Graphs and Combinatorics (2005) 21:161–168
Digital Object Identifier (DOI) 10.1007/s00373-004-0598-4

Graphs and Combinatorics

On Cross-intersecting Families of Sets

Christian Bey

Fakultät für Mathematik, Otto-von-Guericke-Universität, Universitätsplatz 2, 39106 Magdeburg, Germany. e-mail: christian.bey@mathematik.uni-magdeburg.de

Abstract. A family $\mathscr{A}$ of ℓ-element subsets and a family $\mathscr{B}$ of k-element subsets of an n-element set are cross-intersecting if every set from $\mathscr{A}$ has a nonempty intersection with every set from $\mathscr{B}$. We compare two previously established inequalities each related to the maximization of the product $|\mathscr{A}||\mathscr{B}|$, and give a new and short proof for one of them. We also determine the maximum of $|\mathscr{A}|\omega_\ell + |\mathscr{B}|\omega_k$ for arbitrary positive weights ω_ℓ, ω_k.

Key words. Erdős-Ko-Rado Theorem, Cross-intersecting families, Quadratic LYM inequality

1. Introduction

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. A set family $\mathscr{A} \subseteq \binom{[n]}{k}$ is called intersecting if $A \cap B \neq \emptyset$ for all $A, B \in \mathscr{A}$. One of the basic results in extremal set theory is the Erdős-Ko-Rado Theorem:

Theorem 1 ([4]). *Let $\mathscr{A} \subseteq \binom{[n]}{k}$ be intersecting. Then, if $k \leq n/2$,*

$$|\mathscr{A}| \leq \binom{n-1}{k-1}.$$

Equality holds for $k < n/2$ if and only if $\mathscr{A}$ consists of all k-element subsets of $[n]$ containing a fixed element.

Several generalizations of this result are known, among others to pairs of set families. Two families $\mathscr{A}, \mathscr{B} \subseteq 2^{[n]}$ are called cross-intersecting if both $\mathscr{A}$ and $\mathscr{B}$ are non-empty and for all $A \in \mathscr{A}$ and $B \in \mathscr{B}$ we have $A \cap B \neq \emptyset$.

The following generalization of the Erdős-Ko-Rado Theorem was obtained by Matsumoto and Tokushige [11].

Theorem 2 ([11]). *Let $\mathscr{A} \subseteq \binom{[n]}{\ell}$ and $\mathscr{B} \subseteq \binom{[n]}{k}$ be cross-intersecting. Then, if $\ell, k \leq n/2$,*

$$|\mathscr{A}| \cdot |\mathscr{B}| \leq \binom{n-1}{\ell-1}\binom{n-1}{k-1}.$$

Equality holds for $\ell + k < n$ *if and only if* $\mathscr{A}$ and $\mathscr{B}$ *consist of all* ℓ*-element resp.* k*-element sets containing a fixed element.*

This result extends earlier work of Pyper [12] who proved the same if $\ell = k$ or $\ell < k$ and $2k + \ell - 2 \leq n$.

It is known that the inequality in Theorem 2 does not hold for all ℓ, k with $\ell + k \leq n$, see the examples given in [11, 7].

Another generalization of the Erdős-Ko-Rado Theorem also involving the product of two cross-intersecting families was given in [2].

Theorem 3 ([2]). *Let* $\mathscr{A} \subseteq \binom{[n]}{\ell}$ *and* $\mathscr{B} \subseteq \binom{[n]}{k}$ *be cross-intersecting. Then, if* $\ell + k \leq n$,

$$\frac{|\mathscr{A}|}{\binom{n}{\ell}} + \frac{|\mathscr{B}|}{\binom{n}{k}} + \frac{n(n-\ell-k)}{\ell k}\frac{|\mathscr{A}|}{\binom{n}{\ell}}\frac{|\mathscr{B}|}{\binom{n}{k}} \leq 1.$$

Equality holds for $\ell + k < n$ *if and only if* $\mathscr{A}$ and $\mathscr{B}$ *consist of all* ℓ*-element resp.* k*-element sets containing a fixed element.*

A straightforward calculation shows that Theorem 2 yields a slightly stronger inequality than Theorem 3 exactly for cross-intersecting families satisfying $\binom{n-1}{\ell-1} < |\mathscr{A}| < \frac{k}{\ell}\binom{n-1}{\ell-1}$ and $\frac{\ell}{k}\binom{n-1}{k-1} < |\mathscr{B}| < \binom{n-1}{k-1}$ (here we assume $\ell < k$). Thus none of the two theorems implies the other. However, we show in this note how our Theorem 3 can be used to give a new and short proof of the Matsumoto-Tokushige Theorem 2.

