Regularized Quadrature Methods for Fredholm Integral Equations of the First Kind

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Abstract Although quadrature methods for solving ill-posed integral equations of the first kind were introduced just after the publication the of classical papers on the regularization by A.N. Tikhonov and D.L. Phillips, there are still no known results on the convergence rate of such discretization. At the same time, some problems appearing in practice, such as Magnetic Particle Imaging (MPI), allow one only a discretization corresponding to a quadrature method. In the present paper we study the convergence rate of quadrature methods under general regularization scheme in the Reproducing Kernel Hilbert Space setting.

1 Introduction

The so-called direct or discretization methods for solving Fredholm integral equations can be conventionally subdivided into three groups, namely: degenerate-kernel methods, such as Galerkin method and the Sloan iteration [5, 21], collocation methods, and the Nyström or quadrature methods.

For Fredholm equations of the second kind, there exist fairly complete results on the analysis and optimization of methods from all the above mentioned groups. Just to mention a few references, we refer to the books [1, 8, 18, 20].

As to Fredholm equations of the first kind, due to the fact that they are, in general, ill-posed, direct methods for solving them are usually combined with a regulariza-

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tion procedure. Regularized degenerate-kernel methods were studied extensively. A few selected references are [6, 12, 17]. There are also studies on regularized collocation methods [7, 15, 19].

At the same time, quadrature methods have not been investigated enough for ill-posed Fredholm equations of the first kind. In spite of the fact that a quadrature method is the first direct method suggested for such equations, to the best of our knowledge, no results about convergence rates of regularized quadrature methods are known in the literature up to now. Our paper sheds a light on this issue.

The distinguishing feature of a quadrature method for a Fredholm integral equation of the first kind

$$\int_{\Omega} s(t,x)c(x)d\Omega(x) = u(t), \qquad t \in \omega \subset \mathbb{R}^{d_2},\ x \in \Omega \subset \mathbb{R}^{d_1} \tag{1}$$

is that it uses another type of information than Galerkin type or collocation methods. Namely, for a Galerkin type method we should be given Fourier coefficients of the kernel $s(t,x)$ and the right-hand side u. A collocation method uses the values of the right-hand side u at the collocation points $\{t_i\}_{i=1}^{N} \in \omega$ and the information about the kernel $s(t,x)$ in the form

$$\int_{\Omega} s(t_i,x)s(t_j,x)d\Omega(x).$$

However such kind of information is not available for some problems. For example, the information acquisition of Magnetic Particle Imaging (MPI) technology [4, 16] allows an access only to a discretized form of the corresponding equation (1) with respect to x in such a way that one should deal with the following system

$$\sum_{j=1}^{M} w_j s(t,x^j)c(x^j) = u(t), \tag{2}$$

where w_j are some positive weights and $\{x^j\} \subset \Omega$ is a system of knots that can be formed by the so-called Lissajous nodes, for example [4]. Therefore, a quadrature method naturally appears for such problems and further investigation of the properties of this discretization strategy is required.

The paper organized as follows: In the next section we introduce some basic assumptions and definitions. Then, in Section 3 we consider quadrature methods for the discretiaztion of Fredholm integral equations of the first kind and investigate some of their characteristics. In Section 4 we discuss the application of the general regularization scheme for dealing with the ill-posedness of the problem and estimate the rate of convergence of the proposed method. Finally, in the last section the algorithms and some numerical illustrations are presented.

2 Preliminaries

Let $L_2(\Omega)$, $\Omega \subset \mathbb{R}^{d_1}$, and $L_2(\omega)$, $\omega \subset \mathbb{R}^{d_2}$ be the Hilbert spaces of square summable functions on Ω and ω equipped with the standard inner products with respect to the measures $d\Omega(x)$ and $d\omega(t)$. Moreover we also consider the space $C(\Omega)$ of continuous functions on Ω.

The discretization (2) presupposes that the integral operator of (1) acts from the space allowing the evaluation of functions at the points of Ω. It is known that Reproducing Kernel Hilbert Spaces (RKHS) are natural spaces with such property. It is also known that any RKHS, say H_K, can be generated by the corresponding reproducing kernel $K = K(x,y)$, $x,y \in \Omega$, which is a symmetric and positive-defined function. Moreover, H_K is equipped with an inner product $\langle \cdot, \cdot \rangle_{H_K}$ such that for any $f \in H_K$ we have

$$f(x) = \langle f, K_x \rangle_{H_K}, \tag{3}$$

where $K_x = K_x(\cdot) = K(x, \cdot)$.

Let's consider a compact integral operator $S_\Omega : H_K \to L_2(\omega)$ given by

$$S_\Omega c(t) := \int_\Omega s(t,x)c(x)d\Omega(x), \qquad t \in \omega. \tag{4}$$

Further we impose additional assumptions on the space H_K and the kernel $s(t,x)$.

Assumption 1. *Let H_K be compactly embedded in $L_2(\Omega)$ and the kernel $K(x,y)$ be such that*

$$K(x,y) := \sum_l \beta_l T_l(x) T_l(y),$$

where for any l $\beta_l > 0$, $T_l(x) \in C(\Omega)$, and $\{T_l(x)\}_{l=1}^\infty$ is a linearly independent system of functions.

