If n is odd this is the LYM inequality. Let n be even. We may assume that

$$\sum_{i\neq n/2} \frac{|\mathcal{A}_i|}{\binom{n}{i}} > 1 - \frac{1}{2} \frac{\binom{n}{n/2}}{\binom{n}{n/2-1}} = \frac{1}{2} - \frac{1}{n} .$$

But then, with Theorem 1,

$$\begin{aligned}1 - \sum_{i\neq n/2} \frac{|\mathcal{A}_i|}{\binom{n}{i}} \geq \frac{|\mathcal{A}_{n/2}|}{\binom{n}{n/2}} \left(1 + \sum_{\ell<n/2} \left(\frac{n}{\ell} - 2\right) \frac{|\mathcal{A}_\ell|}{\binom{n}{\ell}} + \sum_{k>n/2} \left(\frac{n}{n-k} - 2\right) \frac{|\mathcal{A}_k|}{\binom{n}{k}} \right) \\ \geq \frac{|\mathcal{A}_{n/2}|}{\binom{n}{n/2}} \left(1 + \frac{4}{n-2} \left(\frac{1}{2} - \frac{1}{n} \right) \right) = \frac{|\mathcal{A}_{n/2}|}{\binom{n}{n/2-1}} .\end{aligned}$$

∎

5. The linear lattice

We will establish a q-analogue of the PLYM inequality in Theorem 1 by using a linear algebraic approach. Recall that ranked poset P is called regular if for all i and $p \in P_i$ the numbers $|\Delta_{i-1}(p)|$ and $|\nabla_{i+1}(p)|$ depend only on i. For a regular rank 1 poset $P = P_0 \cup P_1$ we set $d_0 = |\nabla(p)|$ resp. $d_1 = |\Delta(p)|$ for $p \in P_0$ resp. $p \in P_1$, and we call d_0 and d_1 the degrees of P. The adjacency matrix of a rank 1 poset is the adjacency matrix of its bipartite Hasse-graph. Furthermore, P is called connected if its Hasse-graph is so.

An easy double counting argument shows that a regular poset is a LYM poset. The following result establishes a quadratic LYM inequality for regular rank 1 posets. It can be considered as a bipartite version of Hoffman's bound on the largest coclique in a regular graph (cf. [11, Cor. 6.4.1], [20, Thm. 2.1.4.ii]). A slightly more general result was obtained in [20, Thm. 3.1.4] using eigenvalue interlacing.

Proposition 11. *Let $P = P_0 \cup P_1$ be a regular connected rank 1 poset with degrees d_0, d_1. Let λ denote the second largest eigenvalue of the adjacency matrix of P. If $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1$ is a 2-level Sperner family in P, then*

$$\frac{|\mathcal{A}_0|}{W_0} + \frac{|\mathcal{A}_1|}{W_1} + \left(\frac{d_0 d_1}{\lambda^2} - 1\right) \frac{|\mathcal{A}_0|}{W_0} \frac{|\mathcal{A}_1|}{W_1} \leq 1.$$

Proof. Let A be the adjacency matrix of P. We have

$$A = \begin{pmatrix} 0 & W \\ W^\top & 0 \end{pmatrix},$$

where W is the submatrix of A whose rows resp. columns are indexed P_0 resp. P_1. Note that the nonzero eigenvalues of A are exactly the (positive and negative) square roots of the nonzero eigenvalues of $WW^\top$, with equal multiplicities correspondingly. We denote the eigenvalues of the latter matrix by $\lambda_1^2, \lambda_2^2, \ldots$, with $\lambda_1 > \lambda_2 > \cdots \geq 0$. Since P is regular and connected, the largest eigenvalue is $\lambda_1^2 = d_0 d_1$, and the corresponding eigenspace is one-dimensional and generated by the all one vector. In particular, $\lambda_2 = \lambda$ is the second largest eigenvalue.