Both of the above quadratic inequalities have the same equality characterization. We show that maximizing only the linear term $|\mathscr{A}|/\binom{n}{\ell} + |\mathscr{B}|/\binom{n}{k}$ yields to a different optimal configuration. Actually, we have the following more general result.

Theorem 4. *Let* $\mathscr{A} \subseteq \binom{[n]}{\ell}$ *and* $\mathscr{B} \subseteq \binom{[n]}{k}$ *be cross-intersecting, and let* ω_ℓ *and* ω_k *be positive reals. Then, if* $\ell + k \leq n$,

$$|\mathscr{A}|\omega_\ell + |\mathscr{B}|\omega_k \leq \max\left\{\omega_\ell + \binom{n}{k}\omega_k - \binom{n-\ell}{k}\omega_k, \omega_k + \binom{n}{\ell}\omega_\ell - \binom{n-k}{\ell}\omega_\ell\right\}$$

Equality holds for $\ell + k < n$ *if and only if* $\mathscr{A} = \{A\}$ *and* $\mathscr{B} = \{B : B \cap A \neq \emptyset\}$ for some $A \in \binom{[n]}{\ell}$, *or* $\mathscr{B} = \{B\}$ *and* $\mathscr{A} = \{A : A \cap B \neq \emptyset\}$ *for some* $B \in \binom{[n]}{k}$, *except* *in the case $\ell = k = 2$ and $\omega_\ell = \omega_k$ in which there is the additional optimal configuration of all ℓ- resp. k-element sets containing a fixed element.*

The case $\omega_\ell = \omega_k$ in Theorem 4 was proven by Frankl and Tokushige [6].

2. Tools

Shadows

Given a family $\mathscr{A} \subseteq 2^{[n]}$, let $\nabla_k(\mathscr{A})$ denote the upper shadow of $\mathscr{A}$ in level k, that is $\nabla_k(\mathscr{A}) = \{B \in \binom{[n]}{k} : B \supseteq A \text{ for some } A \in \mathscr{A}\}$. Define the reverse lexicographic order $<_L$ on $\binom{[n]}{\ell}$ by $A <_L B$ iff $\max\{i \in A \setminus B\} < \max\{i \in B \setminus A\}$. Let $\mathscr{L}(m, \ell)$ resp. $\mathscr{R}(m, \ell)$ denote the family of the m largest resp. smallest ℓ-element sets in the order $<_L$. It is known and easy to check that $\nabla_k(\mathscr{L}(m, \ell)) = \mathscr{L}(|\nabla_k(\mathscr{L}(m, \ell))|, k)$. The famous Kruskal–Katona Theorem determines the minimum of $|\nabla_k(\mathscr{A})|$ taken over all families $\mathscr{A} \subseteq \binom{[n]}{\ell}$ of fixed size $|\mathscr{A}|$.

Theorem 5 ([10, 8]). *Let $\mathscr{A} \subseteq \binom{[n]}{\ell}$. Then $|\nabla_k(\mathscr{A})| \geq |\nabla_k(\mathscr{L}(|\mathscr{A}|, \ell))|$.*