Assumption 2. *Let W^τ be the so-called space with a given rate of convergence $\tau = \tau(N)$ for the system $\{T_l(x)\}_{l=1}^\infty$ (see details about such spaces in [2]), i.e. W^τ is a normed space embedded in $C(\Omega)$ such that for any $f \in W^\tau$ it holds true*

$$\min_{u \in span\{T_l\}_{l=1}^N} \|f - u\|_{C(\Omega)} \le \tau(N) \|f\|_{W^\tau}. \tag{5}$$

Remark 1. To illustrate Assumption 2, let us consider the space $W_\infty^r(\Omega)$, $\Omega = [-1,1]$, of functions on Ω having absolutely continuous derivatives of order up to $(r-1)$ and $\|f^{(r)}\|_{L_\infty} \le \infty$,

$$\|f\|_{W_\infty^r(\Omega)} = \sum_{l=0}^r \left\| f^{(l)} \right\|_{L_\infty}$$

Consider the system $T_l(x) = \cos(l \arccos x)$, $x \in [-1;1]$, of Chebyshev polynomials of the first kind, that are extensively used in the context of MPI-technology with Lissajous acquisition points [4]. Then from [3] it follows that for any $f \in W_\infty^r(\Omega)$

the condition (5) holds with $\tau(N) = O(N^{-r})$. Thus, $W^r_\infty(\Omega)$ can be seen as W^τ with $\tau(N) = cN^{-r}$.

Assumption 3. *Let the sequence $\{\beta_l\}$ be such that for any N*

$$\sum_{l=N+1}^{\infty} \beta_l \|T_l\|^2_{C(\Omega)} \leq \tau^2(N). \tag{6}$$

Remark 2. In the context of Remark 1 it is enough to assume that $\beta_l = O(l^{-\gamma})$ for $\gamma > 2r+1$.

3 Discretization by a quadrature rule

In this section we introduce quadrature methods for discretizing the operator S_Ω and show that under our assumptions the discretization error is of order $O(\tau(N))$.

Consider a quadrature rule $Q_{w,M}$ such that for any $g(x) \in H_K$

$$Q_{w,M}(g) = \sum_{j=1}^{M} w_j g(x^j), \tag{7}$$

where $\{x^j\}_{j=1}^M \subset \Omega$ and $\{w_j\}_{j=1}^M \subset \mathbb{R}^+$ are the systems of quadrature knots and weights respectively.

Assumption 4. *Let for any natural N there exists $M = M(N)$ such that for $l_1, l_2 = 1, 2, ..., N$ it holds*

$$Q_{w,M}(T_{l_1} T_{l_2}) = \int_\Omega T_{l_1}(x) T_{l_2}(x) d\Omega(x).$$

Remark 3. In the context of Remark 1 the Gaussian quadrature formula $Q_{w,M}$ meets Assumption 4 with $M = N+1$ for $T_{l_i} = \cos(l_i \arccos(x)), i = 1,2$, $x_j = \cos(\frac{2j-1}{2M}\pi)$ and $w_j = \frac{\pi}{M}$. However, in general the set $\{T_{l_1}(x) T_{l_2}(x)\}$ consists of N^2 different functions, and therefore M could be of order $O(N^2)$.

Now we apply the quadrature rule $Q_{w,M}$ for the discretization of the operator S_Ω. According to our notation we have

$$S_{w,M} c = (S_{w,M} c)(t) := Q_{w,M}(s(t,\cdot)c(\cdot)) = \sum_{j=1}^{M} w_j s(t, x^j) c(x^j), \tag{8}$$

where $S_{w,M} : H_K \to L_2(\omega)$.

Assumption 5. *Let $\{e_k\}_{k=1}^\infty$ be an orthonormal basis of $L_2(\omega)$. Assume that*

(i) for any ν the function $S_\nu(\cdot) = \int_\omega e_\nu(t) s(t,\cdot) d\omega(t)$ belongs to W^τ;

(ii) there exists some constant c_1 such that $(\sum_\nu \|S_\nu\|^2_{W^\tau})^{1/2} \leq c_1$;

(iii) for any fixed $t \in \omega$ the function $g(y) = \int_\Omega s(t,x)K_y(x)d\Omega(x)$ belongs to H_K.

Remark 4. We illustrate Assumption 5 (conditions (ii)) in terms of the space $W^\tau = W^r_\infty(\Omega)$ for $\omega = \Omega = [-1;1]$. Suppose that the kernel $s(t,x)$ has bounded continuous 1-periodic mixed derivatives $\frac{\partial^{l+1}}{\partial x^l \partial t} s(t,x)$, $l = 1,\dots,r$, and the orthonormal basis $\{e_k\}_{k=1}^\infty$ is the system of trigonometric functions. From the integration by parts formula it follows that for any 1-periodic continuously differentiable function g

$$\left| \int_\omega g(t) e_\nu(t) d\omega(t) \right| \le c\nu^{-1} \max_{t\in\Omega} \left| g'(t) \right|,$$