Now let φ be the characteristic row vector of $\mathcal{A}_0$ (of length W_0). We have

$$\varphi W (\varphi W)^\top = \sum_{p \in P_1} |\Delta_0(p) \cap \mathcal{A}_0|^2 .$$

Here nonzero summands occur only for $p \in P_1 \setminus \mathcal{A}_1$ since $\mathcal{A}$ is a Sperner family. Applying the Cauchy–Schwarz inequality therefore yields

$$\varphi W(\varphi W)^\top \geq \frac{\left(\sum_{p\in P_1\setminus\mathcal{A}_1} |\Delta_0(p)\cap\mathcal{A}_0|\right)^2}{W_1-|\mathcal{A}_1|} = \frac{|\mathcal{A}_0|^2\, d_0^2}{W_1-|\mathcal{A}_1|} = d_0 d_1 \frac{|\mathcal{A}_0|^2/W_0}{1-|\mathcal{A}_1|/W_1}\,.$$

On the other hand, if $\varphi=\varphi_1+\varphi_2+\dots$ is the orthogonal decomposition of φ according to the eigenspaces of $WW^\top$, then

$$\varphi WW^\top\varphi^\top = \lambda_1^2\varphi_1\varphi_1^\top + \sum_{i\geq 2}\lambda_i^2\varphi_i\varphi_i^\top \leq d_0d_1\varphi_1\varphi_1^\top + \lambda^2(|\mathcal{A}_0| - \varphi_1\varphi_1^\top)\,.$$

Since φ_1 is proportional to the all one vector and since all other eigenvectors φ_i, $i\geq 2$, are orthogonal to φ_1, the sum of all entries of φ_1 equals the sum of all entries of φ which is $|\mathcal{A}_0|$. Thus, φ_1 is the constant vector with entry $|\mathcal{A}_0|/W_0$, and $\varphi_1\varphi_1^\top = |\mathcal{A}_0|^2/|W_0|$. It follows

$$\varphi WW^\top\varphi^\top \leq \lambda^2\,|\mathcal{A}_0|\left(\frac{|\mathcal{A}_0|}{W_0}\left(\frac{d_0d_1}{\lambda^2}-1\right)+1\right).$$

Combining the upper and lower estimation of $\varphi W(\varphi W)^\top$ yields the inequality of the proposition. ∎

Now let V_n be an n-dimensional linear space over $GF(q)$. We denote the set resp. number of all i-dimensional subspaces of V_n by $\begin{bmatrix} V_n \\ i \end{bmatrix}$ resp. $\begin{bmatrix} n \\ i \end{bmatrix}$.

Lemma 12. *Let $0<\ell<k<n$. If $\mathcal{A}=\mathcal{A}_\ell\cup\mathcal{A}_k$ is a Sperner family, $\mathcal{A}_\ell\subseteq\begin{bmatrix} V_n \\ \ell \end{bmatrix}$, $\mathcal{A}_k\subseteq\begin{bmatrix} V_n \\ k \end{bmatrix}$, then*

$$\frac{|\mathcal{A}_\ell|}{\begin{bmatrix} n \\ \ell \end{bmatrix}} + \frac{|\mathcal{A}_k|}{\begin{bmatrix} n \\ k \end{bmatrix}} + \frac{(q^n-1)(q^k-q^\ell)}{(q^\ell-1)(q^n-q^k)}\,\frac{|\mathcal{A}_\ell|}{\begin{bmatrix} n \\ \ell \end{bmatrix}}\,\frac{|\mathcal{A}_k|}{\begin{bmatrix} n \\ k \end{bmatrix}} \leq 1.$$

The following arguments also hold if $q\to 1$, thus yielding a new proof of Lemma 5.

Proof. First note that the quadratic coefficient in the above inequality does not change if ℓ and k are replaced by $n-k$ and $n-\ell$. Thus, using orthogonal complements, we may assume that $\ell\leq n/2$ and $\ell+k\leq n$.