Note that $\mathscr{A} \subseteq \binom{[n]}{\ell}$ and $\mathscr{B} \subseteq \binom{[n]}{k}$ are cross-intersecting ($\ell + k \leq n$) iff the families $\mathscr{A}^c := \{[n] \setminus A : A \in \mathscr{A}\}$ and $\nabla_{n-\ell}(\mathscr{B})$ are disjoint. Then, by Theorem 5, $|\mathscr{A}| \leq \binom{n}{\ell} - |\nabla_{n-\ell}(\mathscr{L}(|\mathscr{B}|, \ell))|$, and replacing $\mathscr{B}$ by $\mathscr{L}(|\mathscr{B}|, \ell)$ and $\mathscr{A}^c$ by $\mathscr{R}(|\mathscr{A}|, n - \ell)$ gives the following result (cf. [5, Thm. 3.6]):

Theorem 6. *If $\mathscr{A} \subseteq \binom{[n]}{\ell}$ and $\mathscr{B} \subseteq \binom{[n]}{k}$ are cross-intersecting, then so are $\mathscr{L}(|\mathscr{A}|, \ell)$ and $\mathscr{L}(|\mathscr{B}|, k)$.*

Corollary 1 ([9]). *Let $\mathscr{A} \subseteq \binom{[n]}{\ell}$ and $\mathscr{B} \subseteq \binom{[n]}{k}$ be cross-intersecting. Then, if $\ell + k \leq n$, $|\mathscr{A}| \leq \binom{n-1}{\ell-1}$ or $|\mathscr{B}| \leq \binom{n-1}{k-1}$.*

Note that Corollary 1 also follows from Theorem 3.

Although we will not need it let us state the following equivalent version of Theorem 3 in terms of shadows.

Theorem 7. *Let $\mathscr{A} \subseteq \binom{[n]}{\ell}$. Then, for $n \geq k \geq \ell$,*

$$\frac{|\nabla_k(\mathscr{A})|}{\binom{n}{k}} \geq \frac{|\mathscr{A}|}{\binom{n}{\ell}} \frac{k(n-\ell)}{\ell(n-k) + n(k-\ell)\frac{|\mathscr{A}|}{\binom{n}{\ell}}}.$$

Theorem 7 follows from Theorem 3 applied to the cross-intersecting families $\mathscr{A} \subseteq \binom{[n]}{\ell}$ and $\mathscr{B} := \left(\binom{[n]}{k} \setminus \nabla_k(\mathscr{A})\right)^c \subseteq \binom{[n]}{n-k}$.

Shifting

We recall the shifting operation from [4]:

Definition 1. For $\mathscr{A} \subseteq 2^{[n]}$ and $1 \leq i < j \leq n$ define the shifted family $s_{i,j}(\mathscr{A})$ by $s_{i,j}(\mathscr{A}) = \{s_{i,j}(A) : A \in \mathscr{A}\}$, where for $A \in \mathscr{A}$

$$s_{i,j}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } j \in A, i \notin A, (A \setminus \{j\}) \cup \{i\} \notin \mathscr{A}, \\ A & \text{otherwise.} \end{cases}$$

A family $\mathscr{A} \subseteq 2^{[n]}$ is called left-shifted, if $s_{i,j}(\mathscr{A}) = \mathscr{A}$ holds for all $1 \leq i < j \leq n$.

Note that by applying successively shift operations a left-shifted family is obtained.

The following properties of the shift operation is well known and easy to check ([4], [5, Lemma 4.2]).

Lemma 1. *Let* $\mathscr{A} \subseteq \binom{[n]}{\ell}$ *and* $\mathscr{B} \subseteq \binom{[n]}{k}$ *be cross-intersecting. Put* $\mathscr{A}' := s_{i,j}(\mathscr{A})$ *and* $\mathscr{B}' := s_{i,j}(\mathscr{B})$. *Then* $\mathscr{A}' \subseteq \binom{[n]}{\ell}$, $\mathscr{B}' \subseteq \binom{[n]}{k}$, $|\mathscr{A}'| = |\mathscr{A}|$, $|\mathscr{B}'| = |\mathscr{B}|$, *and the families* $\mathscr{A}'$ *and* $\mathscr{B}'$ *are cross-intersecting.*

By Lemma 1 we can assume that cross-intersecting families are left-shifted, when dealing with their cardinalities.