where c is come constant that does not depend on ν. Using the above inequality and the definition of the norm in $W^r_\infty(\Omega)$ we have

$$\begin{aligned} \|S_\nu(x)\|^2_{W^r_\infty(\Omega)} &= \left(\sum_{l=0}^r \max_{x\in\Omega} \left| \int_\omega \frac{\partial^l s(t,x)}{\partial x^l} e_\nu(t) d\omega(t) \right| \right)^2 \\ &\le \left(c\nu^{-1} \sum_{l=0}^r \max_{t,x\in[-1,1]} \left| \frac{\partial^{l+1} s(t,x)}{\partial x^l \partial t} \right| \right)^2 \\ &\le \left(c\nu^{-1} r \max_{l\in(1,\dots,r)} \max_{t,x\in[-1,1]} \left| \frac{\partial^{l+1} s(t,x)}{\partial x^l \partial t} \right| \right)^2 . \end{aligned}$$

By summing over all ν we finally obtain

$$\begin{aligned} \sqrt{\sum_\nu \|S_\nu\|^2_{W^\tau}} &\le cr \sqrt{\sum_\nu \nu^{-2}} \max_{l\in(1,..,r)} \max_{t,x\in[-1,1]} \left| \frac{\partial^{l+1} s(t,x)}{\partial x^l \partial t} \right| \\ &= cr \max_{l\in(1,..,r)} \max_{t,x\in[-1,1]} \left| \frac{\partial^{l+1} s(t,x)}{\partial x^l \partial t} \right| \le \infty. \end{aligned}$$

Thus the condition (ii) fulfills.

We need the following lemmas to estimate the accuracy of the quadrature approximation (8) for operator S_Ω.

Lemma 1. *Let $f(x) \in W^\tau$. If Assumptions 2, 4 are satisfied then for $M = M(N)$ and any $l < N$ it holds*

$$\left| \int_\Omega f(x) T_l(x) d\Omega(x) - Q_{w,M}(fT_l) \right| \le 2\mu_\Omega \tau(N) \|f\|_{W^\tau} \|T_l\|_{C(\Omega)},$$

where $\mu_\Omega = \int_\Omega d\Omega(x)$.

Proof. According to Assumption 2 there exist a function $u \in span\{T_l\}_{l=1}^N$ such that (5)is satisfied. Then taking into account Assumption 4 we obtain

$$\left|\int f(x)T_l(x)d\Omega(x) - Q_{w,M}(fT_l)\right| = \left|\int (f(x)-u(x))T_l(x)d\Omega(x) - Q_{w,M}((f-u)T_l)\right|$$
$$\leq 2\mu_\Omega \|f-u\|_{C(\Omega)}\|T_l\|_{C(\Omega)} \leq 2\mu_\Omega \tau(N)\|f\|_{W^\tau}\|T_l\|_{C(\Omega)},$$

which proves the statement. □

Lemma 2. *Let $f \in W^\tau$ and Assumptions 1–4 be satisfied. Then*

$$\left\|\int_\Omega f(x)K_y(x)d\Omega(x) - Q_{w,M}(fK_y)\right\|_{H_K} \leq 2\mu_\Omega\sqrt{1+\tau^2(0)}\|f\|_{W^\tau}\tau(N).$$

Proof.

Using the Assumption 1 and (3) we get

$$\left\|\int_\Omega f(x)K_y(x)d\Omega(x) - \sum_{j=1}^M w_j f(x^j)K_y(x^j)\right\|_{H_K}^2$$
$$= \left\langle \int_\Omega f(x)K_y(x)d\Omega(x) - \sum_{j=1}^M w_j f(x^j)K_y(x^j), \int_\Omega f(\widetilde{x})K_y(\widetilde{x})d\Omega(\widetilde{x}) - \sum_{i=1}^M w_i f(x^i)K_y(x^i)\right\rangle_{H_K}$$
$$= \int_\Omega f(\widetilde{x})\int_\Omega f(x)\left\langle K_y(\widetilde{x}),K_y(x)\right\rangle_{H_K} d\Omega(x)d\Omega(\widetilde{x}) - \int_\Omega f(x)\sum_{i=1}^M w_i f(\widetilde{x}^i)\left\langle K_y(x),K_y(\widetilde{x}^i)\right\rangle_{H_K} d\Omega(x)$$
$$- \int_\Omega f(\widetilde{x})\sum_{j=1}^M w_j f(x^j)\left\langle K_y(\widetilde{x}),K_y(x^j)\right\rangle_{H_K} d\Omega(\widetilde{x}) + \sum_{j=1}^M w_j f(x^j)\sum_{i=1}^M w_i f(x^i)\left\langle K_y(x^j),K_y(x^i)\right\rangle_{H_K}$$
$$= \int_\Omega f(\widetilde{x})\int_\Omega f(x)K(x,\widetilde{x})d\Omega(x)d\Omega(\widetilde{x}) - \int_\Omega f(x)\sum_{i=1}^M w_i f(x^i)K(x,x^i)d\Omega(x)$$
$$- \int_\Omega f(\widetilde{x})\sum_{j=1}^M w_j K(\widetilde{x},x^j)d\Omega(\widetilde{x}) + \sum_{j=1}^M w_j f(x^j)\sum_{i=1}^M w_i f(x^i)K(x^j,x^i)$$
$$= \sum_{l=1}^\infty \beta_l \left[\int_\Omega f(\widetilde{x})T_l(\widetilde{x})d\Omega(\widetilde{x})\int_\Omega f(x)T_l(x)d\Omega(x) - \int_\Omega f(x)T_l(x)d\Omega(x)\sum_{i=1}^M w_i f(x^i)T_l(x^i)\right.$$
$$\left.- \int_\Omega f(\widetilde{x})T_l(\widetilde{x})d\Omega(\widetilde{x})\sum_{j=1}^M w_j f(x^j)T_l(x^j) + \sum_{j=1}^M w_j f(x^j)T_l(x^j)\sum_{i=1}^M w_i f(x^i)T_l(x^i)\right]$$
$$= \sum_{l=1}^\infty \beta_l \left[\int_\Omega f(x)T_l(x)d\Omega(x) - \sum_{i=1}^M w_i f(x^i)T_l(x^i)\right]^2.$$

