On cross-intersecting families of sets

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Abstract

A family $\mathcal{A}$ of ℓ-element subsets and a family $\mathcal{B}$ of k-element subsets of an n-element set are cross-intersecting if every set from $\mathcal{A}$ has a nonempty intersection with every set from $\mathcal{B}$. We compare two previously established inequalities each related to the maximization of the product $|\mathcal{A}|\,|\mathcal{B}|$, and give a new and short proof for one of them. We also determine the maximum of $|\mathcal{A}|\,\omega_\ell + |\mathcal{B}|\,\omega_k$ for arbitrary positive weights ω_ℓ, ω_k.

1 Introduction

Let $[n]$ denote the set $\{1, 2, \dots, n\}$, and $\binom{[n]}{k}$ the set of all k-element subsets of $[n]$. A set family $\mathcal{A} \subseteq \binom{[n]}{k}$ is called intersecting if $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{A}$. One of the basic results in extremal set theory is the Erdős-Ko-Rado Theorem:

Theorem 1 ([4]). *Let $\mathcal{A} \subseteq \binom{[n]}{k}$ be intersecting. Then, if $k \le n/2$,*

$$|\mathcal{A}| \le \binom{n-1}{k-1}.$$

Equality holds for $k < n/2$ if and only if $\mathcal{A}$ consists of all k-element subsets of $[n]$ containing a fixed element.

Several generalizations of this result are known, among others to pairs of set families. Two families $\mathcal{A}, \mathcal{B} \subseteq 2^{[n]}$ are called cross-intersecting if both $\mathcal{A}$ and $\mathcal{B}$ are non-empty and for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \cap B \neq \emptyset$.

The following generalization of the Erdős-Ko-Rado Theorem was obtained by Matsumoto and Tokushige [11].

Theorem 2 ([11]). *Let $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ be cross-intersecting. Then, if $\ell, k \leq n/2$,*

$$|\mathcal{A}| \cdot |\mathcal{B}| \leq \binom{n-1}{\ell-1}\binom{n-1}{k-1}.$$

Equality holds for $\ell + k < n$ if and only if $\mathcal{A}$ and $\mathcal{B}$ consist of all ℓ-element resp. k-element sets containing a fixed element.

This result extends earlier work of Pyper [12] who proved the same if $\ell = k$ or $\ell < k$ and $2k + \ell - 2 \leq n$.

It is known that the inequality in Theorem 2 does not hold for all ℓ, k with $\ell + k \leq n$, see the examples given in [7, 11].

Another generalization of the Erdős-Ko-Rado Theorem also involving the product of two cross-intersecting families was given in [2].

Theorem 3 ([2]). *Let $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ be cross-intersecting. Then, if $\ell + k \leq n$,*

$$\frac{|\mathcal{A}|}{\binom{n}{\ell}} + \frac{|\mathcal{B}|}{\binom{n}{k}} + \frac{n(n-\ell-k)}{\ell k}\frac{|\mathcal{A}|}{\binom{n}{\ell}}\frac{|\mathcal{B}|}{\binom{n}{k}} \leq 1.$$

Equality holds for $\ell + k < n$ if and only if $\mathcal{A}$ and $\mathcal{B}$ consist of all ℓ-element resp. k-element sets containing a fixed element.

A straightforward calculation shows that Theorem 2 yields a slightly stronger inequality than Theorem 3 exactly for cross-intersecting families satisfying $\binom{n-1}{\ell-1} < |\mathcal{A}| < \frac{k}{\ell}\binom{n-1}{\ell-1}$ and $\frac{\ell}{k}\binom{n-1}{k-1} < |\mathcal{B}| < \binom{n-1}{k-1}$ (here we assume $\ell < k$). Thus none of the two theorems implies the other. However, we show in this note how our Theorem 3 can be used to give a new and short proof of the Matsumoto-Tokushige Theorem 2.

Both of the above quadratic inequalities have the same equality characterization. We show that maximizing only the linear term $|\mathcal{A}|/\binom{n}{\ell} + |\mathcal{B}|/\binom{n}{k}$ yields to a different optimal configuration. Actually, we have the following more general result.