We apply Proposition 11 to the rank 1 poset $P=\begin{bmatrix} V_n \\ \ell \end{bmatrix}\cup\begin{bmatrix} V_n \\ k \end{bmatrix}$. It is regular with degrees $\begin{bmatrix} k \\ \ell \end{bmatrix}$ and $\begin{bmatrix} n-\ell \\ k-\ell \end{bmatrix}$. We will show that the second largest eigenvalue λ of P is given by

$$\lambda^2 = \begin{bmatrix} k-1 \\ \ell-1 \end{bmatrix}\begin{bmatrix} n-\ell-1 \\ n-k-1 \end{bmatrix} q^{k-\ell}\,,$$

which shows the inequality of the lemma.

For $0 \le i,j \le n$ let $W_{i,j}$ be the $\left[{n \atop i}\right] \times \left[{n \atop j}\right]$ matrix with rows indexed by $\left[{V_n \atop i}\right]$, columns indexed by $\left[{V_n \atop j}\right]$, and entries $0,1$, where for $I \subseteq \left[{V_n \atop i}\right]$, $J \subseteq \left[{V_n \atop j}\right]$ it holds $(W_{i,j})_{I,J} = 1$ iff $I \subseteq J$.

We look for the second largest eigenvalue of the matrix $W_{\ell,k}W_{\ell,k}^{\top}$. This matrix lies in the Bose–Mesner algebra of the Grassmann scheme $J_q(n,\ell)$, which makes the determination of all its eigenvalues easy. They are given by

$$\begin{bmatrix} k-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n-\ell-j \\ n-k-j \end{bmatrix} q^{j(k-\ell)}, \quad j = 0,\dots,\ell,$$

with corresponding eigenspaces

$$\mathrm{Ker}(W_{j-1,j}^{\top})W_{j,\ell}, \quad j = 0,\dots,\ell,$$

where $\mathrm{Ker}(W_{-1,0}^{\top}) := \mathbb{R}$. Indeed, using the identities (cf. [16])

$$W_{j,\ell}W_{\ell,k} = \begin{bmatrix} k-j \\ \ell-j \end{bmatrix} W_{j,k} \quad \text{for } j \le \ell \le k,$$

$$W_{j,k}W_{\ell,k}^{\top} = \sum_{i=0}^{j} \begin{bmatrix} n-\ell-j \\ n-k-i \end{bmatrix} q^{i(k-\ell+i-j)} W_{i,j}^{\top} W_{i,\ell} \quad \text{for } j \le \ell \le k$$

(where the latter one follows from a q-analogue of Vandermonde's identity, see (4.4) in [16]), we find for fixed j, $0 \le j \le \ell$, and $e \in \mathrm{Ker}(W_{j-1,j}^{\top})W_{j,\ell}$, say $e = fW_{j,\ell}$ with $fW_{j-1,j}^{\top} = 0$ ($f = 1$ if $j = 0$) that

$$eW_{\ell,k}W_{\ell,k}^{\top} = fW_{j,\ell}W_{\ell,k}W_{\ell,k}^{\top} = \begin{bmatrix} k-j \\ \ell-j \end{bmatrix} fW_{j,k}W_{\ell,k}^{\top}$$
$$= \begin{bmatrix} k-j \\ \ell-j \end{bmatrix} \begin{bmatrix} n-\ell-j \\ n-k-j \end{bmatrix} q^{j(k-\ell)} e.$$

A straightforward calculation shows that the above eigenvalues of $W_{\ell,k}W_{\ell,k}^{\top}$ are strictly decreasing in j, the second largest thus attained for $j = 1$. Further, since all eigenvalues are nonzero, $W_{\ell,k}W_{\ell,k}^{\top}$ and hence also $W_{\ell,k}$ have full row rank ([21]). Thus, the dimension of the eigenspace $\mathrm{Ker}(W_{j-1,j}^{\top})W_{j,\ell}$ is $\left[{n \atop j}\right] - \left[{n \atop j-1}\right]$. Since $\sum_{j=0}^{\ell} \left[{n \atop j}\right] - \left[{n \atop j-1}\right] = \left[{n \atop \ell}\right]$, we have given all eigenvalues of the matrix $W_{\ell,k}W_{\ell,k}^{\top}$. ∎

Observe that the quadratic coefficients

$$c_{\ell,k} = \frac{(q^n-1)(q^k-q^\ell)}{(q^\ell-1)(q^n-q^k)}$$

satisfy $(1+c_{\ell,k})(1+c_{k,m})=1+c_{\ell,m}$ for all $0<\ell<k<m<n$. Hence, Lemma 12 and Theorem 3 give the following q-analogue of Theorem 1.