Taking into account Lemma 1 and Assumption 3 we obtain

$$\sum_{l=1}^{\infty}\beta_l\left[\int_\Omega f(x)T_l(x)d\Omega(x)-\sum_{i=1}^{M}w_i f(x^i)T_l(x^i)\right]^2$$
$$=\sum_{l=1}^{N}\beta_l\left[\int_\Omega f(x)T_l(x)d\Omega(x)-\sum_{i=1}^{M}w_i f(x^i)T_l(x^i)\right]^2$$
$$+\sum_{l=N+1}^{\infty}\beta_l\left[\int_\Omega f(x)T_l(x)d\Omega(x)-\sum_{i=1}^{M}w_i f(x^i)T_l(x^i)\right]^2$$
$$\leq\sum_{l=1}^{N}4\beta_l\mu_\Omega^2\tau^2(N)\|f\|_{W^\tau}^2\|T_l\|_{C(\Omega)}^2+\sum_{l=N+1}^{\infty}4\beta_l\mu_\Omega^2\|f\|_{W^\tau}^2\|T_l\|_{C(\Omega)}^2$$
$$\leq 4\mu_\Omega^2\tau^2(N)\|f\|_{W^\tau}^2(1+\tau^2(0))$$

that completes the proof. □

Theorem 1. *Let Assumptions 1 – 5 be satisfied. Then it holds*

$$\|S_\Omega - S_{w,M}\|_{H_K\to L_2(\omega)}\leq c_2\tau(N), \tag{9}$$

where $c_2 = 2\mu_\Omega c_1\sqrt{1+\tau^2(0)}$.

Proof. For the orthonormal basis $\{e_k\}_{k=1}^{\infty}\in L_2(\omega)$ Parseval's identity asserts that for $S_\Omega c - S_{w,M}c \in L_2(\omega)$

$$\|S_\Omega c - S_{w,M}c\|_{H_K\to L_2(\omega)} = \sqrt{\sum_\nu \langle S_\Omega c - S_{w,M}c, e_\nu\rangle_{L_2(\omega)}^2}. \tag{10}$$

Thus, to proof the theorem it is necessary to bound the corresponding inner product in (10).

Since $c(x)$ can be represented by using (3) we have

$$S_\Omega c - S_{w,M}c = \int_\Omega s(t,x)c(x)d\Omega(x) - \sum_{j=1}^{M} w_j s(t,x^j)c(x^j) = \int_\Omega s(t,x)\langle c, K_x\rangle_{H_K} d\Omega(x)$$
$$-\sum_{j=1}^{M} w_j s(t,x^j)\langle c, K_{x^j}\rangle_{H_K} = \left\langle c, \int_\Omega s(t,x)K_x d\Omega(x) - \sum_{j=1}^{M} w_j s(t,x^j)K_{x^j}\right\rangle_{H_K}.$$

Then using Cauchy-Schwarz inequality and Lemma 2 we get

$$\begin{aligned}
\langle S_\Omega c - S_{w,M}c, e_\nu\rangle &= \int_\omega e_\nu(t)\left\langle c, \int_\Omega s(t,x)K_x d\Omega(x) - \sum_{j=1}^M w_j s(t,x^j)K_{x^j}\right\rangle_{H_K} d\omega(t)\\
&= \left\langle c, \int_\Omega S_\nu(x)K_x d\Omega(x) - \sum_{j=1}^M w_j K_{x^j} S_\nu(x^j)\right\rangle_{H_K}\\
&\le \|c\|_{H_K}\left\|\int_\Omega S_\nu(x)K_x d\Omega(x) - \sum_{j=1}^M w_j K_{x^j} S_\nu(x^j)\right\|_{H_K}\\
&\le \|c\|_{H_K} 2\mu_\Omega \sqrt{1+\tau^2(0)}\|S_\nu(x)\|_{W^\tau}\tau(N).
\end{aligned}$$

Substituting the inequality above into (10) and taking into account Assumption 5 we can bound

$$\begin{aligned}
\|S_\Omega c - S_{w,M}c\|_{H_K\to L_2(\omega)} &\le 2\mu_\Omega\sqrt{1+\tau^2(0)}\tau(N)\|c\|_{H_K}\sqrt{\sum_\nu \|S_\nu(x)\|_{W^\tau}}\\
&\le 2\mu_\Omega c_1\sqrt{1+\tau^2(0)}\|c\|_{H_K}\tau(N).
\end{aligned}$$

□

4 Regularization

As we already mentioned in the Introduction, Fredholm integral equations of the first kind are usually ill-posed, and therefore regularization is needed for their stable solving.