Theorem 4. *Let $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ be cross-intersecting, and let ω_ℓ and ω_k be positive reals. Then, if $\ell + k \leq n$,*

$$|\mathcal{A}|\omega_\ell + |\mathcal{B}|\omega_k \leq \max\left\{\omega_\ell + \binom{n}{k}\omega_k - \binom{n-\ell}{k}\omega_k,\ \omega_k + \binom{n}{\ell}\omega_\ell - \binom{n-k}{\ell}\omega_\ell\right\}$$

Equality holds for $\ell + k < n$ if and only if $\mathcal{A} = \{A\}$ and $\mathcal{B} = \{B : B \cap A \neq \emptyset\}$ for some $A \in \binom{[n]}{\ell}$, or $\mathcal{B} = \{B\}$ and $\mathcal{A} = \{A : A \cap B \neq \emptyset\}$ for some

$B \in \binom{[n]}{k}$, *except in the case* $\ell = k = 2$ *and* $\omega_\ell = \omega_k$ *in which there is the additional optimal configuration of all* ℓ*- resp.* k*-element sets containing a fixed element.*

The case $\omega_\ell = \omega_k$ in Theorem 4 was proven by Frankl and Tokushige [6].

2 Tools

Shadows

Given a family $\mathcal{A} \subseteq 2^{[n]}$, let $\nabla_k(\mathcal{A})$ denote the upper shadow of $\mathcal{A}$ in level k, that is $\nabla_k(\mathcal{A}) = \{B \in \binom{[n]}{k} : B \supseteq A \text{ for some } A \in \mathcal{A}\}$. Define the reverse lexicographic order $<_L$ on $\binom{[n]}{\ell}$ by $A <_L B$ iff $\max\{i \in A \setminus B\} < \max\{i \in B \setminus A\}$. Let $\mathcal{L}(m, \ell)$ resp. $\mathcal{R}(m, \ell)$ denote the family of the m largest resp. smallest ℓ-element sets in the order $<_L$. It is known and easy to check that $\nabla_k(\mathcal{L}(m, \ell)) = \mathcal{L}(|\nabla_k(\mathcal{L}(m, \ell))|, k)$. The famous Kruskal–Katona Theorem determines the minimum of $|\nabla_k(\mathcal{A})|$ taken over all families $\mathcal{A} \subseteq \binom{[n]}{\ell}$ of fixed size $|\mathcal{A}|$.

Theorem 5 ([8, 10]). *Let* $\mathcal{A} \subseteq \binom{[n]}{\ell}$*. Then* $|\nabla_k(\mathcal{A})| \geq |\nabla_k(\mathcal{L}(|\mathcal{A}|, \ell))|$.

Note that $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ are cross-intersecting ($\ell + k \leq n$) iff the families $\mathcal{A}^c := \{[n] \setminus A : A \in \mathcal{A}\}$ and $\nabla_{n-\ell}(\mathcal{B})$ are disjoint. Then, by Theorem 5, $|\mathcal{A}| \leq \binom{n}{\ell} - |\nabla_{n-\ell}(\mathcal{L}(|\mathcal{B}|, \ell))|$, and replacing $\mathcal{B}$ by $\mathcal{L}(|\mathcal{B}|, \ell)$ and $\mathcal{A}^c$ by $\mathcal{R}(|\mathcal{A}|, n - \ell)$ gives the following result (cf. [5, Thm. 3.6]):

Theorem 6. *If* $\mathcal{A} \subseteq \binom{[n]}{\ell}$ *and* $\mathcal{B} \subseteq \binom{[n]}{k}$ *are cross-intersecting, then so are* $\mathcal{L}(|\mathcal{A}|, \ell)$ *and* $\mathcal{L}(|\mathcal{B}|, k)$.

Corollary 7 ([9]). *Let* $\mathcal{A} \subseteq \binom{[n]}{\ell}$ *and* $\mathcal{B} \subseteq \binom{[n]}{k}$ *be cross-intersecting. Then, if* $\ell + k \leq n$, $|\mathcal{A}| \leq \binom{n-1}{\ell-1}$ *or* $|\mathcal{B}| \leq \binom{n-1}{k-1}$.

Note that Corollary 7 also follows from Theorem 3.

Although we will not need it let us state the following equivalent version of Theorem 3 in terms of shadows.