Theorem 13. *Let $\mathcal{A}\subseteq V_n$ be a Sperner family. Then we have*

$$\sum_{i=0}^{n}\frac{|\mathcal{A}_i|}{\left[{n \atop i}\right]}+\sum_{I=\{i_1,\dots,i_s\}\subseteq[n-1],s\geq 2}\left(\prod_{j=1}^{s-1}\frac{(q^n-1)(q^{i_{j+1}}-q^{i_j})}{(q^{i_j}-1)(q^n-q^{i_{j+1}})}\right)\left(\prod_{i\in I}\frac{|\mathcal{A}_i|}{\left[{n \atop i}\right]}\right)\leq 1.$$

We remark that all applications of Theorem 1 presented in Section 4 admit q-analogues.

6. Multisets

Let $\alpha_1\geq\alpha_2\geq\cdots\geq\alpha_n$ be positive integers. The set $S(\alpha_1,\dots,\alpha_n)$ of all n–tuples $x=(x_1,\dots,x_n)$ of integers x_i satisfying $0\leq x_i\leq\alpha_i$, $i=1,2,\dots,n$, is equipped with the partial order given by $(x_1,\dots,x_n)\leq(y_1,\dots,y_n)$ iff $x_i\leq y_i$ for all i. With this order, $S(\alpha_1,\dots,\alpha_n)$ is the product of the n chains $0\lessdot 1\lessdot\cdots\lessdot\alpha_i$, $i=1,\dots,n$. It can be regarded as the set of multisubsets of a multiset of $\alpha_1+\alpha_2+\cdots+\alpha_n$ elements, where α_i elements are of type i. The k-th level resp. Whitney number of $S(\alpha_1,\dots,\alpha_n)$ is denoted by $S_k(\alpha_1,\dots,\alpha_n)$ resp. $W_k(\alpha_1,\dots,\alpha_n)$.

It was proved in [2] that $S(\alpha_1,\dots,\alpha_n)$ is a LYM poset. This follows also from the LYM product theorem (cf. [11, Thm. 4.6.2]). We will establish a polynomial LYM inequality only in the special case where $\alpha_1=\alpha_2=\cdots=\alpha_n=\alpha$. Let α^n be a short hand notation for $\alpha,\dots,\alpha$ (n times α).

We again first establish a quadratic LYM inequality for 2-level Sperner families. Our determination of the coefficients $c_{\ell,k}$ is similar as in the Boolean case. Here we will use the Clements–Lindström Theorem [6], which solves the shadow minimization problem in $S(\alpha_1,\dots,\alpha_n)$: Among all families $\mathcal{A}_k\subseteq S_k(\alpha_1,\dots,\alpha_n)$ of the same cardinality $m=|\mathcal{A}_k|$, the one consisting of the first m elements of the k-th level in the antilexicographic ordering has minimum size of shadow in level ℓ. Antilexicographic means that $(x_1,\dots,x_n)$ preceds $(y_1,\dots,y_n)$ iff $x_i<y_i$ for the largest i with $x_i\neq y_i$.