Consider now a noisy version of the equation (1) that can be written as

$$S_\Omega c = u^\delta, \tag{11}$$

where u^δ is such that $\|u - u^\delta\| \le \delta$. We assume that (11) for $\delta = 0$ has solutions and denote by $c^\dagger$ the so-called best approximate of minimal norm solution. It is natural [14] to consider that $c^\dagger \in H_K$ belongs to the range of the operator $\phi(S_\Omega^* S_\Omega)$ for some index function ϕ (i.e. continuous, strictly increasing function and such that $\phi(0) = 0$). By S_Ω^* we denote the adjoint of S_Ω.

Furthermore we assume that the index function ϕ is operator monotone.

Definition 1. The function ϕ is operator monotone on $(0,a)$ if for any pair of self-adjoint operators $A,B : L_2 \to L_2$ with spectrum in $(0,a)$, where $a = \max\{\|A\|,\|B\|\}$, we have $\phi(A) \le \phi(B)$ whenever $A \le B$.

It is known [9] that if ϕ is an operator monotone function on $(0,a)$ then for any pair of self-adjoint operators A,B, $\|A\|,\|B\| \le b$, $b < a$ there exists a constant d_1 such that

$$\|\phi(A) - \phi(B)\| \le d_1\phi(\|A-B\|). \tag{12}$$

Moreover from [13] one can find that the operator monotone functions satisfy the inequality

$$\phi(t)/t \le T\phi(s)/s, \quad \text{whenever} \quad 0 < s < t < a, \tag{13}$$

where T is some constant. Summarizing our discussion above we impose the following assumption.

Assumption 6. *Let $c^\dagger = \phi(S_\Omega^* S_\Omega)v$, where $\|v\|_{H_K} \le 1$, and ϕ is an operator monotone function on the interval $(0, \|S_\Omega^* S_\Omega\|_{L_2(\Omega)}]$.*

For regularization of the equation (11) we consider general regularization scheme combined with the discretization according to (7), namely the solution $c_{\alpha,\delta}^N$ is calculated as $c_{\alpha,\delta}^N := g_\alpha(S_{w,M}^* S_{w,M}) S_{w,M}^* u^\delta$, where $\{g_\alpha(\lambda)\}, 0 < \lambda \le \|S_\Omega^* S_\Omega\|_{L_2(\Omega)}$ is a parametric family of bounded functions with α being a regularization parameter.

Of course, not every family can be used as a regularization.

Definition 2. A family $\{g_\alpha\}$ is called a regularization, if there are constants $\chi_{-1}, \chi_{1/2}$, χ_0 for which

$$\sup_{0<\lambda\le\|S_\Omega^* S_\Omega\|_{L_2(\Omega)}} |1 - \lambda g_\alpha(\lambda)| \le \chi_0$$

$$\sup_{0<\lambda\le\|S_\Omega^* S_\Omega\|_{L_2(\Omega)}} \sqrt{\lambda}\, |g_\alpha(\lambda)| \le \frac{\chi_{1/2}}{\sqrt{\alpha}}$$

$$\sup_{0<\lambda\le\|S_\Omega^* S_\Omega\|_{L_2(\Omega)}} |g_\alpha(\lambda)| \le \frac{\chi_{-1}}{\alpha}.$$

Moreover, due to (13) the following properties of $\{g_\alpha(\lambda)\}, 0 < \lambda \le \|S_\Omega^* S_\Omega\|_{L_2(\Omega)}$ can be derived

$$\sup_{0<\lambda\le\|S_\Omega^* S_\Omega\|_{L_2(\Omega)}} |1 - \lambda g_\alpha(\lambda)|\phi(\lambda) \le \chi\phi(\alpha), \tag{14}$$

Remark 5. For Tiknonov-Phillips regularization, which will be used in the next section, it holds

$$g_\alpha(\lambda) = \frac{1}{\alpha + \lambda},$$

and the above mentioned conditions are satisfied with $\chi_{-1} = \chi_0 = \chi = 1, \chi_{1/2} = 1/2$.

Next we prove the following lemma.

Lemma 3. *Under Assumptions 1-6 it holds*

$$\|c^\dagger - c_{\alpha,\delta}^N\|_{H_K} \le \chi\phi(\alpha) + \chi_0 d_1 \phi(c_2\tau(N)) + \frac{\chi_{1/2}\delta}{2\sqrt{\alpha}} + \frac{\chi_{1/2} c_2 \tau(N) \|c^\dagger\|_{H_K}}{2\sqrt{\alpha}}.$$

Proof. The statement of the lemma follows from [10, formula (4)], but for completeness we present the proof here as well.