Theorem 8. *Let* $\mathcal{A} \subseteq \binom{[n]}{\ell}$*. Then, for* $n \geq k \geq \ell$,

$$\frac{|\nabla_k(\mathcal{A})|}{\binom{n}{k}} \geq \frac{|\mathcal{A}|}{\binom{n}{\ell}} \frac{k(n-\ell)}{\ell(n-k) + n(k-\ell)\frac{|\mathcal{A}|}{\binom{n}{\ell}}}.$$

Theorem 8 follows from Theorem 3 applied to the cross-intersecting families $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} := \left(\binom{[n]}{k} \setminus \nabla_k(\mathcal{A})\right)^c \subseteq \binom{[n]}{n-k}$.

Shifting

We recall the shifting operation from [4]:

Definition. *For $\mathcal{A} \subseteq 2^{[n]}$ and $1 \le i < j \le n$ define the shifted family $s_{i,j}(\mathcal{A})$ by $s_{i,j}(\mathcal{A}) = \{s_{i,j}(A) : A \in \mathcal{A}\}$, where for $A \in \mathcal{A}$*

$$s_{i,j}(A) = \begin{cases} (A \setminus \{j\}) \cup \{i\} & \text{if } j \in A,\ i \notin A,\ (A \setminus \{j\}) \cup \{i\} \notin \mathcal{A}\,, \\ A & \text{otherwise}\,. \end{cases}$$

A family $\mathcal{A} \subseteq 2^{[n]}$ is called left-shifted, if $s_{i,j}(\mathcal{A}) = \mathcal{A}$ holds for all $1 \le i < j \le n$.

Note that by applying successively shift operations a left-shifted family is obtained.

The following properties of the shift operation is well known and easy to check ([4], [5, Lemma 4.2]).

Lemma 9. *Let $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ be cross-intersecting. Put $\mathcal{A}' := s_{i,j}(\mathcal{A})$ and $\mathcal{B}' := s_{i,j}(\mathcal{B})$. Then $\mathcal{A}' \subseteq \binom{[n]}{\ell}$, $\mathcal{B}' \subseteq \binom{[n]}{k}$, $|\mathcal{A}'| = |\mathcal{A}|$, $|\mathcal{B}'| = |\mathcal{B}|$, and the families $\mathcal{A}'$ and $\mathcal{B}'$ are cross-intersecting.*

By Lemma 9 we can assume that cross-intersecting families are left-shifted, when dealing with their cardinalities.

Generating sets

We recall some basic facts from [1], see also [3, Ch. 2.4]. For a family $\mathcal{A} \subseteq 2^{[n]}$ let $\max(\mathcal{A}) := \max\{i : i \in \cup_{A \in \mathcal{A}} A\}$.

Proposition 10. *Given a left-shifted family $\mathcal{A} \subseteq \binom{[n]}{k}$, $\emptyset \ne \mathcal{A} \ne \binom{[n]}{k}$, there is a family $\mathcal{G} \subseteq 2^{[n]}$, called generating family for $\mathcal{A}$, with the following properties:*

(i) $G \nsubseteq H$ *for all* $G, H \in \mathcal{G}$, $G \ne H$

(ii) $\mathcal{A} = \nabla_k(\mathcal{G})$

(iii) $\max(\mathcal{G}) \le \max(\mathcal{G}')$ *for all* $\mathcal{G}' \subseteq 2^{[n]}$ *with* $\nabla_k(\mathcal{G}') = \mathcal{A}$

Put $m := \max(\mathcal{G})$.

(iv) *For all $G \in \mathcal{G}$ with $m \in G$ and all $i < m$ there is a $H \in \mathcal{G}$ with $s_{i,m}(G) \supseteq H$.*

(v) For every $G \in \mathcal{G}$ with $m \in G$ we have

$$\nabla_k(\{G\}) \setminus \nabla_k(\mathcal{G} \setminus \{G\}) = \{A : A \cap [m] = G,\, |A| = k\} \neq \emptyset\,.$$

(vi) For every $G \in \mathcal{G}$ with $m \in G$ we have

$$\nabla_k(\{G \setminus \{m\}\}) \setminus \mathcal{A} = \{A : A \cap [m] = G \setminus \{m\},\, |A| = k\} \neq \emptyset\,.$$

Note that by (i) and (iv) we have $|\mathcal{G}| = 1$ if and only if $[\max(\mathcal{G})] \in \mathcal{G}$.