Note that there might not exist a 2-level Sperner family on levels ℓ and k in $S(\alpha^n)$. Using the Clements–Lindström Theorem it is easily seen that this is the case if and only if $\ell=0$ or $k=\alpha n$ or $k-\ell\geq(n-1)\alpha$. Otherwise, if there is a 2-level Sperner family on levels ℓ and k, then there exists a $0\leq\beta<\alpha$ such that $\{x\in S_\ell(\alpha^n):x_n>\beta\}\cup\{x\in S_k(\alpha^n):x_n\leq\beta\}$ is a 2-level Sperner family too. Thus, in this case, by applying (7) from Remark 4 we see that the quadratic coefficient $c_{\ell,k}$ in (QLYM) satisfies

$$c_{\ell,k} \le \min_{\beta} \left\{ \left(\frac{W_k(\alpha^n)}{W_k(\alpha^{n-1},\beta)} - 1 \right) \Big/ \left(\frac{W_\ell(\alpha^n)}{W_\ell(\alpha^{n-1},\beta)} - 1 \right) - 1 \right\}, \tag{13}$$

where the minimum is taken over all $0 \le \beta < \alpha$ such that $W_k(\alpha^{n-1},\beta) \neq 0$ and $W_\ell(\alpha^{n-1},\beta) \neq W_\ell(\alpha^n)$.

Lemma 14. *Let $\mathcal{A} = \mathcal{A}_\ell \cup \mathcal{A}_k$ be a 2-level Sperner family in $S(\alpha^n)$, $n \ge 2$, and let $c_{\ell,k}$ be given by the RHS of (13). Then we have*

$$\frac{|\mathcal{A}_\ell|}{W_\ell(\alpha^n)} + \frac{|\mathcal{A}_k|}{W_k(\alpha^n)} + c_{\ell,k} \frac{|\mathcal{A}_\ell|}{W_\ell(\alpha^n)} \frac{|\mathcal{A}_k|}{W_k(\alpha^n)} \le 1. \tag{14}$$

Proof. The proof is by induction on n. We write again $c_{\ell,k}^{(n)}$ in order to specify n.

We will use in the induction step the existence of the positive coefficients $c_{\ell,k}^{(n-1)}$. Let us therefore first show that $c_{\ell,k}$ is positive. Indeed, $c_{\ell,k} > 0$ is equivalent to

$$\frac{W_k(\alpha^n)}{W_\ell(\alpha^n)} > \frac{W_k(\alpha^{n-1},\beta)}{W_\ell(\alpha^{n-1},\beta)},$$

or, using $W_i(\alpha_1,\dots,\alpha_n) = \sum_{j=0}^{\alpha_n} W_{i-j}(\alpha_1,\dots,\alpha_{n-1})$, to

$$\frac{\sum_{i=\beta+1}^{\alpha} W_{k-i}(\alpha^{n-1})}{\sum_{i=\beta+1}^{\alpha} W_{\ell-i}(\alpha^{n-1})} > \frac{\sum_{i=0}^{\beta} W_{k-i}(\alpha^{n-1})}{\sum_{i=0}^{\beta} W_{\ell-i}(\alpha^{n-1})}.$$

The last inequality however follows from the strict logarithmic concavity of the Whitney numbers of $S(\alpha^{n-1})$:

$$\frac{W_{k-i}(\alpha^{n-1})}{W_{\ell-i}(\alpha^{n-1})} > \frac{W_{k-j}(\alpha^{n-1})}{W_{\ell-j}(\alpha^{n-1})} \quad \text{for} \quad i > j.$$

Let us now prove inequality (14). Note that (14) holds for all Sperner families

$$\mathcal{A}_\ell \cup \mathcal{A}_k = \{x \in S_\ell(\alpha^n) : x_n > \beta\} \cup \{x \in S_k(\alpha^n) : x_n \le \beta\}, \quad 0 \le \beta < \alpha, \tag{15}$$

by our choice of $c_{\ell,k}$.

In general, by the Clements–Lindström Theorem, we may assume that $\mathcal{A}_k$ is the family of the first $|\mathcal{A}_k|$ elements of $S_k(\alpha^n)$, and $\mathcal{A}_\ell$ is the family of the last $W_\ell(\alpha^n) - |\Delta_\ell(\mathcal{A}_k)|$ elements of $S_\ell(\alpha^n)$.