Taking into account the above properties of the regularization family $g_\alpha(\lambda)$ we have

$$\begin{aligned}\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} &\le \|(I-g_\alpha(S^*_{w,m}S_{w,m})S^*_{w,m}S_{w,m})\phi(S^*_{w,m}S_{w,m})\|_{H_K\to L_2(\omega)}\\ &+\|(I-g_\alpha(S^*_{w,m}S_{w,m})S^*_{w,m}S_{w,m})(\phi(S^*_{w,m}S_{w,m})-\phi(S^*_\Omega S_\Omega)\|_{H_K\to L_2(\omega)}\\ &+\|g_\alpha(S^*_{w,m}S_{w,m})S^*_{w,m}(u-u^\delta)\|_{H_K\to L_2(\omega)}\\ &+\|g_\alpha(S^*_{w,m}S_{w,m})S^*_{w,m}(S_\Omega - S_{w,m})\|_{H_K\to L_2(\omega)}\|c^\dagger\|_{H_K}\\ &\le \chi\phi(\alpha)+\chi_0\|\phi(S^*_{w,m}S_{w,m})-\phi(S^*_\Omega S_\Omega)\|_{L_2(\omega)}+\chi_{1/2}\frac{\delta}{\sqrt{\alpha}}\\ &+\chi_{1/2}\|c^\dagger\|_{H_K}\frac{\|S_{w,m}-S_\Omega\|_{H_K\to L_2(\omega)}}{\sqrt{\alpha}}.\end{aligned}$$

Using the property (12) and Theorem 1 we complete the proof:

$$\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} \le \chi\phi(\alpha)+\chi_0 d_1\phi(c_2\tau(N))+\chi_{1/2}\frac{\delta}{\sqrt{\alpha}}+\chi_{1/2}\|c^\dagger\|_{H_K}\frac{c_2\tau(N)}{\sqrt{\alpha}}.$$

□

Theorem 2. *Let N be the smallest positive integer such that*

$$c_2\tau(N) \le \begin{cases}\alpha, & \phi(t)\ge\sqrt{t}\\ \delta, & \phi(t)<\sqrt{t}\end{cases}.$$

Then for $\alpha=\theta^{-1}(\delta)$, where $\theta(t)=\sqrt{t}\phi(t), \quad t\in[0,\|S_\Omega\|^2]$, we have

$$\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} \le c\phi(\theta^{-1}(\delta)),$$

with some constant c that does not depend on δ, N, M.

Proof.
If $\phi(\lambda)\ge\sqrt{\lambda}$, $t\in[0,\|S_\Omega\|^2]$, then from Lemma 3 we conclude that

$$\begin{aligned}\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} &\le \chi\phi(\alpha)+\chi_0 d_1\phi(\alpha)+\frac{\chi_{1/2}\delta}{2\sqrt{\alpha}}+\frac{\chi_{1/2}\sqrt{\alpha}\|c^\dagger\|_{H_K}}{2}\\ &\le(\chi+\chi_0 d_1+\frac{\chi_{1/2}\|c^\dagger\|_{H_K}}{2})\phi(\alpha)+\frac{\chi_{1/2}\delta}{2\sqrt{\alpha}}.\end{aligned}$$

Thus from the definition of the function $\theta(t)$ it holds that $\delta/\sqrt{\theta^{-1}(\delta)}=\phi(\theta^{-1}(\delta))$ and we obtain

$$\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} \le c\phi(\theta^{-1}(\delta)).$$

For the case $\phi(\lambda)<\sqrt{\lambda}$, $t\in[0,\|S_\Omega\|^2]$, from Lemma 3 the following estimation holds true

$$\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} \le \chi\phi(\alpha) + \chi_0 d_1 \phi(\delta) + \frac{\chi_{1/2}\delta}{2\sqrt{\alpha}} + \frac{\chi_{1/2}\delta\|c^\dagger\|_{H_K}}{2\sqrt{\alpha}}.$$

Taking into account that $\phi(\delta) < \sqrt{\delta}$ and $\alpha = \theta^{-1}(\delta)$ we finally obtain

$$\|c^\dagger - c^N_{\alpha,\delta}\|_{H_K} \le c\phi(\theta^{-1}(\delta)).$$

□

Remark 6. It is known [11] that the error bound $\phi(\theta^{-1}(\delta))$ is optimal by the order for $c^\dagger \in Range(\phi(S^*_\Omega S_\Omega))$, i.e. it can't be improved for the class of solutions under consideration.

5 Algorithms and Numerical Illustrations

In this section we compare two discretization strategies for Fredholm integral equation of the first kind: the regularized collocation method studied in [15] and the regularized quadrature approximation described in the present paper. For reader convenience we describe both algorithm in ready-to-use form below. Note that the presented descriptions correspond to the Tikhonov-Phillips regularization scheme, namely $g_\alpha(\lambda) = 1/(\alpha+\lambda)$.