Generating sets

We recall some basic facts from [1], see also [3, Ch. 2.4]. For a family $\mathscr{A} \subseteq 2^{[n]}$ let $\max(\mathscr{A}) := \max\{i : i \in \cup_{A \in \mathscr{A}} A\}$.

Proposition 1. *Given a left-shifted family* $\mathscr{A} \subseteq \binom{[n]}{k}$, $\emptyset \neq \mathscr{A} \neq \binom{[n]}{k}$, *there is a family* $\mathscr{G} \subseteq 2^{[n]}$, *called generating family for* $\mathscr{A}$, *with the following properties:*

(i) $G \not\subseteq H$ *for all* $G, H \in \mathscr{G}$, $G \neq H$
(ii) $\mathscr{A} = \nabla_k(\mathscr{G})$
(iii) $\max(\mathscr{G}) \leq \max(\mathscr{G}')$ *for all* $\mathscr{G}' \subseteq 2^{[n]}$ *with* $\nabla_k(\mathscr{G}') = \mathscr{A}$
Put $m := \max(\mathscr{G})$.
(iv) For all $G \in \mathscr{G}$ *with* $m \in G$ *and all* $i < m$ *there is a* $H \in \mathscr{G}$ *with* $s_{i,m}(G) \supseteq H$.
(v) For every $G \in \mathscr{G}$ *with* $m \in G$ *we have*

$$\nabla_k(\{G\}) \setminus \nabla_k(\mathscr{G} \setminus \{G\}) = \{A : A \cap [m] = G, |A| = k\} \neq \emptyset.$$

(vi) For every $G \in \mathscr{G}$ *with* $m \in G$ *we have*

$$\nabla_k(\{G \setminus \{m\}\}) \setminus \mathscr{A} = \{A : A \cap [m] = G \setminus \{m\}, |A| = k\} \neq \emptyset.$$

Note that by (i) and (iv) we have $|\mathscr{G}| = 1$ if and only if $[\max(\mathscr{G})] \in \mathscr{G}$.

3. A Short Proof of Theorem 2

We may suppose that $\ell \leq k$.

Case 1. $|\mathscr{A}| \leq \binom{n-1}{\ell-1}$ and $|\mathscr{A}|/\binom{n}{\ell} + |\mathscr{B}|/\binom{n}{k} < (\ell + k)/n$. Then

$$\frac{|\mathscr{A}|}{\binom{n}{\ell}} \frac{|\mathscr{B}|}{\binom{n}{k}} < \frac{|\mathscr{A}|}{\binom{n}{\ell}} \left(\frac{\ell + k}{n} - \frac{|\mathscr{A}|}{\binom{n}{\ell}} \right) \leq \frac{\ell}{n} \frac{k}{n}.$$

Case 2. $|\mathscr{A}|/\binom{n}{\ell} + |\mathscr{B}|/\binom{n}{k} \geq (\ell + k)/n$. Then, by Theorem 3,

$$\frac{|\mathscr{A}|}{\binom{n}{\ell}} \frac{|\mathscr{B}|}{\binom{n}{k}} \leq \frac{\ell k}{n(n - \ell - k)} \left(1 - \frac{\ell + k}{n}\right) = \frac{\ell}{n} \frac{k}{n}.$$

Case 3. $|\mathscr{A}| > \binom{n-1}{\ell-1}$.

By Theorem 6 we may assume that $\mathscr{A} = \mathscr{L}(|\mathscr{A}|, \ell)$ and $\mathscr{B} = \mathscr{L}(|\mathscr{B}|, k)$. Let $\mathscr{A}^0 := \{A \in \mathscr{A} : n \notin A\}$ and $\mathscr{B}^1 := \{B \setminus \{n\} : n \in B \in \mathscr{B}\}$. Then $|\mathscr{A}| = |\mathscr{A}^0| + \binom{n-1}{\ell-1}$ and $|\mathscr{B}| = |\mathscr{B}^1|$, and $\mathscr{A}^0 \subseteq \binom{[n-1]}{\ell}$ and $\mathscr{B}^1 \subseteq \binom{[n-1]}{k-1}$ are cross-intersecting.