3 A short proof of Theorem 2

We may suppose that $\ell \leq k$.
Case 1: $|\mathcal{A}| \leq \binom{n-1}{\ell-1}$ and $|\mathcal{A}|/\binom{n}{\ell} + |\mathcal{B}|/\binom{n}{k} < (\ell + k)/n$.
Then

$$\frac{|\mathcal{A}|}{\binom{n}{\ell}} \frac{|\mathcal{B}|}{\binom{n}{k}} < \frac{|\mathcal{A}|}{\binom{n}{\ell}} \left(\frac{\ell + k}{n} - \frac{|\mathcal{A}|}{\binom{n}{\ell}} \right) \leq \frac{\ell}{n} \frac{k}{n}\,.$$

Case 2: $|\mathcal{A}|/\binom{n}{\ell} + |\mathcal{B}|/\binom{n}{k} \geq (\ell + k)/n$.
Then, by Theorem 3,

$$\frac{|\mathcal{A}|}{\binom{n}{\ell}} \frac{|\mathcal{B}|}{\binom{n}{k}} \leq \frac{\ell k}{n(n - \ell - k)} \left(1 - \frac{\ell + k}{n} \right) = \frac{\ell}{n} \frac{k}{n}\,.$$

Note that in view of Corollary 7 the case $\ell = k$ is already settled. We now suppose that $\ell < k$.
Case 3: $|\mathcal{A}| > \binom{n-1}{\ell-1}$.
By Theorem 6 we may assume that $\mathcal{A} = \mathcal{L}(|\mathcal{A}|, \ell)$ and $\mathcal{B} = \mathcal{L}(|\mathcal{B}|, k)$. Let $\mathcal{A}^0 := \{A \in \mathcal{A} : n \notin A\}$ and $\mathcal{B}^1 := \{B \setminus \{n\} : n \in B \in \mathcal{B}\}$. Then $|\mathcal{A}| = |\mathcal{A}^0| + \binom{n-1}{\ell-1}$ and $|\mathcal{B}| = |\mathcal{B}^1|$, and $\mathcal{A}^0 \subseteq \binom{[n-1]}{\ell}$ and $\mathcal{B}^1 \subseteq \binom{[n-1]}{k-1}$ are cross-intersecting.

Since $2k \leq n$ we have $(k-1)(k-\ell) < (n-k)(n-\ell-k)$ and thus

$$\frac{n - 2\ell}{\ell} < \frac{(n-1)(n-\ell-k)}{\ell(k-1)}\,.$$

Applying Theorem 3 to $\mathcal{A}^0$ and $\mathcal{B}^1$ therefore gives

$$\frac{|\mathcal{B}^1|}{\binom{n-1}{k-1}} \left(1 + \frac{(n-2\ell)}{\ell} \frac{|\mathcal{A}^0|}{\binom{n-1}{\ell}} \right) < 1 - \frac{|\mathcal{A}^0|}{\binom{n-1}{\ell}}\,.$$

Multiplying with $1 + |\mathcal{A}^0|/\binom{n-1}{\ell-1}$ yields

$$\left(1 + \frac{|\mathcal{A}^0|}{\binom{n-1}{\ell-1}} \right) \frac{|\mathcal{B}^1|}{\binom{n-1}{k-1}} < 1\,,$$

and thus $|\mathcal{A}|\,|\mathcal{B}| < \binom{n-1}{\ell-1}\binom{n-1}{k-1}$.

4 Proof of Theorem 4

If $\ell + k = n$ then an easy argument using complements shows the validity of Theorem 4. Hence we assume that $\ell + k < n$.

Let $\mathcal{A} \subseteq \binom{[n]}{\ell}$ and $\mathcal{B} \subseteq \binom{[n]}{k}$ be cross-intersecting families for which $|\mathcal{A}|\omega_\ell + |\mathcal{B}|\omega_k$ is maximum. By Lemma 9 we may assume that $\mathcal{A}$ and $\mathcal{B}$ are both left-shifted. Let $\mathcal{G}$ and $\mathcal{H}$ be generating families for $\mathcal{A}$ and $\mathcal{B}$, respectively, according to Proposition 10. Then $\mathcal{G}$ and $\mathcal{H}$ are cross-intersecting too. Indeed, otherwise there are $G \in \mathcal{G}$ and $H \in \mathcal{H}$ with $G \cap H = \emptyset$, and since $\ell + k \le n$, we find $A \supseteq G$ and $B \supseteq H$ with $A \cap B = \emptyset$, a contradiction.