5.1 Regularized Collocation Method

According to [15], the solution $c^N_{\alpha,\delta}$ can be represented as

$$c^N_{\alpha,\delta} = \sum_{j=1}^{M} c_j w_j s(t_j,\cdot),$$

where the vector $\mathbf{c} \in \mathbb{R}^M$ of the coefficients c_j can be found from the system

$$\alpha\mathbf{c} + A\mathbf{c} = \mathbf{u}^\delta,$$

with $A = MW$,

$$W = diag(\omega_1,\ldots,\omega_M),\ \ M = [m_{ij}],\ \ m_{ij} = \int_0^1 s(t_j,t)s(t_i,t)dt,$$
$$\mathbf{u}^\delta = (u^\delta(t_1),\ldots,u^\delta(t_M)).$$

5.2 Regularized Quadrature Method

Recall that for a regularization of the equation (11) we use Tikhonov-Phillips method combined with the discretization according to (7), namely

$$\alpha c + S^*_{w,M} S_{w,M} c = S^*_{w,M} u^\delta, \tag{15}$$

where for any $b(t) \in L_2(\omega)$

$$(S^*_{w,M} b)(\cdot) = \sum_{j=1}^{M} w_j K(\cdot, x^j) \int_\omega s(t, x^j) b(t) d\omega(t). \tag{16}$$

Note that the solution of (15) belongs to $Range(S^*_{w,M})$. It means that due to (16) the element $c^N_{\alpha,\delta}$ can be represented as

$$c^N_{\delta,\alpha} = \sum_{k=1}^{M} c_k K(\cdot, x^k).$$

Thus, the solution of the equation (15) is derived from the system of linear equations with respect to $c_k, k = 1,..,M$, namely the values c_k can be found from the system

$$\alpha c_k + \sum_{p=1}^{M} c_p s_{k,p} = u^\delta_k, \quad k = 1..M,$$

where $u^\delta_k = \int_\omega s(t, x^k) u^\delta(t) d\omega(t)$ and

$$s_{k,p} = \sum_{\mu=1}^{M} w_k w_\mu K(x^\mu, x^p) \int_\omega s(t, x^\mu) s(t, x^k) d\omega(t).$$

5.3 Numerical Comparison

In our numerical tests we put both algorithms side-by-side for integral equations (1) defined on $\omega = \Omega = [0,1]$ with kernels

$$s(t,x) = \sum_{l=1}^{D} \sum_{m=1}^{D} d_{lm} \frac{\cos(2\pi l t)\cos(2\pi m x)}{(2\pi l)^p (2\pi m)^q},$$

and the exact solutions

$$c^\dagger(t) = \sum_{k=1}^{D} c^\dagger_k \frac{\cos(2\pi k t)}{(2\pi k)^\nu},$$

where $d_{lm}, l, m = 1, \ldots, D$ and $c^\dagger_k, k = 1, \ldots, D$ are uniformly distributed random numbers from $(0,1)$. In our experiments $D = 100$, $p = 1$, $q = 3$, and $\nu = 2$.

We choose such values of the parameters p and q to demonstrate the advantage in the sense of the accuracy of the quadrature method over the collocation in the case when kernels $s(t,x)$ are smoother with respect to integration variables.

It is clear that for such kernels and solutions the right-hand sides $u(t)$ can be calculated explicitly as well:

$$u(t) = \sum_{l=1}^{D}\sum_{m=1}^{D} d_{lm}\frac{\cos(2\pi lt)c_k}{2(2\pi l)^p(2\pi m)^{q+\nu}}.$$

Then we generate the noisy data u^δ as follows

$$u^\delta(t) = \sum_{l=1}^{D}\cos(2\pi lt)\sum_{m=1}^{D}\left(d_{lm}\frac{c_k}{2(2\pi l)^p(2\pi m)^{q+\nu}} + \delta\xi_l\right),$$

where δ is the noise intensity and $\xi_l, l = 1,\dots,D$ are uniformly distributed random numbers from $(-1,1)$.

Both considered methods are tested with $t_i = x^i = \frac{i}{M}, i = 1,2,\dots,M, M = 100$. Moreover, we consider a quadrature rule with equal weights.

Note that for the realization of the quadrature method one should also define the RKHS H_K and for simplicity we consider $K(x,x') = \sum_{j=1}^{M} l_j(x)l_j(x')$, where $l_j, j = 1,2,\dots,M$ are the fundamental interpolation functions associated with the knots $x^j, j = 1,2,\dots,M$. In this case $K(x^i,x^j) = \delta_{ij}$.

We employed Tikhonov-Phillips regularization with the optimal choice of α from the set $\alpha_i = 10^{-20}\cdot 1.08^{i-1}, i = 1,2\dots,1000$.

Table 1 reports mean relative error over 10 simulations as described above. Figure 1 shows results of a particular simulation. The presented results demonstrate the advantage of a quadrature method in the case when kernels of integral operator are more smooth with respect to the integration variables.