Since $2k \leq n$ we have $(k - 1)(k - \ell) < (n - k)(n - \ell - k)$ and thus

$$\frac{n - 2\ell}{\ell} < \frac{(n - 1)(n - \ell - k)}{\ell(k - 1)}.$$

Applying Theorem 3 to $\mathscr{A}^0$ and $\mathscr{B}^1$ therefore gives

$$\frac{|\mathscr{B}^1|}{\binom{n-1}{k-1}}\left(1+\frac{(n-2\ell)}{\ell}\frac{|\mathscr{A}^0|}{\binom{n-1}{\ell}}\right)<1-\frac{|\mathscr{A}^0|}{\binom{n-1}{\ell}}.$$

Multiplying with $1+|\mathscr{A}^0|/\binom{n-1}{\ell-1}$ yields

$$\left(1+\frac{|\mathscr{A}^0|}{\binom{n-1}{\ell-1}}\right)\frac{|\mathscr{B}^1|}{\binom{n-1}{k-1}}<1,$$

and thus $|\mathscr{A}||\mathscr{B}|<\binom{n-1}{\ell-1}\binom{n-1}{k-1}$.

4. Proof of Theorem 4

If $\ell+k=n$ then an easy argument using complements shows the validity of Theorem 4. Hence we assume that $\ell+k<n$.

Let $\mathscr{A}\subseteq\binom{[n]}{\ell}$ and $\mathscr{B}\subseteq\binom{[n]}{k}$ be cross-intersecting families for which $|\mathscr{A}|\omega_\ell+|\mathscr{B}|\omega_k$ is maximum. By Lemma 1 we may assume that $\mathscr{A}$ and $\mathscr{B}$ are both left-shifted. Let $\mathscr{G}$ and $\mathscr{H}$ be generating families for $\mathscr{A}$ and $\mathscr{B}$, respectively, according to Proposition 1. Then $\mathscr{G}$ and $\mathscr{H}$ are cross-intersecting too. Indeed, otherwise there are $G\in\mathscr{G}$ and $H\in\mathscr{H}$ with $G\cap H=\emptyset$, and since $\ell+k\leq n$, we find $A\supseteq G$ and $B\supseteq H$ with $A\cap B=\emptyset$, a contradiction.

If $\max(\mathscr{H})<\max(\mathscr{G})=:m$ then choose $G\in\mathscr{G}$ with $m\in G$ and consider $\mathscr{G}':=\mathscr{G}\cup\{G\setminus\{m\}\}$. Note that $G\neq\{m\}$. Thus $\mathscr{G}'$ and $\mathscr{H}$ are cross-intersecting. But $|\nabla_\ell(\mathscr{G}')|>|\mathscr{A}|$ by Proposition 1 (vi), contradicting the optimality of $\mathscr{A}$ and $\mathscr{B}$. Hence $\max(\mathscr{H})=\max(\mathscr{G})=m$.

Suppose now that $\mathscr{G}$ contains a set G with $m\in G$ and $1<|G|<m$. Then $|\mathscr{G}|>1$. Now $\mathscr{H}$ contains a set H with $G\cap H=\{m\}$ since otherwise the families $\mathscr{G}':=\mathscr{G}\cup\{G\setminus\{m\}\}$ and $\mathscr{H}$ are cross-interseting, yielding a contradiction as above. Further, we have $G\cup H=[m]$ since if there exists $i\in[m]$ with $i\notin G\cup H$ then $s_{i,m}(G)\cap H=\emptyset$, which in view of Proposition 1 (iv) contradicts the fact that $\mathscr{G}$ and $\mathscr{H}$ are cross-intersecting. It follows $1<|H|<m$ and $|\mathscr{H}|>1$. Now consider the families

$$\mathscr{G}_1:=\mathscr{G}\setminus\{G\},\quad \mathscr{H}_1:=\mathscr{H}\cup\{H\setminus\{m\}\},\ \mathscr{H}_2:=\mathscr{H}\setminus\{H\},\quad \mathscr{G}_2:=\mathscr{G}\cup\{G\setminus\{m\}\}.$$