For $n = 2$ there is then nothing to show, since every such 2-level Sperner family is one of the families in (15).

Let $\beta \geq 0$ be the smallest integer such that $|\mathcal{A}_k| \leq W_k(\alpha^{n-1}, \beta)$, and set $\mathcal{A}_k^{(\beta)} = \{x \in \mathcal{A}_k : x_n = \beta\}$ and $\mathcal{A}_\ell^{(\beta)} = \{x \in \mathcal{A}_\ell : x_n = \beta\}$. We will use the notations $W_i = W_i(\alpha^n)$, $W_i^{(\gamma)} = W_{i-\gamma}(\alpha^{n-1})$, $W_i^{(<\gamma)} = W_i(\alpha^{n-1}, \gamma-1)$, $W_i^{(>\gamma)} = W_{i-\gamma-1}(\alpha^{n-1}, \alpha-\gamma-1)$. By our choice of $\mathcal{A}_\ell \cup \mathcal{A}_k$ we have $|\mathcal{A}_k| = |\mathcal{A}_k^{(\beta)}| + W_k^{(<\beta)}$ and $|\mathcal{A}_\ell| = |\mathcal{A}_\ell^{(\beta)}| + W_\ell^{(>\beta)}$. If $\mathcal{A}_\ell^{(\beta)} = \emptyset$ then $\mathcal{A}_\ell \cup \mathcal{A}_k$ is one of the families in (15). Let $\mathcal{A}_\ell^{(\beta)} \neq \emptyset$. We get a 2-level Sperner family in $S(\alpha^{n-1})$ by deleting the n-th coordinate in all elements of $\mathcal{A}_\ell^{(\beta)} \cup \mathcal{A}_k^{(\beta)}$. By induction, we have

$$\frac{|\mathcal{A}_\ell^{(\beta)}|}{W_\ell^{(\beta)}} + \frac{|\mathcal{A}_k^{(\beta)}|}{W_k^{(\beta)}} + c_{\ell-\beta,k-\beta}^{(n-1)} \frac{|\mathcal{A}_\ell^{(\beta)}|}{W_\ell^{(\beta)}} \frac{|\mathcal{A}_k^{(\beta)}|}{W_k^{(\beta)}} \leq 1. \tag{16}$$

We want to show that

$$\frac{|\mathcal{A}_\ell^{(\beta)}| + W_\ell^{(>\beta)}}{W_\ell} + \frac{|\mathcal{A}_k^{(\beta)}| + W_k^{(<\beta)}}{W_k} + c_{\ell,k}^{(n)} \frac{\left(|\mathcal{A}_\ell^{(\beta)}| + W_\ell^{(>\beta)}\right)}{W_\ell} \frac{\left(|\mathcal{A}_k^{(\beta)}| + W_k^{(<\beta)}\right)}{W_k} \leq 1.$$

Using the upper estimation of $|\mathcal{A}_\ell^{(\beta)}||\mathcal{A}_k^{(\beta)}|$ in (16), the last inequality will follow from

$$\begin{aligned} &\frac{|\mathcal{A}_\ell^{(\beta)}|}{W_\ell^{(\beta)}} \left(\frac{W_\ell^{(\beta)}}{W_\ell} + c_{\ell,k}^{(n)} \frac{W_\ell^{(\beta)}}{W_\ell} \frac{W_k^{(<\beta)}}{W_k} - \frac{c_{\ell,k}^{(n)}}{c_{\ell-\beta,k-\beta}^{(n-1)}} \frac{W_\ell^{(\beta)}}{W_\ell} \frac{W_k^{(\beta)}}{W_k} \right) \\ &\quad + \frac{|\mathcal{A}_k^{(\beta)}|}{W_k^{(\beta)}} \left(\frac{W_k^{(\beta)}}{W_k} + c_{\ell,k}^{(n)} \frac{W_\ell^{(>\beta)}}{W_\ell} \frac{W_k^{(\beta)}}{W_k} - \frac{c_{\ell,k}^{(n)}}{c_{\ell-\beta,k-\beta}^{(n-1)}} \frac{W_\ell^{(\beta)}}{W_\ell} \frac{W_k^{(\beta)}}{W_k} \right) \\ &\quad \leq 1 - \frac{W_\ell^{(>\beta)}}{W_\ell} - \frac{W_k^{(<\beta)}}{W_k} - c_{\ell,k}^{(n)} \frac{W_\ell^{(>\beta)}}{W_\ell} \frac{W_k^{(<\beta)}}{W_k} - \frac{c_{\ell,k}^{(n)}}{c_{\ell-\beta,k-\beta}^{(n-1)}} \frac{W_\ell^{(\beta)}}{W_\ell} \frac{W_k^{(\beta)}}{W_k}. \end{aligned}$$