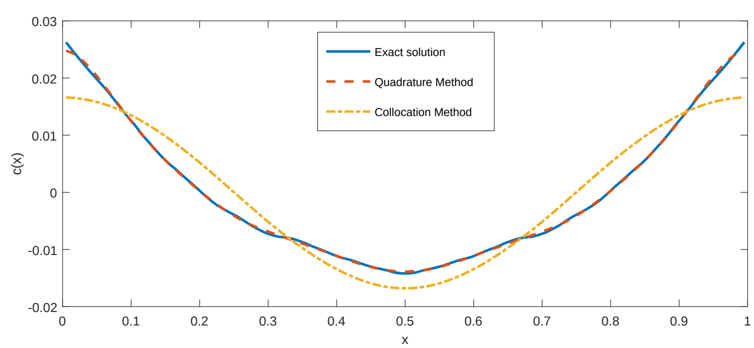

Fig. 1 Reconstruction of $c^\dagger$ (Exact solution) by the Collocation and Quadrature Methods, $\delta = 10^{-9}$

Table 1 Average errors of collocation and quadrature methods over 10 simulations of the noisy data .

	Mean Relative Error				
	$\delta = 10^{-5}$	$\delta = 10^{-6}$	$\delta = 10^{-7}$	$\delta = 10^{-8}$	$\delta = 10^{-9}$
Collocation Method	0.4615	0.1764	0.1542	0.1547	0.1547
Quadrature Method	0.4021	0.1749	0.1040	0.0722	0.0380

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References

1. Atkinson K.E.: A survey of numerical methods for the solution of Fredholm integral equations of the second kind. SIAM, Philadelphia, 1976, 230 pp.
2. Cohen, A., DeVore, R., Kerkyacharian, G., Picard, D.: Maximal spaces with given rate of convergence for thresholding algorithms. Appl. Comput. Harmon. Anal. **11**, 167–191 (2001)
3. Dzyadyk, V. K., Shevchuk, I. A.: Theory of uniform approximation of functions by polynomials. Walter de Gruyter GmbHCo. KG, Berlin, (2008)
4. Erb, W., Kaethner, C., Ahlborg, M., Buzug, T. M.: Bivariate Lagrange interpolation at the node points of non-degenerate Lissajous curves. Numer. Math. **133**, 685–705 (2016)
5. Graham, I.G., Atkinson, K.E.: On the Sloan iteration applied to integral equations of the first kind. IMA J. Numer. Anal. **13**, 29–41 (1993)
6. Groetsch, C.W.: Convergence analysis of a regularized degenerate kernel methods for Fredholm integral equations of the first kind. Integral Equations and Operator Theory **13**, 63–75 (1990)
7. Krebs, J., Louis, A.K., Wendland, H.: Sobolev error estimates and a priori parameter selection for semi-discrete Tikhonov regularization. J. Inv. Ill-Posed Problems, **17**, 845–869 (2009)
8. Kress R.: Linear integral equations. Springer, Heidelberg, 1989, 398 pp.
9. Mathé, P., Pereverzev, S.V. Moduli of continuity for operator valued functions Numer. Funct. Anal. Optim. **23**, 623–631 (2002)
10. Mathé, P., Pereverzev, S.V.: Discretization strategy for ill-posed problems in variable Hilbert scales Inverse Problems. **19**, 1263–1277 (2003)
11. Mathé, P., Pereverzev, S.V.: Geometry of linear ill-posed problems in variable Hilbert scales Inverse Problems. **19**, 789–803 (2003)
12. Maas, P., Pereverzev, S.V., Ramlau R., Solodky, S.G.: An adaptive discretization for Tikhonov-Phillips regularization with a posteriori parameter selection. Numer. Math. **87**, 485–502 (2001)
13. Mathé, P., Pereverzev, Sergei V.: Regularization of some linear ill-posed problems with discretized random noisy data. Math. Comp. **75**, 1913–1929 (2006)
14. Mathé, P., Hofmann, B.: How general are general source conditions? Inverse Problems, **24**, 1–5 (2008)
15. Nair, M.T., Pereverzev, S.: Regularized collocation method for Fredholm integral equations of the first kind. J. Complex. **23**, 454–467 (2007)
16. N. Panagiotopoulos, R. Duschka, M. Ahlborg, G. Bringout, C. Debbeler, M. Graeser, C. Kaethner, K. Lüdtke-Buzug, H. Medimagh, J. Stelzner, T. M. Buzug, J. Barkhausen, F. M. Vogt, and J. Haegele: Magnetic particle imaging – current developments and future directions. International Journal of Nanomedicine, **10**, 3097–3114 (2015)

17. Plato, R., Vainikko, G.: On the regularization of projection methods for solving ill-posed problems. Numer. Math. **57**, 63–79 (1990)
18. Pereverzev S.V.: Optimization of methods for approximate solution of operator equations. Nova Science Publishers, New York, 1996, 352 pp.
19. Pereverzev, S.V., Solodky, S.G., Volynets, E.A.: The balancing principle in solving semi-discrete inverse problems in Sobolev scales by Tikhonov method. Applicable Analysis, **91**, 435–446 (2012)
20. Reinhard H.-J.: Analysis of Approximation methods for differential and integral equations. Springer, New York, 1985, 398 pp.
21. Sloan, Ian H.: Error analysis for a class of degenerate kernel methods. Numer. Math. **25**, 231–238 (1976)