By our choice of G and H it is clear that $\mathscr{G}_i$ and $\mathscr{H}_i$ are cross-intersecting, $i=1,2$. Hence $\mathscr{A}_i:=\nabla_\ell(\mathscr{G}_i)$ and $\mathscr{B}_i:=\nabla_k(\mathscr{H}_i)$ are cross-intersecting competitors for $\mathscr{A}$ and $\mathscr{B}$, $i=1,2$. Let $g:=|G|$ and $h:=|H|$ and recall $g+h-1=m$. Using Proposition 1 (v), (vi) we get

$$|\mathscr{A}_1|\omega_\ell + |\mathscr{B}_1|\omega_k - (|\mathscr{A}|\omega_\ell + |\mathscr{B}|\omega_k) = \binom{n-m}{k-h+1}\omega_k - \binom{n-m}{\ell-g}\omega_\ell,$$
$$|\mathscr{A}_2|\omega_\ell + |\mathscr{B}_2|\omega_k - (|\mathscr{A}|\omega_\ell + |\mathscr{B}|\omega_k) = \binom{n-m}{\ell-g+1}\omega_\ell - \binom{n-m}{k-h}\omega_k.$$

Since $\mathscr{A}$ and $\mathscr{B}$ are optimal, none of the two numbers are positive. This gives

$$\frac{\binom{n-m}{k-h+1}}{\binom{n-m}{k-h}} \le \frac{\binom{n-m}{\ell-g}}{\binom{n-m}{\ell-g+1}} = \frac{\binom{n-m}{n-\ell-h+1}}{\binom{n-m}{n-\ell-h}},$$

which in view of $k-h < n-\ell-h$ contradicts the strict logarithmic concavity of the binomial coefficients.

It follows that either $\mathscr{G} = \{[m]\}$ or $\mathscr{G} = \{\{1\},\dots,\{m\}\}$. In the first case we have $\mathscr{H} = \{\{1\},\dots,\{m\}\}$ and

$$|\mathscr{A}|\omega_\ell + |\mathscr{B}|\omega_k = \binom{n-m}{\ell-m}\omega_\ell + \left(\binom{n}{k} - \binom{n-m}{k}\right)\omega_k =: L(m),$$

with $1 \le m \le \ell$, in the second case necessarily $\mathscr{H} = \{[m]\}$ and

$$|\mathscr{A}|\omega_\ell + |\mathscr{B}|\omega_k = \binom{n-m}{k-m}\omega_k + \left(\binom{n}{\ell} - \binom{n-m}{\ell}\right)\omega_\ell =: K(m),$$

with $1 \le m \le k$.

Now one easily verifies that $L(m-1) \le L(m)$ is equivalent to

$$\binom{n-m}{n-\ell-1}\omega_\ell \le \binom{n-m}{k-1}\omega_k \tag{1}$$

which in view of $k-1 < n-\ell-1$ shows that $\max_{1\le m\le \ell} L(m) = \max\{L(1), L(\ell)\}$. Analogously, we have $\max_{1\le m\le k} K(m) = \max\{K(1), K(k)\}$. Using (1) one easily verifies that if $\omega_k/\omega_\ell \ne \binom{n-2}{\ell-1}/\binom{n-2}{k-1}$ then either $L(1) < L(2)$ or $K(1) < K(2)$. Further, if $\omega_k/\omega_\ell = \binom{n-2}{\ell-1}/\binom{n-2}{k-1}$ and $\ell \ge 3$ or $k \ge 3$ then $L(1) = L(2) < L(3)$ resp. $K(1) = K(2) < K(3)$.

Thus we have $\mathscr{G} = \{[\ell]\}$ and $\mathscr{H} = \{\{1\},\dots,\{\ell\}\}$, or $\mathscr{G} = \{\{1\},\dots,\{k\}\}$ and $\mathscr{H} = \{[k]\}$, or, if $\ell = k = 2$ and $\omega_\ell = \omega_k$, $\mathscr{G} = \mathscr{H} = \{\{1\}\}$, which completes the proof.

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Received: December 22, 2003
Final version received: October 20, 2004