If $\max(\mathcal{H}) < \max(\mathcal{G}) =: m$ then choose $G \in \mathcal{G}$ with $m \in G$ and consider $\mathcal{G}' := \mathcal{G} \cup \{G \setminus \{m\}\}$. Note that $G \neq \{m\}$. Thus $\mathcal{G}'$ and $\mathcal{H}$ are cross-intersecting. But $|\nabla_\ell(\mathcal{G}')| > |\mathcal{A}|$ by Proposition 10 (vi), contradicting the optimality of $\mathcal{A}$ and $\mathcal{B}$. Hence $\max(\mathcal{H}) = \max(\mathcal{G}) = m$.

Suppose now that $\mathcal{G}$ contains a set G with $m \in G$ and $1 < |G| < m$. Then $|\mathcal{G}| > 1$. Now $\mathcal{H}$ contains a set H with $G \cap H = \{m\}$ since otherwise the families $\mathcal{G}' := \mathcal{G} \cup \{G \setminus \{m\}\}$ and $\mathcal{H}$ are cross-interseting, yielding a contradiction as above. Further, we have $G \cup H = [m]$ since if there exists $i \in [m]$ with $i \notin G \cup H$ then $s_{i,m}(G) \cap H = \emptyset$, which in view of Proposition 10 (iv) contradicts the fact that $\mathcal{G}$ and $\mathcal{H}$ are cross-intersecting. It follows $1 < |H| < m$ and $|\mathcal{H}| > 1$. Now consider the families

$$\begin{aligned} \mathcal{G}_1 &:= \mathcal{G} \setminus \{G\}, \quad \mathcal{H}_1 := \mathcal{H} \cup \{H \setminus \{m\}\} \\ \mathcal{H}_2 &:= \mathcal{H} \setminus \{H\}, \quad \mathcal{G}_2 := \mathcal{G} \cup \{G \setminus \{m\}\}\,. \end{aligned}$$

By our choice of G and H it is clear that $\mathcal{G}_i$ and $\mathcal{H}_i$ are cross-intersecting, $i = 1, 2$. Hence $\mathcal{A}_i := \nabla_\ell(\mathcal{G}_i)$ and $\mathcal{B}_i := \nabla_k(\mathcal{H}_i)$ are cross-intersecting competitors for $\mathcal{A}$ and $\mathcal{B}$, $i = 1, 2$. Let $g := |G|$ and $h := |H|$ and recall $g + h - 1 = m$. Using Proposition 10 (v), (vi) we get

$$\begin{aligned} |\mathcal{A}_1|\omega_\ell + |\mathcal{B}_1|\omega_k - (|\mathcal{A}|\omega_\ell + |\mathcal{B}|\omega_k) &= \binom{n-m}{k-h+1}\omega_k - \binom{n-m}{\ell-g}\omega_\ell\,, \\ |\mathcal{A}_2|\omega_\ell + |\mathcal{B}_2|\omega_k - (|\mathcal{A}|\omega_\ell + |\mathcal{B}|\omega_k) &= \binom{n-m}{\ell-g+1}\omega_\ell - \binom{n-m}{k-h}\omega_k\,. \end{aligned}$$

Since $\mathcal{A}$ and $\mathcal{B}$ are optimal, none of the two numbers are positive. This gives

$$\frac{\binom{n-m}{k-h+1}}{\binom{n-m}{k-h}} \le \frac{\binom{n-m}{\ell-g}}{\binom{n-m}{\ell-g+1}} = \frac{\binom{n-m}{n-\ell-h+1}}{\binom{n-m}{n-\ell-h}}\,,$$

which in view of $k - h < n - \ell - h$ contradicts the strict logarithmic concavity of the binomial coefficients.