Using that (14) holds for all families in (15), we see that the RHS of the last inequality is not less than the coefficient of $|\mathcal{A}_\ell^{(\beta)}|/W_\ell^{(\beta)}$ and of $|\mathcal{A}_k^{(\beta)}|/W_k^{(\beta)}$ on the LHS. Hence, this inequality follows from (LYM) applied to the Sperner family $\mathcal{A}_\ell^{(\beta)} \cup \mathcal{A}_k^{(\beta)} \subseteq S(\alpha^{n-1})$. ∎

Remark 15. For general $S(\alpha_1, \ldots, \alpha_n)$ there are 2-level Sperner families for which we have equality in (LYM) (cf. [10, 19]). The corresponding levels give zero coefficients $c_{\ell,k}$, which make it impossible to apply the above argument. In fact, the statement of the lemma is false for general $S(\alpha_1, \ldots, \alpha_n)$, also in cases in which (LYM) is strict for all 2-level Sperner families. For example, in $S(4,3,2)$ with $\ell = 3$, $k = 4$, the two families in (15) yield the coefficients

2/5 (for $\beta=0$) and 5/16 (for $\beta=1$), whereas the family $\mathcal{A}_3\cup\mathcal{A}_4=(S(4,3,2)\setminus\{(3,0,0)\})\cup\{(4,0,0)\}$ yields the optimal coefficient $c_{3,4}=1/4$.

We note that for $k-\ell>1$ the optimal coefficients $c_{\ell,k}$ in Lemma 14 are in general strictly larger than those given by (3) in Theorem 3. For example, in $S(3,3,3)$ we have $1+c_{2,3}=3/2$ (attained for $\beta=0$), $1+c_{3,4}=5/3$ (for $\beta=1$) and $1+c_{2,4}=3$ (for $\beta=0$). Nevertheless, Lemma 14 and Theorem 3 yield the following polynomial LYM inequality for $S(\alpha^n)$ which generalizes Theorem 1.

Theorem 16. *Let $n\geq 2$, $\alpha\geq 1$, and*

$$c_{i-1,i}=\min_{0\leq\beta<\alpha}\left\{\left(\frac{W_i(\alpha^n)}{W_i(\alpha^{n-1},\beta)}-1\right)\Big/\left(\frac{W_{i-1}(\alpha^n)}{W_{i-1}(\alpha^{n-1},\beta)}-1\right)-1\right\}$$

for all $1<i<n\alpha$. Furthermore, let $c_{\ell,k}$ for $0<\ell<k<n\alpha$, $k-\ell>1$, be given by (3), and c_I for $I\subseteq[1,n\alpha-1]$, $|I|\geq 3$, by (4).

If $\mathcal{A}\subseteq S(\alpha^n)$ is a Sperner family, then

$$\sum_{i=0}^{n\alpha}\frac{|\mathcal{A}_i|}{W_i(\alpha^n)}+\sum_{I\subseteq[1,n\alpha-1],|I|\geq 2}c_I\left(\prod_{i\in I}\frac{|\mathcal{A}_i|}{W_i(\alpha^n)}\right)\leq 1\,.$$

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