It follows that either $\mathcal{G} = \{[m]\}$ or $\mathcal{G} = \{\{1\}, \ldots, \{m\}\}$. In the first case we have $\mathcal{H} = \{\{1\}, \ldots, \{m\}\}$ and

$$|\mathcal{A}|\omega_\ell + |\mathcal{B}|\omega_k = \binom{n-m}{\ell-m}\omega_\ell + \left(\binom{n}{k} - \binom{n-m}{k}\right)\omega_k =: L(m)\,,$$

with $1 \leq m \leq \ell$, in the second case necessarily $\mathcal{H} = \{[m]\}$ and

$$|\mathcal{A}|\omega_\ell + |\mathcal{B}|\omega_k = \binom{n-m}{k-m}\omega_k + \left(\binom{n}{\ell} - \binom{n-m}{\ell}\right)\omega_\ell =: K(m)\,,$$

with $1 \leq m \leq k$.

Now one easily verifies that $L(m-1) \leq L(m)$ is equivalent to

$$\binom{n-m}{n-\ell-1}\omega_\ell \leq \binom{n-m}{k-1}\omega_k \tag{1}$$

which in view of $k-1 < n-\ell-1$ shows that $\max_{1\leq m\leq \ell} L(m) = \max\{L(1), L(\ell)\}$. Analogously, we have $\max_{1\leq m\leq k} K(m) = \max\{K(1), K(k)\}$. Using (1) one easily verifies that if $\omega_k/\omega_\ell \neq \binom{n-2}{\ell-1}/\binom{n-2}{k-1}$ then either $L(1) < L(2)$ or $K(1) < K(2)$. Further, if $\omega_k/\omega_\ell = \binom{n-2}{\ell-1}/\binom{n-2}{k-1}$ and $\ell \geq 3$ or $k \geq 3$ then $L(1) = L(2) < L(3)$ resp. $K(1) = K(2) < K(3)$.

Thus we have $\mathcal{G} = \{[\ell]\}$ and $\mathcal{H} = \{\{1\}, \ldots, \{\ell\}\}$, or $\mathcal{G} = \{\{1\}, \ldots, \{k\}\}$ and $\mathcal{H} = \{[k]\}$, or, if $\ell = k = 2$ and $\omega_\ell = \omega_k$, $\mathcal{G} = \mathcal{H} = \{\{1\}\}$, which completes the proof.

References

[1] R. Ahlswede and L. H. Khachatrian. The complete intersection theorem for systems of finite sets. *European J. Combin.*, **18**, 125–136 (1997).

[2] Ch. Bey. Polynomial LYM inequalities. *Combinatorica*, to appear.

[3] K. Engel. *Sperner Theory.* Cambridge University Press, Cambridge (1997).

[4] P. Erdős and C. Ko and R. Rado. Intersection theorems for systems of finite sets. *Quart. J. Math. Oxford (2)*, **12**, 313–320 (1961).

[5] P. Frankl. Extremal set systems. In R.L. Graham, M. Grötschel and L. Lovász, editors, *Handbook of Combinatorics, Vol. 2*, pages 1293–1329, Elsevier, Amsterdam (1995).

[6] P. Frankl and N. Tokushige. Some best possible inequalities concerning cross-intersecting families. *J. Combin. Theory A*, **61**, 87–97 (1992).

[7] P. Frankl and N. Tokushige. Some inequalities concerning cross-intersecting families. *Combinatorics, Probability and Computing*, **7**, 247–260 (1998).

[8] Gy. O. H. Katona. A theorem of finite sets. In P. Erdős and G. Katona, editors, *Theory of Graphs*, pages 187–207, Akadémiai Kiadó and Academic Press, Budapest and New York (1968).

[9] D. J. Kleitman. On a conjecture of Milner on k-graphs with non-disjoint edges. *J. Combin. Theory*, **5**, 153–156 (1968).

[10] J. B. Kruskal. The number of simplices in a complex. In R. Bellman, editor, *Mathematical Optimization Techniques*, pages 251–278, Univ. of California Press., Berkeley-Los Angeles (1963).

[11] M. Matsumoto and N. Tokushige. The exact bound in the Erdős-Ko-Rado Theorem for cross-intersecting families. *J. Combin. Theory A*, **52**, 90–97 (1989).

[12] L. Pyber. A new generalization of the Erdős-Ko-Rado Theorem. *J. Combin. Theory A*, **43**, 85–90 (1986).